INFINITE SEQUENCES WITH SIGN-SYMMETRIC LIOUVILLE DISTRIBUTIONS

BY

RAMESHWAR D. GUPTA* (St. John), JOLANTA K. MISIEWICZ (Wroclaw),
AND DONALD ST. P. RICHARDS** (Charlottesville, Virginia)

Abstract. We define and study the sign-symmetric Dirichlet-type and Liouville-type distributions on n-dimensional space. We give a complete stochastic representation for infinite sequences of random variables \( \{X_n: n \in \mathbb{N}\} \) with the property that for every \( n \in \mathbb{N} \) the random vector \((X_1, \ldots, X_n)\) has a sign-symmetric Liouville-type distribution.

1. Introduction. In this paper we study a family of multivariate distributions which contains as special cases many well-known classes of distributions, including the spherically symmetric distributions [3]. The distributions treated here may be obtained by the following construction. Let \( Z_1, \ldots, Z_n \) be mutually independent, real-valued, random variables, where the probability density function of \( Z_i \) is

\[
\frac{\alpha_i}{2^\beta_i \Gamma(\beta_i)} |z_i|^{\beta_i - 1} \exp(-|z_i|^{\alpha_i})
\]

for \( z_i \in \mathbb{R} \) and positive parameters \( \alpha_i \) and \( \beta_i, i = 1, \ldots, n \). Further, define

\[
U_i = \frac{Z_i}{\left(\sum_{j=1}^{n} |Z_j|^{\alpha_j}\right)^{1/\alpha_i}}
\]

for \( i = 1, \ldots, n \). This construction is similar to a well-known procedure for constructing the Dirichlet distributions using gamma random variables [11]. The distribution of the random vector \((U_1, \ldots, U_n)\) is called the sign-symmetric Dirichlet-type distribution.

We shall also study the distribution of the random vector

\[
(X_1, \ldots, X_n) = (U_1 \Theta^{1/\alpha_1}, \ldots, U_n \Theta^{1/\alpha_n}),
\]

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where \( \Theta \) is a positive random variable independently distributed of \((U_1, \ldots, U_n)\). We call the distribution of the random vector \((X_1, \ldots, X_n)\) the sign-symmetric Liouville-type distribution. This construction is similar to a well-known procedure for constructing the multivariate Liouville distributions using Dirichlet random variables (cf. [5], [7]—[10], [18]).

When \( \alpha_i = 2 \), the distribution of \((X_1, \ldots, X_n)\) has been considered in [4]; in addition, when \( \beta_i = 1 \), this class reduces further to the spherically symmetric (or rotationally invariant) distributions, a family that is widely utilized in probability and statistics both in finite-dimensional settings (cf. [3], [5], [12], [17]) and infinite-dimensional applications (cf. [13], [14], [17]).

When \( \alpha_i = \alpha \) and \( \beta_i = 1 \), the distribution of \((X_1, \ldots, X_n)\) reduces to the \( l_\alpha \)-isotropic distributions (cf. [1], [2], [15]), so-called because they have the nice property that the level curves of their densities are spheres in the \( l_\alpha \)-norm on \( \mathbb{R}^n \). It has been shown recently [16] that, within Bayesian analysis, the \( l_\alpha \)-isotropic distributions satisfy certain robustness properties known to hold for independent, identically distributed data.

To explain our choice of terminology, note that the distribution of the random vector \((X_1, \ldots, X_n)\) is sign-symmetric if it has the same distribution as \((r_1X_1, \ldots, r_nX_n)\), where \( r_1, \ldots, r_n \) is the Rademacher sequence of independent (and independent of \((X_1, \ldots, X_n)\)) random variables with \( P(r_i = 1) = P(r_i = -1) = 1/2 \). We use the terms “sign-symmetric Dirichlet-type distribution” and “sign-symmetric Liouville-type distribution” to underline that our distributions differ from the Dirichlet and Liouville distributions not only by their sign-symmetry but that they also differ by the shape of their supports.

In Section 2 we derive certain preliminary results for the sign-symmetric Dirichlet-type distribution. In Section 3 we give some basic properties of the sign-symmetric Liouville-type distribution, including marginal and conditional distributions, and formulas for moments and correlations. We also deduce the extent to which a sign-symmetric Liouville-type random vector \((X_1, \ldots, X_n)\) has a unique representation of the form (1.3) for some sign-symmetric Dirichlet-type random vector \((U_1, \ldots, U_n)\) and positive random variable \( \Theta \).

Section 4 contains the main results of this paper. We prove that an infinite sequence of sign-symmetric Dirichlet-type random variables exists if and only if \( \sum_{i=1}^{\infty} \beta_i/\alpha_i < \infty \), and in that case we construct the sequence. In both cases, \( \sum_{i=1}^{\infty} \beta_i/\alpha_i < \infty \) or \( \sum_{i=1}^{\infty} \beta_i/\alpha_i = \infty \), we derive a stochastic representation for infinite sequences of sign-symmetric Liouville-type random variables. It turns out that in the former case, the sequence of sign-symmetric Liouville-type random variables is a scale mixture of a related sign-symmetric Dirichlet-type sequence; while in the latter case, it is a scale mixture of a sequence of independent random variables each having density (1.1). This situation is completely unlike that of the exchangeable \( l_\alpha \)-isotropic sequences [1], [2], [15]; in that context it is not possible to obtain more than one type of stochastic
representation for exchangeable \( l_x \)-isotropic sequences of random variables, for then we have \( \sum_{i=1}^{n} \beta_i/\alpha_i \to \infty \) as \( n \to \infty \).

Finally, it should be noted that all the techniques used in this paper also apply to the corresponding distributions on the positive orthant; that is, the distributions of the random vectors \((U_1, \ldots, U_n)\) defined as in (1.2) but where the random variables \( Z_i \) have densities

\[ \frac{\alpha_i}{\Gamma(\beta_i/\alpha_i)} z_i^{\beta_i-1} \exp(-z_i) \]

for \( z_i > 0, i = 1, \ldots, n \). Therefore all our results have natural analogues for the classical Dirichlet and Liouville random vectors as well as for the distributions considered in [19].

2. The sign-symmetric Dirichlet-type distributions. Throughout \( \alpha_1, \ldots, \alpha_n \) and \( \beta_1, \ldots, \beta_n \) are positive parameters. We use the notation \( p_i = \sum_{j=1}^{i} \beta_j/\alpha_j \) for \( i = 1, \ldots, n \). We also let \((a)_+ = a \) or \( 0 \) according as \( a > 0 \) or \( a \leq 0 \), respectively.

2.1. Lemma. We have

\[ \int_{\mathbb{R}^{n-1}} \left(1 - \sum_{i=1}^{n} |u_i|^\alpha_i\right)^{\beta_n/\alpha_n - 1} \prod_{i=1}^{n-1} |u_i|^\beta_i - 1 \, du_i = 2^{\alpha_1 - 1} \frac{\Gamma(\beta_n/\alpha_n)}{\Gamma(p_n) \prod_{i=1}^{n-1} \alpha_i} \frac{\Gamma(\beta_i/\alpha_i)}{\Gamma(\beta_i/\alpha_i)} \prod_{i=1}^{n-1} \alpha_i \]

Proof. To prove this result, we use the symmetry of the integrand to reduce the domain of integration from \( \mathbb{R}^{n-1} \) to \( \mathbb{R}^+ \) (where \( \mathbb{R}^+ = [0, \infty) \)); replace each \( u_i \) by \( u_i^{1/\alpha_i} \); and then apply the classical Dirichlet integral [5], [7].

2.2. Definition. A random vector \((U_1, \ldots, U_n)\) is said to have a sign-symmetric Dirichlet-type distribution with parameters \( \alpha_1, \ldots, \alpha_n \) and \( \beta_1, \ldots, \beta_n \) if

(i) \( U_n \) is a symmetric random variable;

(ii) \( \sum_{i=1}^{n} |U_i|^\alpha_i = 1 \) almost surely; and

(iii) the joint density function of \((U_1, \ldots, U_{n-1})\) is

\[ \frac{\Gamma(p_n)}{\Gamma(\beta_n/\alpha_n) \prod_{i=1}^{n-1} \alpha_i} \frac{\Gamma(\beta_i/\alpha_i)}{\Gamma(\beta_i/\alpha_i)} \prod_{i=1}^{n-1} \alpha_i \]

(2.1)

Whenever Definition 2.2 holds we write

\[ (U_1, \ldots, U_n) \sim \mathcal{D}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n). \]

If \( \alpha_i \equiv 1 \), then the distribution of \((U_1, \ldots, U_n)\) may be called the sign-symmetric Dirichlet distribution. If \( \alpha_i \equiv 2 \) and \( \beta_i \equiv 1 \), then the random vector \((U_1, \ldots, U_n)\) is uniformly distributed on the unit sphere \( \{ (x_1, \ldots, x_n) : x_1^2 + \ldots + x_n^2 = 1 \} \) in \( \mathbb{R}^n \) (see [3] and [12]). If \( \alpha_i \equiv x \) and \( \beta \equiv 1 \), then \((U_1, \ldots, U_n)\) is \( l_x \)-isotropic [1], [2], [15].
We now collect some basic properties of the distributions \( \mathcal{U} \mathcal{D}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n) \). We will first deal with the properties of the marginal distributions.

2.3. Proposition. Suppose that \((U_1, \ldots, U_n) \sim \mathcal{U} \mathcal{D}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)\) and \(1 \leq k < n\).

(i) The marginal density function of \((U_1, \ldots, U_k)\) is

\[
\frac{\Gamma(p_n)}{\Gamma(p_n - p_k)} \prod_{i=1}^{k} \frac{\alpha_i}{2\Gamma(\beta_i/\alpha_i)} |u_i|^{\beta_i - 1} \left(1 - \sum_{i=1}^{k} |u_i|^{2\alpha_i}\right)^{p_n - p_k - 1}.
\]

(ii) Let \(e\) be a random variable taking values \(\pm 1\) each with probability \(1/2\), and independent of \((U_1, \ldots, U_n)\). Then for any \(\alpha > 0\)

\[
(U_1, \ldots, U_k, e \left( \sum_{i=k+1}^{n} |U_i|^{2\alpha_i} \right)^{1/\alpha}) \sim \mathcal{U} \mathcal{D}(\alpha_1, \ldots, \alpha_k, \alpha; \beta_1, \ldots, \beta_k, (p_n - p_k)\alpha).
\]

Proof. (i) The marginal density function of \((U_1, \ldots, U_k)\) is

\[
\frac{\Gamma(p_n)}{\Gamma(p_n - p_k)} \prod_{i=1}^{k} \frac{\alpha_i}{2\Gamma(\beta_i/\alpha_i)} \left|u_i\right|^{\beta_i - 1} \left(\sum_{j=k+1}^{n} \left(1 - \sum_{i=1}^{k} \left|u_i\right|^{2\alpha_i}\right)^{-1} \prod_{i=k+1}^{k} |u_i|^{\beta_i - 1} du_i\right).
\]

The desired result can be obtained by substituting

\[
u_j = \left(1 - \sum_{i=1}^{k} |u_i|^{2\alpha_i}\right)^{1/\alpha} y_j
\]

for \(j = k+1, \ldots, n-1\), and then using Lemma 2.1.

(ii) It is not difficult to see that the conditions (i) and (ii) of Definition 2.2 are satisfied. As for the third condition, this follows from the formula obtained above for the marginal distribution of \((U_1, \ldots, U_k)\).

2.4. Remark. As a simple application of (2.2) we obtain the following result. Suppose that \(\alpha\) and \(\beta\) are positive numbers. Also suppose that the \((k+1)\)-dimensional random vector \((V_1, \ldots, V_{k+1}) \sim \mathcal{U} \mathcal{D}(\alpha_1, \ldots, \alpha_k, \alpha; \beta_1, \ldots, \beta_k, \beta)\). If

\[
\beta/\alpha = p_n - p_k,
\]

then it follows from (2.2) that the vectors \((U_1, \ldots, U_k)\) and \((V_1, \ldots, V_k)\) have the same distribution.

2.5. Proposition. Suppose that \((U_1, \ldots, U_n) \sim \mathcal{U} \mathcal{D}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)\) and \(1 < k < n\). Then the conditional density function of \((U_1, \ldots, U_{k-1})\) given \(\{U_{k+1} = u_{k+1}, \ldots, U_n = u_n\}\) is

\[
\frac{\Gamma(p_k)}{\Gamma(p_k - p_n)} \prod_{i=1}^{k-1} \frac{\alpha_i}{2\Gamma(\beta_i/\alpha_i)} |u_i|^{\beta_i - 1} \left(1 - \sum_{i=1}^{k-1} \left|u_i\right|^{2\alpha_i}\right)^{p_k - p_n - 1} \left(1 - \sum_{i=k+1}^{n} \left|u_i\right|^{2\alpha_i}\right)^{1-\alpha_k}.
\]

Proof. This is obtained in the standard way by dividing the joint density function of \((U_1, \ldots, U_{k-1}, U_{k+1}, \ldots, U_n)\) by the marginal density function of \((U_{k+1}, \ldots, U_n)\).
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2.6. Remark. By rewriting the conditional density function of \((U_1, \ldots, U_{k-1})\), given \(\{U_{k+1} = u_{k+1}, \ldots, U_n = u_n\}\) in (2.3), in the form

\[
\frac{\Gamma (p_k) \prod_{i=1}^{k-1} \alpha_i \beta_i^{u_i}}{\prod_{i=k+1}^{n} \Gamma (\beta_i/\alpha_i)} \left( \prod_{i=1}^{k-1} \left( 1 - \sum_{i=1}^{k-1} \alpha_i R^{1/\alpha_i} \right) \right)^{R^{A_i}}.
\]

where \(R = 1 - \sum_{i=1}^{k} |U_i|^\alpha_i\), and \(A_i = -\sum_{i=1}^{k-1} \alpha_i^{-1}\), we see that (2.4) or (2.3) is also the conditional density function of the random vector \((U_1 R^{1/\alpha_1}, \ldots, U_{k-1} R^{1/\alpha_{k-1}})\), given \(R\), where

\[(U_1, \ldots, U_k) \sim \mathcal{D}(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_k).\]

2.7. Proposition. (i) For \(h_1, \ldots, h_n > 0\),

\[
E \left( \prod_{i=1}^{n} |U_i|^{h_i} \right) = \frac{\Gamma (p_n) \prod_{i=1}^{n} \Gamma (\beta_i/\alpha_i + h_i/\alpha_i)}{\Gamma (p_n + \sum_{i=1}^{n} h_i/\alpha_i) \prod_{i=1}^{n} \Gamma (\beta_i/\alpha_i)}.
\]

(ii) For \(i, j = 1, \ldots, n, i \neq j\),

\[
E(U_j) = \text{Cov}(U_i, U_j) = 0 \quad \text{and} \quad \text{Var}(U_i) = \frac{\Gamma (p_n) \Gamma (\beta_i/\alpha_i + 2/\alpha_i)}{\Gamma (p_n + 2/\alpha_i) \Gamma (\beta_i/\alpha_i)}.
\]

(iii) For \(i = 1, \ldots, k\) and \(h > 0\),

\[
E(|U_i|^h \mid U_{k+1} = u_{k+1}, \ldots, U_n = u_n)
\]

\[
= \frac{\Gamma (p_n) \Gamma (\beta_i/\alpha_i + h/\alpha_i)}{\Gamma (p_n + h/\alpha_i) \Gamma (\beta_i/\alpha_i)} \left( 1 \sum_{j=k+1}^{n} |U_j|^\alpha_j \right)^{h/\alpha_i}.
\]

Proof. These formulas follow immediately from Lemma 2.1. ■

2.8. Remark. In the Introduction it was implied that if \(Z_1, \ldots, Z_n\) are independent random variables with density function (1.1), and \(U_1, \ldots, U_n\) are defined as in (1.2), then

\[(U_1, \ldots, U_n) \sim \mathcal{D}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n).\]

This claim can be verified by observing that \(U_n\) is a symmetric random variable; that \(\sum_{i=1}^{n} |U_i|^\alpha_i = 1\) with probability 1; and that the joint density function of \((U_1, \ldots, U_{n-1})\), which is calculated from the joint density of \(Z_1, \ldots, Z_n\), is of the form (2.1).

3. The sign-symmetric Liouville-type distributions.

3.1. Definition. A random vector \((X_1, \ldots, X_n)\) is said to have a sign-symmetric Liouville-type distribution if there exist a sign-symmetric Dirichlet-type random vector \((U_1, \ldots, U_n) \sim \mathcal{D}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)\) and a nonnegative random variable \(\Theta\), independent of \((U_1, \ldots, U_n)\), such that

\[(X_1, \ldots, X_n) \cong (U_1 \Theta^{1/\alpha_1}, \ldots, U_n \Theta^{1/\alpha_n}).\]
Whenever Definition 3.1 holds we will use the notation

$$(X_1, \ldots, X_n) \sim \mathcal{D}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta).$$

**3.2. PROPOSITION.** Suppose that $(X_1, \ldots, X_n) \sim \mathcal{D}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta)$ and $\Theta$ is absolutely continuous with probability density function $g$. Then the joint density function of $(X_1, \ldots, X_n)$ is

$$f(x_1, \ldots, x_n) = \frac{\lambda_n}{2^n \prod_{i=1}^n |x_i|^\beta_i} \exp \left\{ -\frac{1}{2} \sum_{i=1}^n |x_i|^\alpha_i \right\} g \left( \sum_{i=1}^n |x_i|^\alpha_i \right),$$

where $x_1, \ldots, x_n \in \mathbb{R}$.

**Proof.** Since $\sum_{i=1}^n |U_i|^\alpha_i = 1$ almost surely, it follows from Definition 3.1 that $\Theta = \sum_{i=1}^n |X_i|^\alpha_i$ almost surely. Then the result is obtained using the standard method of transformations. ♦

**3.3. EXAMPLES.** (i) Let $\alpha_i \equiv 2$ and $\beta_i \equiv 1$. Then the random vector

$$(X_1, \ldots, X_n) \sim \mathcal{D}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta)$$

has a spherically symmetric distribution [3], [12].

(ii) If $\alpha_i \equiv 1$, then $(X_1, \ldots, X_n)$ may be viewed as a sign-symmetric Liouville distribution.

(iii) If $\alpha_i \equiv \alpha > 0$ and $\beta_i \equiv 1$, then $(X_1, \ldots, X_n)$ has an $l_\alpha$-isotropic distribution [15]. In the case where $(X_1, \ldots, X_n)$ is absolutely continuous, the density function is constant on the $l_\alpha$-spheres $\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^\alpha = c \}$, $c$ constant.

(iv) Suppose that $\Theta$ has a beta distribution with parameters $p_n$ and $\beta/\alpha$, where $\alpha, \beta > 0$. Applying Proposition 3.2 and Proposition 2.3 (i) we deduce that the distribution of $(X_1, \ldots, X_n)$ is the same as the distribution of $(Y_1, \ldots, Y_n)$, where $(Y_1, \ldots, Y_n)$ is a subvector of the vector

$$(Y_1, \ldots, Y_{n+1}) \sim \mathcal{D}(\alpha_1, \ldots, \alpha_n, \beta; \beta_1, \ldots, \beta_n).$$

Note that the distribution of $(X_1, \ldots, X_n)$ depends on $\alpha$ and $\beta$ only through the ratio $\beta/\alpha$; cf. Remark 2.4.

(v) Similarly to the independence properties of the Liouville distributions [7], the random variables $X_1, \ldots, X_n$ are mutually independent if and only if each $X_i$ has density function

$$f_i(x_i) = \frac{\alpha_i b^{\beta_i/\alpha_i}}{2^n \prod_{i=1}^n |x_i|^\beta_i} \exp \left\{ -\frac{1}{2} \sum_{i=1}^n |x_i|^\alpha_i \right\} \quad \text{for } x_i \in \mathbb{R}, \ i = 1, \ldots, n.$$

In that case, $\Theta$ has density function

$$f(\Theta) = \frac{b^{p_n}}{\Gamma(p_n)} r^{p_n - 1} e^{-br} \quad \text{for } r > 0.$$
The marginal distributions of $(X_1, \ldots, X_n) \sim \mathcal{S}\mathcal{L}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta)$ are given in the following result.

3.4. PROPOSITION. Suppose that $(X_1, \ldots, X_n) \sim \mathcal{S}\mathcal{L}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta)$ and $1 \leq k < n$. Then

$$(X_1, \ldots, X_k) \sim \mathcal{S}\mathcal{L}(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_k; \Theta_k),$$

where

$$\Theta_k = (1 - \sum_{i=k+1}^{n} \left| U_i \right|^{\alpha_i}) \Theta.$$

Moreover, $\Theta_k$ is absolutely continuous (with respect to Lebesgue measure) and the density function $g_k$ of $\Theta_k$ is

$$g_k(r) = \frac{\Gamma(p_k)}{\Gamma(p_k) \Gamma(p_n - p_k)} \int_0^\infty (t - r)^{p_n - p_k - 1} t^{1 - p_n} \lambda(dt)$$

for $r > 0$, where $\lambda$ is the distribution function of $\Theta$.

Proof. By Definition 3.1 we have

$$(X_1, \ldots, X_k) = (U_1 \Theta^{1/\alpha_1}, \ldots, U_k \Theta^{1/\alpha_k}).$$

Letting $R_k = 1 - \sum_{i=k+1}^{n} \left| U_i \right|^{\alpha_i}$ and $U_i = U_i R_k^{-1/\alpha_i}$, $i = 1, \ldots, k$, then we obtain

$$\Theta R_k = \Theta_k \text{ and } (X_1, \ldots, X_k) = (U'_1 (R_k \Theta)^{1/\alpha_1}, \ldots, U'_k (R_k \Theta)^{1/\alpha_k}).$$

Moreover, by applying the techniques used to establish Proposition 2.3 we deduce that $R_k$ has a beta distribution with parameters $p_k$ and $p_n - p_k$.

Next, it follows from the definition of the sign-symmetric Dirichlet-type distributions that $(U'_1, \ldots, U'_k) \sim \mathcal{D}\mathcal{D}(\beta_1, \ldots, \beta_k)$ and is independent of $R_k \Theta = \Theta_k$. Finally, the density function of $\Theta_k$ is calculated directly using the representation $\Theta_k = R_k \Theta$ and the known distribution of $R_k$. $\blacksquare$

We now study the uniqueness of the representation $\mathcal{S}\mathcal{L}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta)$ for the sign-symmetric Liouville-type distributions.

3.5. PROPOSITION. The representation $(X_1, \ldots, X_n) \sim \mathcal{S}\mathcal{L}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta)$ of a sign-symmetric Liouville-type random vector $(X_1, \ldots, X_n)$ is unique if

(i) $n = 2$ and $\Theta$ has a continuous density function, or
(ii) $n \geq 3$.

Proof. We will only prove (ii) since the proof of (i) is similar. Thus suppose that

$$(X_1, \ldots, X_n) \sim \mathcal{S}\mathcal{L}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta)$$
and also that
\[(X_1, \ldots, X_n) \sim \mathcal{D} (\alpha_1', \ldots, \alpha_n'; \beta_1', \ldots, \beta_n'; \Theta).
\]
Now consider the random vector \((X_1, \ldots, X_{n-1})\). By Proposition 3.4 there exist positive random variables \(\Theta_{n-1}\) and \(\Theta_{n-1}'\) such that
\[(X_1, \ldots, X_{n-1}) \sim \mathcal{D} (\alpha_1, \ldots, \alpha_{n-1}; \beta_1, \ldots, \beta_{n-1}; \Theta_{n-1})
\]
and
\[(X_1, \ldots, X_{n-1}) \sim \mathcal{D} (\alpha_1', \ldots, \alpha_{n-1}; \beta_1', \ldots, \beta_{n-1}' ; \Theta_{n-1}).
\]
Moreover, \(\Theta_{n-1}\) and \(\Theta_{n-1}'\) are absolutely continuous with continuous density functions, say, \(g_{n-1}\) and \(g_{n-1}'\), respectively. Applying Proposition 3.2 with \(\Theta_{n-1}\) and \(\Theta_{n-1}'\) we obtain two representations for the density function of \((X_1, \ldots, X_{n-1})\). Setting
\[f_{n-1} (r) = r^{1-p_{n-1}} g_{n-1} (r) \quad \text{and} \quad \bar{f} (r) = r^{1-p_{n-1}} g_{n-1}' (r) \quad \text{for} \quad r > 0,
\]
and then comparing the two representations for the density of \((X_1, \ldots, X_{n-1})\) we obtain the identity
\[f_{n-1} \left( \sum_{i=1}^{n-1} |x_i|^\alpha_i \right) \prod_{i=1}^{n-1} |x_i|^{\beta_i-1} = c_1 \bar{f}_{n-1} \left( \sum_{i=1}^{n-1} |x_i|^\alpha_i \right) \prod_{i=1}^{n-1} |x_i|^{\beta_i-1},
\]
where \(c_1\) is a constant.

Now suppose that there exists \(j < n\) such that \(\beta_j \neq \beta_j'\), and without loss of generality assume that \(\beta_j > \beta_j'\); by renumbering the variables in (3.2) we can even assume that \(j = 1\). Then (3.2) reduces to
\[f_{n-1} \left( \sum_{i=1}^{n-1} |x_i|^\alpha_i \right) |x_1|^{\beta_1-\beta_1'} \prod_{i=2}^{n-1} |x_i|^{\beta_i-1} = c_1 \bar{f}_{n-1} \left( \sum_{i=1}^{n-1} |x_i|^\alpha_i \right) \prod_{i=2}^{n-1} |x_i|^{\beta_i-1}.
\]
As \(|x_1| \to 0\), the left-hand side of (3.3) tends to zero for every fixed \(x_2, \ldots, x_{n-1}\). This implies that the right-hand side of (3.3) is identically zero, which eventually leads to the conclusion \(g_{n-1} = 0\). This is clearly a contradiction. Therefore \(\beta_1 \leq \beta_1'\). If \(\beta_1 < \beta_1'\), then a similar argument leads to the conclusion \(g_{n-1} = 0\), again a contradiction. Therefore \(\beta_1 = \beta_1'\). By considering the random vector \((X_2, \ldots, X_n)\) instead of \((X_1, \ldots, X_{n-1})\) and following the above procedure, we eventually obtain \(\beta_j = \beta_j'\) for all \(j = 1, \ldots, n\).

Applying this conclusion to (3.3), we are now faced with the identity
\[f_{n-1} \left( \sum_{i=1}^{n-1} |x_i|^\alpha_i \right) = c_2 \bar{f}_{n-1} \left( \sum_{i=1}^{n-1} |x_i|^\alpha_i \right),
\]
where \(c_2\) is a constant. By equating the level curves of both sides of this identity we find that for any \(r > 0\)
\[
\{(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}; \sum_{i=1}^{n-1} |x_i|^\alpha_i = r\} = \{(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}; \sum_{i=1}^{n-1} |x_i|^\alpha_i = c_3 r\},
\]
where the constant $c_3$ is possibly dependent on $r$. It is well known that this last identity holds if and only if $\alpha_i = \alpha'_i$ for all $i = 1, \ldots, n-1$. As before, we also deduce that $\alpha_n = \alpha'_n$ by working with the random vector $(X_2, \ldots, X_n)$.

In conclusion, the parameters $\alpha_i$ and $\beta_i$ are uniquely determined by the distribution of $(X_1, \ldots, X_n)$. In turn, the distribution of $\Theta$ is also uniquely determined by the constraint $\Theta = \sum_{i=1}^{n} |X_i|^\alpha_i$ almost surely.

We now turn to the behavior of the conditional distributions of sign-symmetric Liouville-type random vectors.

**3.6. Proposition.** Suppose that $(X_1, \ldots, X_n) \sim \mathcal{L}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta)$ where $\Theta$ is absolutely continuous with density function $g$, and $1 \leq k < n$. Then the conditional distribution of $(X_1, \ldots, X_k)$ given $\{X_{k+1}, \ldots, X_n\}$ is $\mathcal{L}(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_k; \Theta_c)$, where $\Theta_c$ is the conditional random variable $\sum_{i=1}^{k} |X_i|^{\alpha_i}$ given $\sum_{i=1}^{n} |X_i|^{\alpha_i}$.

**Proof.** Since $\Theta$ is absolutely continuous, the joint density function of $(X_1, \ldots, X_n)$ is given by Proposition 3.2. Also the marginal density function of $(X_{k+1}, \ldots, X_n)$ is obtained from Proposition 3.4. Dividing the joint density by the marginal density, we find that the conditional density of $(X_1, \ldots, X_k)$ given $\{X_{k+1} = x_{k+1}, \ldots, X_n = x_n\}$ may be written as

$$
\Gamma(p_k) \prod_{i=1}^{k} \frac{\alpha_i}{2\Gamma(\beta_i/\alpha_i)} |x_i|^{\beta_i - 1} \left(\sum_{i=1}^{k} |x_i|^{\alpha_i}\right)^{1-p_k}
$$

$$
\times \left(\sum_{i=1}^{k} |x_i|^{\alpha_i}\right)^{p_k} \left(\sum_{k+1}^{n} |x_i|^{\alpha_i}\right)^{p_n-p_k-1} \left(\sum_{i=1}^{n} |x_i|^{\alpha_i}\right)^{1-p_n} \frac{g(\sum_{i=1}^{k} |x_i|^{\alpha_i})}{g_{n-k}(\sum_{k+1}^{n} |x_i|^{\alpha_i})},
$$

where $g_k$ is defined in Proposition 3.4.

Noting the general expression for the density given in Proposition 3.2, we see that the above expression for the conditional density is the same as the joint density of $\mathcal{L}(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_k; \Theta_c)$, where $\Theta_c$ has the density

$$
X^{|X_{k+1}|^{\alpha_1} \ldots X^{|X_n|^{\alpha_k}}} (x) \frac{g(x+r)}{g_{n-k}(r)}
$$

and $r = \sum_{i=k+1}^{n} |x_i|^{\alpha_i}$. It is straightforward to verify that this last expression is the density of the random variable $\sum_{i=1}^{k} |X_i|^{\alpha_i}$ given $\sum_{k+1}^{n} |X_i|^{\alpha_i} = r$.

In ending this section, we note that the calculation of the moments of a sign-symmetric Liouville-type random vector $(X_1, \ldots, X_n)$ proceeds directly from Definition 3.1 and the formulas for the moments of the sign-symmetric Dirichlet-type random variables. In particular, if $E(\Theta^{2k+1}) < \infty$, then, for any $k \in \mathbb{N}$, $E(X_i^{2k+1}) = 0$, $i = 1, \ldots, n$.

**4. Infinite sequences with sign-symmetric Liouville-type distributions.** In this section we treat the properties of infinite sequences of random variables for which every finite subsequence follows a sign-symmetric Liouville-type distributions.
4.1. **Definition.** An infinite sequence of random variables \( \{U_i : i \in \mathbb{N}\} \) is said to have a *sign-symmetric Dirichlet-type distribution* if for every \( n \in \mathbb{N} \) there exists a symmetric random variable \( W_n \) such that the random vector \((U_1, \ldots, U_n, W_n)\) has a sign-symmetric Dirichlet-type distribution.

4.2. **Proposition.** Suppose that the sequence \( \{U_i : i \in \mathbb{N}\} \) has a sign-symmetric Dirichlet-type distribution. Then there exist sequences of positive numbers \( \alpha_i, \beta_i, \gamma_i \) and \( \delta_i, i \in \mathbb{N} \), such that for every \( n \in \mathbb{N} \) the random vector \((U_1, \ldots, U_n, W_n) \sim \mathcal{D}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n, \delta_n)\). Moreover, the sequences \( \alpha_i, \beta_i \) and \( \delta_i/\gamma_i \) are uniquely determined by the distribution of \( \{U_i\} \).

**Proof.** Choose an \( n \in \mathbb{N} \). By an application of Proposition 2.3 (i) we find that the density function of \((U_1, \ldots, U_n)\) exists and is proportional to

\[
(1 - \sum_{i=1}^{n} |u_i|^{\alpha_i})^{-1} \prod_{i=1}^{n} |u_i|^{\beta_i - 1}.
\]

Since \( n \) is arbitrary, this also establishes the existence of the sequences \( \alpha_i, \beta_i, \gamma_i \) and \( \delta_i \). Moreover, the uniqueness of these sequences follows from Proposition 3.5.

4.3. **Definition.** An infinite sequence of random variables \( \{X_i : i \in \mathbb{N}\} \) is said to have a *sign-symmetric Liouville-type distribution* if for every \( n \in \mathbb{N} \) the random vector \((X_1, \ldots, X_n)\) has a sign-symmetric Liouville-type distribution.

4.4. **Proposition.** Suppose that \( \{X_i : i \in \mathbb{N}\} \) has a sign-symmetric Liouville-type distribution. Then there exist sequences of positive numbers \( \alpha_i \) and \( \beta_i, i \in \mathbb{N} \), and an increasing sequence of random variables \( \Theta_1 \leq \Theta_2 \leq \ldots \) such that, for every \( n \in \mathbb{N} \),

\[
(X_1, \ldots, X_n) \sim \mathcal{L}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta_n).
\]

**Proof.** The existence of the sequences \( \alpha_i \) and \( \beta_i, i \in \mathbb{N} \), follows as in Proposition 4.2.

Next, again choose \( n \in \mathbb{N} \). By the definition of the sign-symmetric Liouville-type vector, there exists a sign-symmetric Dirichlet-type vector \((U_1, \ldots, U_n) \sim \mathcal{D}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)\) and a nonnegative random variable \( \Theta_n \) such that

\[
(X_1, \ldots, X_n) = (U_1 \Theta_1^{1/\alpha_1}, \ldots, U_n \Theta_n^{1/\alpha_n}).
\]

Moreover, this representation also holds for \((X_1, \ldots, X_{n+1})\), so that there exists a sign-symmetric Dirichlet-type random vector \((U_1', \ldots, U_{n+1}')\), with distribution \( \mathcal{D}(\alpha'_1, \ldots, \alpha'_{n+1}; \beta'_1, \ldots, \beta'_{n+1}) \), and a nonnegative random variable \( \Theta_{n+1} \) such that

\[
(4.1) \quad (X_1, \ldots, X_{n+1}) = (U_1' \Theta_{n+1}^{1/\alpha'_1}, \ldots, U_{n+1}' \Theta_{n+1}^{1/\alpha'_{n+1}}).
\]
Now we calculate the marginal distribution of \((X_1, \ldots, X_n)\) starting from (4.1). Then we obtain

\[(X_1, \ldots, X_n) = (V_1 (R\Theta_{n+1})^{1/\alpha_1}, \ldots, V_n (R\Theta_{n+1})^{1/\alpha_n}),\]

where \(R = (1 - |U_{n+1}|^{\beta_{n+1}})\) and \(V_i = R^{1/\alpha_i} U_i, i = 1, \ldots, n\). Then it is evident that

\[(V_1, \ldots, V_n) \sim \mathcal{D}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n).\]

Next, it follows from Proposition 3.5 that \(\alpha_i = \alpha'_i\) and \(\beta_i = \beta'_i, i = 1, \ldots, n\), and also that

\[\Theta_n = (1 - |U_{n+1}|^{\beta_{n+1}}) \Theta_{n+1} \leq \Theta_{n+1}. \]

We will use the notation \(\{X_i: i \in \mathbb{N}\} \sim \mathcal{D}(\alpha_i; \beta_i; \Theta_i)\) whenever a sequence of random variables \(X_1, X_2, \ldots\) has a sign-symmetric Liouville-type distribution as described in Definition 4.3. If the associated random variables \(\Theta_i\) each has a beta distribution with parameters \(p_i\) and \(\delta_{i/\gamma_i}, i = 1, 2, \ldots\), then it follows as in the proof of Proposition 4.4 that the distribution of the sequence \(\{X_i\}\) reduces to a sign-symmetric Dirichlet-type distribution, which will be written as \(\{X_i: i \in \mathbb{N}\} \sim \mathcal{D}(\alpha_i; \beta_i)\).

**4.5. Theorem.** Suppose that \(\{X_i: i \in \mathbb{N}\}\) is a sequence of random variables such that

\[\{X_i: i \in \mathbb{N}\} \sim \mathcal{D}(\alpha_i; \beta_i; \Theta_i),\]

where \(\{\alpha_i\}\) and \(\{\beta_i\}\) satisfy \(\sum_{i=1}^{\infty} \beta_i/\alpha_i = \infty\). Then there exists a probability measure \(\lambda\) on \(\mathbb{R}_+\) such that, for all \(n \in \mathbb{N}\), the joint density function of the random vector \((X_1, \ldots, X_n)\) is

\[\prod_{i=1}^{n} \frac{\alpha_i}{2\Gamma(\beta_i/\alpha_i)} |x_i|^{\beta_i - 1} \int_0^\infty \exp \left\{ -s \sum_{i=1}^{n} |x_i|^{\alpha_i} \right\} \lambda(ds).\]

Equivalently, there exists a nonnegative random variable \(\Theta\) and a sequence of mutually independent random variables \(\{Y_i: i \in \mathbb{N}\}\), all independent of \(\Theta\), where each \(Y_i\) has density function (1.1), and such that

\[(X_1, \ldots, X_n) = (Y_1 \Theta^{1/\alpha_1}, \ldots, Y_n \Theta^{1/\alpha_n}) \quad \text{for all } n \in \mathbb{N}.\]

**Proof.** The method of proof will be similar to the one used in Theorem 2 of [15]. Thus, choose \(k \in \mathbb{N}\). Then for any \(n > k\) we may regard the distribution of \((X_1, \ldots, X_k)\) as a \(k\)-dimensional marginal distribution of \((X_1, \ldots, X_n)\); this defines the random variable \(\Theta_n\) for any \(n > k\). By Proposition 3.4,

\[\Theta_n = (1 - |U_{n+1}|^{\beta_{n+1}}) \Theta_{n+1};\]

hence \(\Theta_n\) is absolutely continuous and with a continuous density function \(g_n\). Furthermore, the density function \(g_k\) of \(\Theta_k\) is given in Proposition 3.4.
Letting \( f_i(r) \equiv r^{1-p_i} g_i(r) \), \( i \in N \), we can write the formula for \( f_k \) as

\[
f_k(r) = \frac{\Gamma(p_n)}{\Gamma(p_k) \Gamma(p_n - p_k)} \int_0^\infty (t-r)^{p_n-p_k-1} f_n(t) \, dt.
\]

By [20] it follows that the function \( f_k \) is \((p_n - p_k)\)-monotone. Letting \( n \to \infty \) and noting that \( p_n - p_k = \sum_{i=k+1}^{n} \beta_i / \alpha_i \to \infty \), it also follows that \( f_k \) is completely monotone. That is, there exists a positive measure \( \lambda_k \) on \( \mathbb{R}_+ \) such that \( f_k \) is the Laplace transform of \( \lambda_k \),

\[
f_k(r) = \int_0^\infty e^{-sr} \lambda_k(ds)
\]

for \( r > 0 \). Now choose \( n = k + 1 \) in (4.2), substitute (4.3) into the right-hand side of the resulting formula, and interchange the order of integration. Then we obtain

\[
f_k(r) = \frac{\Gamma(p_{k+1})}{\Gamma(p_k) \Gamma(p_{k+1}/\alpha_k + 1)} \int_0^\infty t^{(p_{k+1}/\alpha_k + 1)-1} f_{k+1}(r+t) \, dt
\]

\[
= \frac{\Gamma(p_{k+1})}{\Gamma(p_k) \Gamma(p_{k+1}/\alpha_k + 1)} \int_0^\infty t^{(p_{k+1}/\alpha_k + 1)-1} \int_0^\infty e^{-s(r+t)} \lambda_{k+1}(ds) \, dt
\]

\[
= \frac{\Gamma(p_{k+1})}{\Gamma(p_k)} \int_0^\infty s^{-(p_{k+1}/\alpha_k + 1)} e^{-sr} \lambda_{k+1}(ds).
\]

Comparing (4.4) with (4.3) and applying the uniqueness of the Laplace transform, we get

\[
\lambda_{k+1}(ds) = \frac{\Gamma(p_k)}{\Gamma(p_{k+1})} s^{p_{k+1}/\alpha_k + 1} \lambda_k(ds).
\]

Using this recurrence relation we obtain

\[
\lambda_k(ds) = \left( \prod_{i=2}^{k} \frac{\Gamma(p_{i-1})}{\Gamma(p_i)} s^{\beta_i/\alpha_i} \right) \lambda_1(ds) = \frac{\Gamma(p_1)}{\Gamma(p_k)} s^{p_k-p_1} \lambda_1(ds).
\]

Since \( g_1(r) = r^{p_1-1} f_1(r) \), \( r > 0 \), is the density function of \( |X_1|^p_1 \), we have

\[
1 = \int_0^\infty r^{p_1-1} f_1(r) \, dr = \int_0^\infty r^{p_1-1} \int_0^\infty e^{-sr} \lambda_1(ds) \, dr = \Gamma(p_1) \int_0^\infty s^{-p_1} \lambda_1(ds),
\]

proving that the measure \( \lambda \) defined by \( \lambda(ds) = \Gamma(p_1) s^{-p_1} \lambda_1(ds) \), \( s > 0 \), is a probability measure.

Now we express \( \lambda_k \) in terms of \( \lambda \), and then, by (4.3), we obtain

\[
f_k(r) = \frac{1}{\Gamma(p_k)} \int_0^\infty s^{p_k} e^{-sr} \lambda(ds).
\]

This completes the proof of the first part of the theorem.
To see that the second statement in the theorem implies the first one, it is enough to notice that the desired measure $\lambda$ is the distribution of $\Theta$. The converse follows by defining a random variable $\Theta$ independent of $\{Y_i: i \in \mathbb{N}\}$ and with distribution $\lambda$. 

4.6. **Corollary.** If the $\alpha_i$ and $\beta_i$ satisfy $\sum_{i=1}^{\infty} \beta_i/\alpha_i = \infty$, then there exists no infinite sequence $\{U_i: i \in \mathbb{N}\} \sim \mathcal{D} \{\{\alpha_i\}; \{\beta_i\}\}$, a sign-symmetric Dirichlet-type sequence. In particular, there exists no exchangeable infinite sequence of positive random variables such that every finite subsequence has a Dirichlet distribution.

**Proof.** Assume that such a sequence exists. Then from Proposition 4.2 we obtain a formula for the marginal density of $U_1$. Equating this formula for the marginal density of $U_1$ with the one obtained from Theorem 4.5, we have

$$\frac{\alpha_1}{2} \frac{\Gamma(\beta_1/\alpha_1 + \delta_1/\gamma_1)}{\Gamma(\beta_1/\alpha_1) \Gamma(\delta_1/\gamma_1)} |x|^{\beta_1-1} (1 - |x|^{\alpha_1})^{(\beta_1/\gamma_1)-1}$$

$$= \frac{\alpha_1}{2 \Gamma(\beta_1/\alpha_1)} |x|^{\beta_1-1} \int_0^\infty s^{|\beta_1/\alpha_1|} \exp \{ -s |x|^{\alpha_1} \} \lambda(ds)$$

for some probability measure $\lambda$ on $\mathbb{R}_+$. But the left-hand side of this last formula is a density function with support in the interval $[-1, 1]$, while the right-hand side has unbounded support on $\mathbb{R}_+$. Therefore we have a contradiction.

Finally, the second statement is proved by choosing $\alpha_i \equiv \alpha$ and $\beta_i \equiv \beta$ for $\alpha, \beta > 0$. 

Now we turn to the case where the infinite sequences $\alpha_i$ and $\beta_i$ satisfy

$$\sum_{i=1}^{\infty} \beta_i/\alpha_i = p < \infty.$$ That is, $p_n \rightarrow p < \infty$ as $n \rightarrow \infty$.

4.7. **Remark.** Suppose that $p_n \rightarrow p < \infty$ as $n \rightarrow \infty$. Define a consistent family of measures $\{\mu_n: n \in \mathbb{N}\}$ such that $\mu_n$ on $\mathbb{R}^{[1,\ldots,n]}$ has density function

$$\frac{\Gamma(p)}{\Gamma(p-p_n)} \frac{\prod_{i=1}^{n} \alpha_i}{\prod_{i=1}^{n} \Gamma(\beta_i/\alpha_i)} |x|^{\beta_1-1} (1 - \sum_{i=1}^{n} |x_i|^\alpha)^{p-p_n-1}.$$

By Kolmogorov's extension theorem ([6], p. 121) there exists a sequence of random variables $U_1, U_2, \ldots$ such that for every $n \in \mathbb{N}$ the measure $\mu_n$ is the distribution of the random vector $(U_1, \ldots, U_n)$. Then there exists an infinite sequence of random variables with a sign-symmetric Dirichlet-type distribution.

Consider now the sequence of random variables $R_n = \sum_{i=1}^{n} |U_i|^\alpha$. Then it follows from (4.5) that $E(R_n) = p_n/p \rightarrow 1$ and

$$\text{Var}(R_n) = \frac{p_n(p-p_n)}{p^2(p+1)} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$
Therefore $R_n \to 1$ in probability and also in distribution. Since the sequence $R_n$ is increasing, we also have $R_n \to 1$ almost surely.

Define the infinite sequence of random variables $\{X_i: i \in \mathbb{N}\} = \{U_i \Theta^{1/\alpha_i}: i \in \mathbb{N}\}$, where $\{U_i: i \in \mathbb{N}\}$ is a sequence of sign-symmetric Dirichlet-type random variables, $\Theta$ is a positive random variable with distribution $\lambda$, and $\Theta$ is independent of $\{U_i: i \in \mathbb{N}\}$. Then for each $n \in \mathbb{N}$ the joint density function of the random vector $(X_1, \ldots, X_n)$ is

$$
\frac{\Gamma(p)}{\Gamma(p-p_n)} \prod_{i=1}^{n} \frac{\alpha_i}{2} \frac{x_i^{\beta_i-1}}{\Gamma(\beta_i/\alpha_i)} s^{p-1} \int_0^\infty \left(s - \sum_{i=1}^{n} |x_i|^{\alpha_i}\right)^{p-p_n-1} s^{1-p} \lambda(ds).
$$

This proves that the sequence $\{X_i: i \in \mathbb{N}\} \sim S_L (\{\alpha_i\}; \{\beta_i\}; \{\Theta_i\})$. Moreover, as $n \to \infty$, $\Theta_n = \sum_{i=1}^{n} |X_i|^{\alpha_i} = R_n \Theta \to \Theta$ almost surely.

Note also that $(U_1, \ldots, U_n) \sim S_L (\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; R_n)$, with the random variable $R_n = \sum_{i=1}^{n} |U_i|^{\alpha_i} = 1 - \sum_{n+1}^\infty |U_i|^{\alpha_i}$ almost surely.

The following result provides a converse to the construction of the sign-symmetric Liouville-type distributions given in the previous example.

**4.8. Theorem.** Suppose that $\{X_i: i \in \mathbb{N}\}$ is a sequence of random variables such that

$$
\{X_i: i \in \mathbb{N}\} \sim S_L (\{\alpha_i\}; \{\beta_i\}; \{\Theta_i\}),
$$

where $\{\alpha_i\}$ and $\{\beta_i\}$ satisfy $p_n \to p < \infty$ as $n \to \infty$. Then there exist a sign-symmetric Dirichlet-type sequence $\{U_i: i \in \mathbb{N}\} \sim S_D (\{\alpha_i\}; \{\beta_i\})$ and a non-negative random variable $\Theta_n$ independent of $\{U_i: i \in \mathbb{N}\}$, such that $\{X_i: i \in \mathbb{N}\} = \{U_i \Theta^{1/\alpha_i}: i \in \mathbb{N}\}$.

**Proof.** From Definition 4.3 we infer that for every $n \in \mathbb{N}$ there exists a sign-symmetric Dirichlet-type random vector

$$
(W_{1,n}, \ldots, W_{n,n}) \sim S_D (\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)
$$

and a positive random variable $\Theta_n$ independent of $(W_{1,n}, \ldots, W_{n,n})$ such that

$$
(X_1, \ldots, X_n) = (W_{1,n} \Theta^{1/\alpha_1}, \ldots, W_{n,n} \Theta^{1/\alpha_n}).
$$

Consider now the sign-symmetric Dirichlet-type sequence of random variables constructed as in Remark 4.7. For this sequence, we know that for each $n \in \mathbb{N}$

$$
\left(\frac{U_1}{(\sum_{j=1}^{n} |U_j|^{\alpha_1})^{1/\alpha_1}}, \ldots, \frac{U_n}{(\sum_{j=1}^{n} |U_j|^{\alpha_n})^{1/\alpha_n}}\right) \sim S_D (\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n).
$$

Now, for $i = 1, \ldots, n$, we choose

$$
W_{i,n} = \frac{U_i}{(\sum_{j=1}^{n} |U_j|^{\alpha_i})^{1/\alpha_i}}.
$$
Therefore, for each \( n \in \mathbb{N} \)

\[
(4.6) \quad (X_1, \ldots, X_n) = \left( U_1 \left( \frac{\Theta_n}{\sum_{j=1}^{n} |U_j|^{\alpha_j}} \right)^{1/\alpha_1}, \ldots, U_n \left( \frac{\Theta_n}{\sum_{j=1}^{n} |U_j|^{\alpha_j}} \right)^{1/\alpha_n} \right)
\]

almost surely. Writing out (4.6) for two positive integers \( n \) and \( k, k < n \), we have

\[
(4.7) \quad \Theta_n = \left( \frac{\sum_{j=1}^{k} |U_j|^{\alpha_j}}{\sum_{j=1}^{n} |U_j|^{\alpha_j}} \right) \Theta_k
\]

almost surely. Next, we let \( n \to \infty \) with \( k \) fixed, and apply the result (in Remark 4.7) that \( \sum_{i=1}^{\infty} |U_i|^{\alpha_i} \to 1 \) almost surely. Then the right-hand side of (4.6) converges almost surely. Therefore there exists a random variable \( \Theta \) such that \( \Theta_n \) converges to \( \Theta \) almost surely. Then (4.7) implies

\[
\Theta_k = \Theta \sum_{j=1}^{k} |U_j|^{\alpha_j}
\]

almost surely. Consequently, it follows from (4.6) (with \( n \) replaced by \( k \)) that

\[
(X_1, \ldots, X_k) = (U_1 \Theta^{1/\alpha_1}, \ldots, U_k \Theta^{1/\alpha_k})
\]

almost surely.

It remains to prove that \( \Theta \) is independent of \( \{U_i; i \in \mathbb{N}\} \). To see this, consider \( (X_1, \ldots, X_k) \) as a \( k \)-dimensional marginal subvector of the random vector

\[
(X_1, \ldots, X_n) \sim \mathcal{L} \mathcal{L} (\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta_n), \quad k < n.
\]

By Proposition 3.4, the density function of \( \Theta_k \) is

\[
g_k(r) = \frac{\Gamma(p_n)}{\Gamma(p_k) \Gamma(p_n-p_k)} r^{p_k-1} \int_0^\infty (s-r)^{p_n-p_k-1} s^{1-p_n} \lambda_n(ds) \quad \text{for } r > 0,
\]

where \( \lambda_n \) is the distribution function of \( \Theta_n \). Letting \( n \to \infty \), since \( \Theta_n \to \Theta \), we get \( \lambda_n \to \lambda \), where \( \lambda \) is the distribution function of \( \Theta \). Since \( p_n \to p \), letting \( n \to \infty \) in the above formula for \( g_k \), we obtain

\[
g_k(r) = \frac{\Gamma(p)}{\Gamma(p_k) \Gamma(p-p_k)} r^{p_k-1} \int_0^\infty (s-r)^{p-p_k-1} s^{1-p} \lambda(ds)
\]

for \( r > 0 \). Then the joint density function of \( (X_1, \ldots, X_k) \) is

\[
g_k(x) = \frac{\Gamma(p)}{\Gamma(p_k) \Gamma(p-p_k)} \prod_{i=1}^{k} \frac{x_i^{\alpha_i-1}}{2\Gamma(\beta_i/\alpha_i)} \int_0^\infty \left( \sum_{i=1}^{k} |x_i|^{\alpha_i} \right)^{p-p_k-1} s^{1-p} \lambda(ds)
\]

for \( x_1, \ldots, x_k \in \mathbb{R} \); and by Remark 4.7 this is also the marginal density of \( (U_1 \Theta^{1/\alpha_1}, \ldots, U_k \Theta^{1/\alpha_k}) \), where \( \Theta \) is independent of \( \{U_i; i \in \mathbb{N}\} \). Since \( k \) was chosen arbitrarily, the proof is complete. \( \blacksquare \)
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Rameshwar D. Gupta
Department of Mathematics
Statistics & Computer Science
University of New Brunswick, St. John
New Brunswick, Canada E2L 4L5

Jolanta K. Misiewicz
Institute of Mathematics
Technical University of Wroclaw
50-370 Wroclaw, Poland

Donald St. P. Richards
Division of Statistics, University of Virginia
Charlottesville, Virginia 22903, U.S.A.

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