LAWS OF LARGE NUMBERS ON SIMPLY CONNECTED STEP 2-NILPOTENT LIE GROUPS

BY

DANIEL NEUENSCHWANDER (Biel)

Abstract. The Strong Law of Large Numbers due to Marcinkiewicz and Zygmund is carried over to simply connected step 2-nilpotent Lie groups. Moreover, for such groups, we prove analogues of the classical theorems of Hsu–Robbins–Erdős, respectively Baum–Katz, giving information on the rate of convergence in Laws of Large Numbers.

1. Introduction. Simply connected step 2-nilpotent Lie groups are groups which arise as follows: Let \([\cdot, \cdot]: R^d \times R^d \to R^d(d \geq 0)\) be a skew-symmetric bilinear map such that \([[R^d, R^d], R^d] = \{0\}\). Then \(G = R^d\), equipped with the multiplication

\[x \cdot y = x + y + \frac{1}{2}[x, y],\]

is a group of the above-mentioned type. Clearly, \(e = 0\) and \(x^{-1} = -x\). The best-known (noncommutative) examples are the Heisenberg groups \(H^d\) given by

\[H^d = R^{2d+1} = R^d \times R^d \times R\]

and

\[[x, y] = (0, 0, \langle x^{(1)}, y^{(2)} \rangle - \langle x^{(2)}, y^{(1)} \rangle) \in R^d \times R^d \times R = H^d,\]

where

\[x = (x^{(1)}, x^{(2)}, x^{(3)}), \quad y = (y^{(1)}, y^{(2)}, y^{(3)}) \in R^d \times R^d \times R = H^d.\]

The so-called groups of type \(H\), which arise in the context of composition of quadratic forms, all belong to this class (cf. Kaplan [5]). See also Folland and Stein [3].

For \(G = R\), it was shown by Hsu–Robbins–Erdős that in the Law of Large Numbers complete convergence is equivalent to the finiteness of the second moment. Baum and Katz strengthened the Marcinkiewicz–Zygmund Law of Large Numbers in the sense that there is not only strong convergence, but convergence of certain series which implies complete convergence. Both theorems due to Hsu–Robbins–Erdős, respectively Baum–Katz, may be interpreted
as results concerning the rate of convergence in the corresponding Laws of Large Numbers.

In this paper we will prove an analogue of the Marcinkiewicz–Zygmund Law of Large Numbers (which contains the Kolmogorov Strong Law of Large Numbers for random variables with expectation as a special case) for simply connected step 2-nilpotent Lie groups. Moreover, we carry over the theorems of Hsu–Robbins–Erdős, respectively Baum–Katz, to this context.

2. Preliminaries and notation. For real-valued functions \( f, g \), the notation \( f \lesssim g \) means that there is a constant \( K > 0 \) such that \( f(x) \leq Kg(x) \) for all \( x \).

Let \( G \) be a simply connected step 2-nilpotent Lie group. Let \( V_2 = [G, G] \) be the center of \( G \), and \( V_1 \) a complement of \( V_2 \), i.e.

\[
G = V_1 \oplus V_2.
\]

The notation \( x = (x', x'') \in G \) will always be understood with respect to (1), i.e. \( x' \in V_1, x'' \in V_2 \). For \( a > 0 \), \( x \in G \) put

\[
\delta_a(x) = (ax', a^2 x'').
\]

Clearly, \( \delta_a \) is an automorphism of \( G \). A homogeneous gauge on \( G \) is a continuous function \( |\cdot| : G \to [0, \infty[ \) satisfying

\[
|0| = 0, \quad |x| > 0 \quad (x \in G \setminus \{0\}),
\]

\[
|\delta_a(x)| = a|x| \quad (a > 0, x \in G)
\]

(cf. Goodman [4]). By a compactness argument we have

\[
|−x| \lesssim |x|
\]

(cf. Goodman [4], Lemma 2) and

\[
|x \cdot y| \lesssim |x| + |y| \quad (x, y \in G).
\]

An example is

\[
|x|_1 = (\|x'\|^4 + \|x''\|^2)^{1/4}.
\]

By Goodman [4], Lemma 1, all homogeneous gauges \( |\cdot| \) are equivalent (i.e. \( |\cdot| \lesssim |\cdot|_1 \lesssim |\cdot| \)).

Let \( p > 0 \), \( c \in G \), and let \( X \) be a \( G \)-valued random variable. Since

\[
E|X \cdot c|^p \lesssim E(|X| + |c|)^p,
\]

it follows that

\[
E|X|^p < \infty \Rightarrow E|X \cdot c|^p < \infty.
\]
A sequence \( \{X_n\}_{n \geq 1} \) of random variables is said to converge completely to the random variable \( X \) if for every \( \varepsilon > 0 \)

\[
\sum_{n=1}^{\infty} P(|X_n - X| > \varepsilon) < \infty.
\]

By the Borel–Cantelli Lemma, complete convergence implies a.s. convergence.

3. The Marcinkiewicz–Zygmund Law of Large Numbers. The following lemma is well known:

**Lemma 1.** We have \( ||X'_n||, \sqrt{||X'_n||} \leq |x| \).

Now we prove the following analogue of the classical Marcinkiewicz-Zygmund Strong Law of Large Numbers (see e.g. Chow and Teicher [2], Theorem 5.2.2):

**Theorem 1.** Let \( G \) be a simply connected step 2-nilpotent Lie group, and \( | \cdot | \) an arbitrary homogeneous gauge on \( G \). Assume \( X_1, X_2, \ldots \) are i.i.d. \( G \)-valued random variables defined on some common probability space \((\Omega, \mathcal{B}, P)\). Then for any \( p \in ]0, 2[ \)

\[
\delta_{n-1/p} \left( \prod_{j=1}^{n} (X_j, c) \right) \overset{a.s.}{\to} 0 \quad (n \to \infty)
\]

for some \( c \in G \) iff \( E|X_1|^p < \infty \). If so, then in the case \( 1 \leq p < 2 \) we have

\[
\delta = -EX'_1,
\]

while \( \delta' \) and, in the case \( 0 < p < 1 \), also \( \delta' \) can be chosen arbitrarily.

For the proof of the case \( 1 \leq p < 2 \) we need the following lemma, which is similar to Kronecker’s Lemma (cf. Chow and Teicher [2], Lemma 5.1.2).

**Lemma 2.** For any sequence \( \{a_n\}_{n \geq 1} \subset \mathbb{R}^d \) and \( \{b_n\}_{n \geq 1} \subset ]0, \infty[ \) such that \( b_{n+1} \geq b_n \) \((n \geq 1)\), \( b_n \to \infty \) \((n \to \infty)\), and

\[
\frac{1}{b_n} \sum_{j=1}^{n} a_j \to 0 \quad (n \to \infty)
\]

we have

\[
\frac{1}{b_n^2} \sum_{j=1}^{n} a_j \to 0 \quad (n \to \infty).
\]

**Proof.** For \( \varepsilon > 0 \), choose \( N \in \mathbb{N} \) such that

\[
\frac{1}{b_n} \left\| \sum_{j=1}^{n} a_j \right\| \leq \frac{\varepsilon}{3} \quad (n \geq N).
\]
Then, by summation by parts, we get

$$\frac{1}{b_n^2} \left\| \sum_{j=1}^{n} a_j \right\| = \frac{1}{b_n^2} \left\| \sum_{j=1}^{n} b_j a_j \right\| \leq \frac{1}{b_n^2} \left\| \sum_{j=1}^{n} b_j \right\| + \frac{1}{b_n^2} \left\| \sum_{j=1}^{n-1} (b_{j+1}-b_j) \sum_{i=1}^{j} a_i \right\|$$

$$\leq \frac{\varepsilon}{3} + \frac{1}{b_n^2} \left\| \sum_{j=1}^{N-1} (b_{j+1}-b_j) \sum_{i=1}^{j} a_i \right\| + \frac{\varepsilon}{3} \leq \varepsilon$$

for $n$ large enough. \(\blacksquare\)

**Proof of Theorem 1.1.** The “only if” part may be proved similarly to that in the classical situation (cf. Chow and Teicher [2], p. 122): Since

$$\delta_{n-1/p}(X_n \cdot c) = \delta_{n-1/p} \left( - \prod_{j=1}^{n-1} (X_j \cdot c) \cdot \prod_{j=1}^{n} (X_j \cdot c) \right)$$

$$= \delta_{(n/(n-1))^{-1/p}} \left( \delta_{(n-1)^{-1/p}} \left( - \prod_{j=1}^{n-1} (X_j \cdot c) \right) \cdot \delta_{n^{-1/p}} \left( \prod_{j=1}^{n} (X_j \cdot c) \right) \right)$$

$$\overset{a.s.}{\rightarrow} 0 \quad (n \rightarrow \infty)$$

by (4), it follows from the Borel–Cantelli Lemma that

$$\sum_{n=1}^{\infty} P(|X_n \cdot c| > n^{1/p}) < \infty,$$

which implies $E|X_1 \cdot c|^p < \infty$ by Corollary 4.1.3 of Chow and Teicher [2]. So $E|X_1 \cdot c|^p < \infty$.

2. Assume $0 < p < 1$, $E|X_1|^p < \infty$, and let $c \in G$ be arbitrary. By considering $X_j \cdot c$ instead of $X_j$, we may without loss of generality assume $c = 0$. By Lemma 1 and the classical Marcinkiewicz–Zygmund Strong Law of Large Numbers we have

$$n^{-1/p} \left( \sum_{j=1}^{n} X_j \right) \overset{a.s.}{\rightarrow} 0 \quad (n \rightarrow \infty),$$

so we have to prove

$$n^{-2/p} \left( \prod_{j=1}^{n} X_j \right) \overset{a.s.}{\rightarrow} 0 \quad (n \rightarrow \infty).$$

This is equivalent to

$$n^{-2/p} \left( \sum_{j=1}^{n} X_j \right) + \frac{1}{2} n^{-2/p} \left( \sum_{1 \leq i < j \leq n} [X_i, X_j] \right) \overset{a.s.}{\rightarrow} 0 \quad (n \rightarrow \infty).$$
The first summand tends to 0 a.s. by Lemma 1 and the classical Marcinkiewicz–Zygmund Strong Law of Large Numbers. So it remains to show

\begin{equation}
 n^{-2/p} \left( \sum_{1 \leq i < j \leq n} [X_i, X_j] \right)^{\prime\prime} \overset{a.s.}{\to} 0 \quad (n \to \infty).
\end{equation}

But for this we have

\[ \| n^{-2/p} \left( \sum_{1 \leq i < j \leq n} [X_i, X_j] \right)^{\prime\prime} \| = O \left( \left( n^{-(1/p)} \sum_{j=1}^{n} \| X_j \| \right)^2 \right) \overset{a.s.}{\to} 0 \quad (n \to \infty) \]

by Lemma 1 and the classical Marcinkiewicz–Zygmund Strong Law of Large Numbers, which proves (7), and thus (6).

3. Assume $1 \leq p < 2$, $E|X_1|^p < \infty$, and again without loss of generality $c = 0$, $EX_1 = 0$. Let $c' \in V_2$ be arbitrary. Hence, as above, in order to prove (4), it remains to show (7). Define the $V_2$-valued random variables

\[ Z_n := [n^{-1/p} \sum_{j=1}^{n-1} X_j, X_n]^{\prime\prime} \]

\[ Z_n := Z_n \cdot 1_{A_j}, \quad \text{where} \quad A_j = \{ n^{-1/p} \left( \sum_{j=1}^{n-1} X_j \right)^{\prime} \| \leq 1 \}. \]

Without loss of generality, we may assume that there is a $G$-valued random variable $X$ on $(\Omega, \mathcal{B}, P)$ which is distributed like $X_1$ and independent of $\{X_n\}_{n \geq 1}$. By Lemma 1, $E \| X' \| < \infty$, so since for every projection $p$ onto some coordinate subspace of $V_2$

\[ P (\| p(Z_n) \| \geq x \left| X_1, X_2, \ldots, X_{n-1} \right.) \leq P (\| Z_n \| \geq x \left| X_1, X_2, \ldots, X_{n-1} \right.) \]

\[ \leq P (\| X' \| \geq Kx \left| X_1, X_2, \ldots, X_{n-1} \right.) = P (\| X' \| \geq Kx \left| X_1, X_2, \ldots, X_{n-1} \right.) \]

\[ (0 \leq x < \infty) \]

for some fixed $K > 0$ a.s., the Theorem in Chatterji [1] yields

\[ n^{-1/p} \sum_{j=1}^{n} (Z_j - \alpha_j) \overset{a.s.}{\to} 0 \quad (n \to \infty), \]

where $\alpha_j$ is a $V_2$-valued random variable on $(\Omega, \mathcal{B}, P)$ consisting of the components

\[ E (p(Z_n) | p(Z_1), p(Z_2), \ldots, p(Z_{n-1}) ) \text{ a.s.} \]

By Lemma 1 and the classical Marcinkiewicz–Zygmund Strong Law of Large Numbers, there is a.s. an $N$ (random) such that a.s.
\[ Z_n = Z_n \quad (n \geq N) \]
and
\[ E(Z_n | X_1, X_2, \ldots, X_{n-1}) = E(Z_n | X_1, X_2, \ldots, X_{n-1}) = 0 \quad (n \geq N) \]
(since \( EX'_1 = 0 \)); hence a.s.

\[ \alpha_n = 0 \quad (n \geq N). \]

Thus, by (8) and (9),
\[ n^{-1/p} \sum_{j=1}^{n} Z_j \xrightarrow{a.s.} 0 \quad (n \to \infty). \]

By Lemma 2, putting
\[ b_n := n^{1/p} \quad \text{and} \quad \alpha_n := b_n Z_n = \left[ \sum_{j=1}^{n-1} X_j, X_n \right]^\prime, \]
we obtain (7).

4. The fact that \( c' \) is uniquely determined in the case \( 1 \leq p < 2 \) follows from (5). \( \blacksquare \)

4. Rates of convergence. We will use the following consequence of the Hölder Inequality:

**Lemma 3.** Assume \( x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in G. \) Then for some \( K > 0 \) not depending on \( p, q \)
\[ \| \sum_{i=1}^{n} [x_i, y_i] \| \leq K \cdot \max_{1 \leq i \leq n} \| x_i \| \cdot \sum_{i=1}^{n} \| y_i \|. \]

**Proof.** By Hölder’s Inequality,
\[ \| \sum_{i=1}^{n} [x_i, y_i] \| \leq K \left( \sum_{i=1}^{n} \| x_i \|^p \right)^{1/p} \left( \sum_{i=1}^{n} \| y_i \|^q \right)^{1/q} \]
for \( p, q > 1, \ 1/p + 1/q = 1. \) Now \( q \to 1 \) yields the assertion. \( \blacksquare \)

First we carry over the theorem of Hsu–Robbins–Erdős (cf. Chow and Teicher [2], Corollary 10.4.2):

**Theorem 2.** Let \( G \) be a simply connected step 2-nilpotent Lie group, \( | \cdot | \) a homogeneous gauge on \( G, \) and assume \( \{X_n\}_{n \geq 1} \) are i.i.d. \( G \)-valued random variables. Then

\[ \sum_{n=1}^{\infty} P \left( \left| \delta_{n-1} \left( \prod_{i=1}^{n} (X_i : c) \right) \right| \geq \varrho \right) < \infty \quad \text{for every } \varrho > 0 \]

iff
\[ E|X_1|^2 < \infty, \quad c' = -EX'_1. \]
Proof. As in the proof of Theorem 1, we may assume without loss of
generality that \( c = 0, \sigma X_1 = 0 \). We first prove the “if” direction: By Lemma 3,
\[
|\delta_n^{-1}(\prod_{i=1}^{n} X_i)| \leq Kn^{-1} \left\| \sum_{j=1}^{n} X_j \right\| + Kn^{-2} \left\{ \sum_{j=1}^{n} \left\| X_j \right\| \right\}^{1/2} \\
+ (K/\sqrt{2}) \left\{ n^{-1} \max_{1 \leq j \leq n} \left\| X_j \right\| \cdot n^{-1} \sum_{j=1}^{n} \left\| X_j \right\| \right\}^{1/2} \\
= KT_n^{(1)} + K \sqrt{T_n^{(2)}} + (K/\sqrt{2}) \sqrt{T_n^{(3)} \cdot T_n^{(4)}}.
\]

Suppose \( \sigma > 0 \). We have

\[
\sum_{n=1}^{\infty} P(T_n^{(1)} \geq \sigma) < \infty
\]

by Corollary 10.4.2 of Chow and Teicher [2], and

\[
\sum_{n=1}^{\infty} P(T_n^{(2)} \geq \sigma) < \infty
\]

by Theorem 10.4.1 of Chow and Teicher [2] (with \( \alpha = 2, p = \gamma = 1 \); it is easy to see that the theorem is also valid in the case \( \sigma X_1 \neq 0, \sigma > 1 \)). For \( T_n^{(3)} \) we get

\[
\sum_{n=1}^{\infty} P(T_n^{(3)} \geq \sigma) \leq \sum_{n=1}^{\infty} nP(\left\| X_1 \right\| \geq \sigma n) \\
\leq 1 + \int_{1}^{\infty} (t+1) P(\left\| X_1 \right\| ^2 \geq \sigma^2 t^2) \, dt \\
\leq 1 + H \int_{1}^{\infty} P(\left\| X_1 \right\| ^2 \geq s) \, ds \leq 1 + H E \left\| X_1 \right\| ^2 < \infty.
\]

Again by Corollary 10.4.2 of Chow and Teicher [2] we have

\[
\sum_{n=1}^{\infty} P(T_n^{(4)} - E \left\| X_1 \right\| \geq \sigma) < \infty.
\]

Now without loss of generality \( E \left\| X_1 \right\| > 0 \), for otherwise (12) proves the “if”
direction. Since

\[
P\left(|\delta_n^{-1}(\prod_{i=1}^{n} X_i)| \geq \sigma\right) \leq P\left(T_n^{(1)} \geq \frac{\sigma}{3K}\right) + P\left(T_n^{(2)} \geq \left(\frac{\sigma}{3K}\right)^2\right) \\
+ P\left(T_n^{(3)} \geq \left(\frac{\sigma}{3K}\right)^2 \cdot \frac{1}{E \left\| X_1 \right\|}\right) + P(T_n^{(4)} - E \left\| X_1 \right\| \geq E \left\| X_1 \right\|),
\]

inequalities (11)–(14) yield the assertion.
As far as the "only if" part is concerned, first observe that, by (10) and the Borel–Cantelli Lemma,

\[ \frac{1}{n} \sum_{i=1}^{n} (X'_i + c') \overset{a.s.}{\to} 0; \]

hence by the classical Marcinkiewicz–Zygmund Law of Large Numbers it follows that \( EX'_1 = -c' \). Now we show that \( E |X_1|^2 < \infty \). For this, it suffices to prove, by Corollary 10.4.2 of Chow and Teicher [2], that \( E \|X'_1\| < \infty \). Observe that

\[
\prod_{i=1}^{n} (X'_i \cdot c) + \prod_{i=1}^{n} (X_{n+1-i} \cdot c)
\]

\[
= \sum_{i=1}^{n} (X'_i \cdot c) + \frac{1}{2} \sum_{1 \leq i < j \leq n} [X'_i \cdot c, X'_j \cdot c] + \sum_{i=1}^{n} (X_{n+1-i} \cdot c)
\]

\[
+ \frac{1}{2} \sum_{1 \leq i < j \leq n} [X_{n+1-i} \cdot c, X_{n+1-j} \cdot c]
\]

\[
= 2 \sum_{i=1}^{n} (X'_i \cdot c) + \frac{1}{2} \sum_{1 \leq i < j \leq n} ([X'_i \cdot c, X'_j \cdot c] + [X'_j \cdot c, X'_i \cdot c]) = 2 \sum_{i=1}^{n} (X'_i \cdot c).
\]

Now we may proceed as in the proof of Corollary 10.4.2 in Chow and Teicher [2]: Put \( h = \dim Y_2 \), let \( \{X'_n\}_{n \geq 1} \) be an independent copy of the process \( \{X_n\}_{n \geq 1} \), and put \( Y_n = (X'_n \cdot c) - (X'_n \cdot c) \). Then, by (15), the symmetry of \( Y_n \), and Lévy's Inequality (cf. Chow and Teicher [2], Lemma 3.3.5), we get

\[
1 - P^n(\|Y'_n\| < \beta) = P(\max_{1 \leq i \leq n} \|Y'_n\| \geq \beta) \leq P(\max_{1 \leq i \leq n} \|\sum_{j=1}^{i} Y'_j\| \geq \beta/2)
\]

\[
\leq 2hP(\|\sum_{i=1}^{n} Y'_i\| \geq \beta/2h) \leq 4hP(\|\sum_{i=1}^{n} (X'_i \cdot c)'\| \geq \beta/4h)
\]

\[
\leq 8hP(\|\prod_{i=1}^{n} (X'_i \cdot c)'\| \geq \beta/4h) \leq 8hP(\|\prod_{i=1}^{n} (X'_i \cdot c)\| \geq L \sqrt{\beta/4h})
\]

for some constant \( L > 0 \); hence for \( \gamma = L/\sqrt{4h} \), by (10) and (16) we get

\[
\sum_{n=1}^{\infty} \left( 1 - P^n(\|Y'_n\| < n^2) \right) = \sum_{n=1}^{\infty} P(\|Y'_n\| \geq n^2) \sum_{j=0}^{n-1} P^j(\|Y'_n\| < n^2)
\]

\[
\geq \sum_{n=1}^{\infty} nP(\|Y'_n\| \geq n^2) \left[ \frac{1}{n} \sum_{j=0}^{n-1} (1 - 8hP(\|\prod_{i=1}^{j} (X'_i \cdot c)\| \geq \gamma)) \right].
\]
The expression [...] tends to 1 as $n \to \infty$ by (10), and
\[ \sum_{n=1}^{\infty} n P(\|Y_i\| > n^3) \geq \int_1^{\infty} t P(\|Y_i\| > (t+1)^2) dt \]
\[ \geq \frac{1}{4} \int_1^{\infty} P(\|Y_i\| > s) ds \geq \frac{1}{4}(E\|Y_i\| - 1),\]
so $E\|Y_i\| < \infty$, and thus, by Lemma 10.1.1 of Chow and Teicher [2], $E\|(X_1 \cdot c)\| < \infty$. Since $EX_1' = 0$, it follows that
\[ E(X_1 \cdot c)' = EX_1' + c',\]
so $E\|X_1'\| < \infty$. Hence we have $E|X_1|^2 < \infty$. \(\blacksquare\)

Now we formulate an analogue of the Baum–Katz Theorem (cf. Chow and Teicher [2], Theorem 5.2.7):

**Theorem 3.** Let $G$ be a simply connected step 2-nilpotent Lie group, $|\cdot|$ a homogeneous gauge on $G$, and assume that $\{X_{ni}\}_{n \in \mathbb{N}}$ is i.i.d. $G$-valued random variables. Suppose $0 < p < 2$, $E|X_1|^p < \infty$, and let $c \in G$ be such that $c' = -EX_1'$ in the case $1 < p < 2$. Then if $ap \geq 1$, we have
\[ \sum_{n=1}^{\infty} n^{ap-2} P\left( \max_{1 \leq i \leq n} |\delta_{n^{-1/p}}(\prod_{j=1}^{i} (X_j \cdot c))| > q \right) < \infty \]
for every $q > 0$.

**Proof.** The proof is the same as in the classical case (cf. Chow and Teicher [2], p. 130): Again, without loss of generality, $c = 0$, $EX_1' = 0$. By Theorem 1,
\[ \delta_{n^{-1/p}}(\prod_{i=1}^{n} X_i) \xrightarrow{a.s.} 0 \quad (n \to \infty), \]
so
\[ \max_{1 \leq i \leq n} |\delta_{n^{-1/p}}(\prod_{j=1}^{i} X_j)| \xrightarrow{a.s.} 0 \quad (n \to \infty). \]
Thus, by (2) and (3), we obtain
\[ \max_{n+1 \leq i \leq 2n} |\delta_{n^{-1/p}}(\prod_{j=n+1}^{i} X_j)| = \max_{n+1 \leq i \leq 2n} |\delta_{n^{-1/p}}((-\prod_{j=1}^{n} X_j) \cdot \prod_{j=1}^{i} X_j)| \]
\[ \leq |\delta_{n^{-1/p}}(\prod_{j=1}^{n} X_j)| + \max_{1 \leq i \leq 2n} |\delta_{(2n)^{-1/p}}(\prod_{j=1}^{i} X_j)| \xrightarrow{a.s.} 0 \quad (n \to \infty). \]

**Case 1:** $ap = 1$. Put
\[ q' = 2^{-2a} q. \]
Since the random variables

\[
\{ \max_{2^n+1 \leq i \leq 2^{n+1}} \left| \prod_{j=2^n+1}^{i} X_j \right| \}_{n \geq 1}
\]

are independent, it follows from (17) and the Borel–Cantelli Lemma that

\[
\begin{align*}
\sum_{n=1}^{\infty} \mathbb{P} \left( \max_{2^n+1 \leq i \leq 2^{n+1}} \left| \prod_{j=2^n+1}^{i} X_j \right| \geq 2^n q' \right) \\
= \sum_{n=1}^{\infty} \mathbb{P} \left( \max_{1 \leq i \leq 2^n} \left| \prod_{j=1}^{i} X_j \right| \geq 2^n q' \right) \\
\geq \int_{0}^{\infty} \mathbb{P} \left( \max_{1 \leq i \leq [x]} \left| \prod_{j=1}^{i} X_j \right| \geq 2^x q' x^a \right) dx \\
\geq (\log 2)^{-1} \sum_{n=1}^{\infty} \frac{1}{2^n} \mathbb{P} \left( \max_{1 \leq i \leq n} \left| \prod_{j=1}^{i} X_j \right| \geq 2^a q' (2n)^a \right) \\
\geq (2 \log 2)^{-1} \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P} \left( \max_{1 \leq i \leq n} \left| \prod_{j=1}^{i} X_j \right| \geq qn^a \right).
\end{align*}
\]

Case 2: \( \alpha p > 1 \). Put

\[
q' = 2^{-a^2 p/(\alpha p - 1)} q.
\]

Since for \( n \geq 1 \)

\[
(n+1)^{a p/(\alpha p - 1)} \geq n^{a p/(\alpha p - 1)} + \frac{\alpha p}{\alpha p - 1} n^{1/(\alpha p - 1)} \geq n^{a p/(\alpha p - 1)} + n^{1/(\alpha p - 1)},
\]

the random variables

\[
\{ \max_{n^{a p/(\alpha p - 1)} + 1 \leq i \leq n^{a p/(\alpha p - 1)} + n^{1/(\alpha p - 1)}} \left| \prod_{j=n^{a p/(\alpha p - 1)} + 1}^{i} X_j \right| \}_{n \geq 1}
\]

are independent and, by (17),

\[
\begin{align*}
\max_{n^{a p/(\alpha p - 1)} + 1 \leq i \leq n^{a p/(\alpha p - 1)} + n^{1/(\alpha p - 1)}} \left| \prod_{j=n^{a p/(\alpha p - 1)} + 1}^{i} X_j \right| \\
\leq \max_{n^{a p/(\alpha p - 1)} + 1 \leq 2 n^{a p/(\alpha p - 1)}} \left| \prod_{j=[n^{a p/(\alpha p - 1)} + 1]}^{i} X_j \right| \xrightarrow{a.s.} 0 \quad (n \to \infty),
\end{align*}
\]

\[
\begin{align*}
\begin{aligned}
\max_{n^{a p/(\alpha p - 1)} + 1 \leq i \leq 2 n^{a p/(\alpha p - 1)}} \left| \prod_{j=[n^{a p/(\alpha p - 1)} + 1]}^{i} X_j \right| \\
\leq \max_{n^{a p/(\alpha p - 1)} + 1 \leq n^{a p/(\alpha p - 1)} + n^{1/(\alpha p - 1)}} \left| \prod_{j=[n^{a p/(\alpha p - 1)} + 1]}^{i} X_j \right| \xrightarrow{a.s.} 0 \quad (n \to \infty),
\end{aligned}
\end{align*}
\]

\[
\begin{align*}
\begin{aligned}
\max_{n^{a p/(\alpha p - 1)} + 1 \leq i \leq 2 n^{a p/(\alpha p - 1)}} \left| \prod_{j=[n^{a p/(\alpha p - 1)} + 1]}^{i} X_j \right| \\
\leq \max_{n^{a p/(\alpha p - 1)} + 1 \leq n^{a p/(\alpha p - 1)} + n^{1/(\alpha p - 1)}} \left| \prod_{j=[n^{a p/(\alpha p - 1)} + 1]}^{i} X_j \right| \xrightarrow{a.s.} 0 \quad (n \to \infty),
\end{aligned}
\end{align*}
\]

\[
\begin{align*}
\begin{aligned}
\max_{n^{a p/(\alpha p - 1)} + 1 \leq i \leq 2 n^{a p/(\alpha p - 1)}} \left| \prod_{j=[n^{a p/(\alpha p - 1)} + 1]}^{i} X_j \right| \\
\leq \max_{n^{a p/(\alpha p - 1)} + 1 \leq n^{a p/(\alpha p - 1)} + n^{1/(\alpha p - 1)}} \left| \prod_{j=[n^{a p/(\alpha p - 1)} + 1]}^{i} X_j \right| \xrightarrow{a.s.} 0 \quad (n \to \infty),
\end{aligned}
\end{align*}
\]

\[
\begin{align*}
\begin{aligned}
\max_{n^{a p/(\alpha p - 1)} + 1 \leq i \leq 2 n^{a p/(\alpha p - 1)}} \left| \prod_{j=[n^{a p/(\alpha p - 1)} + 1]}^{i} X_j \right| \\
\leq \max_{n^{a p/(\alpha p - 1)} + 1 \leq n^{a p/(\alpha p - 1)} + n^{1/(\alpha p - 1)}} \left| \prod_{j=[n^{a p/(\alpha p - 1)} + 1]}^{i} X_j \right| \xrightarrow{a.s.} 0 \quad (n \to \infty),
\end{aligned}
\end{align*}
\]

\[
\begin{align*}
\begin{aligned}
\max_{n^{a p/(\alpha p - 1)} + 1 \leq i \leq 2 n^{a p/(\alpha p - 1)}} \left| \prod_{j=[n^{a p/(\alpha p - 1)} + 1]}^{i} X_j \right| \\
\leq \max_{n^{a p/(\alpha p - 1)} + 1 \leq n^{a p/(\alpha p - 1)} + n^{1/(\alpha p - 1)}} \left| \prod_{j=[n^{a p/(\alpha p - 1)} + 1]}^{i} X_j \right| \xrightarrow{a.s.} 0 \quad (n \to \infty),
\end{aligned}
\end{align*}
\]
so, by the Borel–Cantelli Lemma,

$$\sum_{n=1}^{\infty} P\left( \max_{1 \leq i \leq n^{1/(ap-1)}} \left| \prod_{j=1}^{i} X_j \right| \geq n^{2/(ap-1)} q' \right)$$

$$= \sum_{n=1}^{\infty} P\left( \max_{1 \leq i \leq n^{1/(ap-1)}} \left| \prod_{j=1}^{i} X_j \right| \geq n^{2/(ap-1)} q' \right)$$

$$\geq \int P\left( \max_{1 \leq i \leq \lfloor t/(ap-1) \rfloor} \left| \prod_{j=1}^{i} X_j \right| \geq (t+1)^{2/(ap-1)} q' dt \right)$$

$$\geq (ap-1) \int x^{ap-2} P\left( \max_{1 \leq i \leq \lfloor x \rfloor} \left| \prod_{j=1}^{i} X_j \right| \geq 2^{2/(ap-1)} q' x^a \right) dx$$

$$\geq A \sum_{n=1}^{\infty} n^{2p-2} P\left( \max_{1 \leq i \leq n} \left| \prod_{j=1}^{i} X_j \right| \geq 2^{2/(ap-1)} q' (2n)^a \right)$$

$$= A \sum_{n=1}^{\infty} n^{2p-2} P\left( \max_{1 \leq i \leq n} \left| \prod_{j=1}^{i} X_j \right| \geq qn^a \right)$$

for some constant $A > 0$. □

REFERENCES
