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THE ORNSTEIN–UHLENBECK PROCESS ASSOCIATED WITH THE LÉVY LAPLACIAN AND ITS DIRICHLET FORM

BY

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Abstract. We prove the existence of an Ornstein–Uhlenbeck type process associated with the Lévy Laplacian. Like the classical case, the law of the Lévy Brownian motion at time 1 is an invariant probability of this process. The corresponding semigroup is explicitly described and the related Dirichlet form is constructed. There exist other parallels with the classical situation such as the hypercontractivity of the semigroup, an analogue of the Cameron–Martin space, etc. However, unlike the classical case in our setting the cylindrical functions do not form a core of the Dirichlet form, in fact the form is identically zero on them. In this sense the Lévy Ornstein–Uhlenbeck process provides an example of a new type of a *gradient-type* (or classical) Dirichlet form which is essentially infinite dimensional.

1. The Ornstein–Uhlenbeck semigroup associated with the Lévy Brownian motion. Let E be a nuclear Fréchet space, E^* its topological dual, and H a separable Hilbert space continuously and densely embedded into E^*. Since E can be identified with the dual of E^* (endowed with the strong topology, see Schaeffer [26], Chapter IV, § 5) and the embedding $H \to E^*$ defines the injection $E = E^{**} \to H$, we may assume that E is continuously and densely injected into H in such a way that

$$\langle k, v\rangle = (k, v)_H \quad \text{for all } v \in E \text{ and all } k \in H,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality $\langle E^*, E\rangle$, and $(\cdot, \cdot)_H$ the scalar product in H.

This gives a standard triple $E \subset H \subset E^*$ in the terminology of the theory of generalized random processes.

Let $e = \{e_n\}_{n=-\infty}^{\infty}$ be an orthonormal basis in H consisting of elements of E. We assume that for every $x \in E$ the series $\sum_n (x, e_n)_H e_n$ converges to x in the topology of E (such a choice is always possible under our assumptions).

The goal of this paper is to study the Markov processes with formal generators

$$\frac{1}{2}\Delta_L := \frac{1}{2}\lim_{n\to\infty}\frac{1}{2n+1}\sum_{i=-n}^{n}\partial_{e_i}^2, \tag{1.1}$$

$$\frac{1}{2}(\Delta_L - x\nabla) := \frac{1}{2}\lim_{n\to\infty}\left(\frac{1}{2n+1}\sum_{i=-n}^{n}\partial_{e_i}^2 - \sum_{i=1}^{n} x_i\partial_{e_i}\right) \tag{1.2}$$

and, in particular, their finite-dimensional approximations.

Unlike the case of the Volterra–Gross Laplacian (see, e.g., Hida et al. [18] and Kuo [20]), in which the corresponding processes exist in E^*, in our setting there are no processes in E^* with generators of the form (1.1), (1.2). It was shown in Accardi et al. [5] that the parabolic equation associated with the differential operator (1.1) can be given a rigorous meaning in an appropriate function space and that, in this space, the equation has a unique solution for every initial datum. Moreover, the Markov process with generator $\Delta_L/2$ exists in a suitable "compactification" of E. This process is called the *Lévy Brownian motion.* Let us recall this construction. We shall assume that there is a continuous linear homeomorphism Ψ: $E^* \to E^*$, called the *e-shift* (or, simply, the *shift* in the following), such that $\Psi(H) \subset H$ and $\Psi e_j = e_{j+1}$ (clearly, this is a restriction on both the triple and the basis). All the assumptions are satisfied for E consisting of all two-sided sequences (x_n) having finite seminorms $p_k(x) = \sup_k |x_n|(|n|+1)^k$ for all $k > 0$ with the topology defined by these seminorms if one chooses for H the two-sided l^2 with its standard basis $\{e_n\}$.

Denote by $\mathcal{M}$ the space of bounded complex-valued Radon measures μ on E^* satisfying the following two conditions:

$$\mu\circ\Psi^{-1} = \mu \quad (\Psi\text{-invariance}), \tag{1.3}$$

$$\int_{E^*} x_i^2\,|\mu|(dx) < \infty \quad \text{for all } i, \tag{1.4}$$

where $x_i(x) = \langle x, e_i\rangle$. Clearly, for Ψ-invariant measures it suffices to require (1.4) only for $i = 1$. In the following by *measures* we mean countably additive ones.

Note $\Re\mu\circ\Psi^{-1} = \Re\mu$ and $|\mu|\circ\Psi^{-1} = |\mu|$ for each measure $\mu\in\mathcal{M}$.

An important example of an element in the class $\mathcal{M}$ is the Gaussian measure ν on E^* with Fourier transform $\exp\bigl(-(x, x)_H/2\bigr)$ $(x\in E)$.

According to the ergodic theorem (see Parthasarathy [23]), for any measure $\mu\in\mathcal{M}$ the limit

$$\|x\|_\mu^2 := g(x) := \lim_{n\to\infty}\frac{1}{2n+1}\sum_{i=-n}^{n} x_i^2 \tag{1.5}$$

exists μ-a.e. and

$$\int_{E^*} g(x)\,\mu(dx) = \int_{E^*} x_1^2\,\mu(dx).$$

For the function g we use the alternative notation $\|\cdot\|_\mu^2$ in order to express the fact that the domain of convergence of the series above depends on μ and to underline that $\|\cdot\|_\mu$ is a seminorm in an appropriate function space (see Accardi and Obata [4]).

Denote by $\mathscr{F}$ the space of the Fourier transforms of the measures from the class $\mathscr{M}$. Recall that for any measure μ on E^* its *Fourier transform* $\tilde{\mu}$ is a complex function on E defined by

$$\tilde{\mu}(v) = \int_{E^*} \exp(i\langle x, v\rangle)\,\mu(dx).$$

It is easy to check that the class $\mathscr{M}$ is a linear space stable under taking convolutions of measures. Hence, $\mathscr{F}$ is an algebra of bounded functions. In addition, $\mathscr{F}$ is stable under complex conjugation, since for every $f = \tilde{\mu} \in \mathscr{F}$ the conjugate function $\bar{f}$ coincides with the Fourier transform of the measure ν defined by $\nu(B) = \bar{\mu}(-B)$ (clearly, $\nu \in \mathscr{M}$).

The completion $\mathscr{C}$ of the space $\mathscr{F}$ with the sup-norm has a natural structure of a C^*-algebra (see Kadison and Ringrose [19]). Therefore, it is algebraically and topologically isomorphic to the algebra $C(S)$ of all continuous complex functions on a certain compact space S (see Theorem 4.4.3 in [19]). In fact, one can take for S the spectrum of $\mathscr{C}$ so that we may assume that E is continuously embedded (but not injectively) into S. Let us recall that the *spectrum* S of $\mathscr{C}$ is the subset of the unit sphere in the dual of $\mathscr{C}$ which consists of all multiplicative functionals (called also *pure states*, see Proposition 4.4.1 in [19]) and is equipped with the weak-* topology (this set is known to be weak*-compact unlike the whole unit sphere; see Proposition 3.2.20 in [19]). Then for every $x \in E$ the functional $\chi = \Pi(x) \in S$ is defined by $\chi(f) = f(x)$.

Put $\theta := \Pi(0)$.

It is important that the above-mentioned mapping $\Pi\colon E \to S$ agrees with the isomorphism between $\mathscr{C}$ and $C(S)$: denoting temporarily by f^* the element in $C(S)$ corresponding to $f \in \mathscr{C}$ we have $f^*(\Pi(x)) = f(x)$ for all $x \in E$. In particular, for any measure σ on E and any $f \in \mathscr{C}$ we obtain

$$\int_E f(x)\,\sigma(dx) = \int_S f^*(s)\,\sigma \circ \Pi^{-1}(ds).$$

In the sequel we often identify $\mathscr{C}$ and $C(S)$.

Recall that, for a homogeneous Markov process with state space M and transition probabilities $P(t, x, \cdot)$, the transition semigroup (on bounded measurable functions ϕ) is defined by the formula

$$Q_t \phi(x) = \int_M \phi(y)\,P(t, x, dy).$$

Now, let $\{w_n(t)\}$ be a sequence of independent real standard Wiener processes on a probability space (Ω, P). Recall that the standard Ornstein–Uhlenbeck process $\xi^x(t)$ is defined as the solution of the stochastic differential equation

(1.6) $$d\xi^x(t) = dw(t) - \tfrac{1}{2}\xi^x(t)\,dt, \qquad \xi^x(0) = x.$$

$\xi^x(t)$ is a Gaussian process which admits the representation

(1.7) $$\xi^x(t) = e^{-t}x + \int_0^t e^{t-s}\,dw_s,$$

and whose transition semigroup $\{T_t\}$ is given by the formula

$$T_t = T^0_{t/2}, \qquad T^0_t f(x) = \int_{R^1} f(e^{-t}x + \sqrt{1-e^{-2t}}\,y)\,\gamma(dy),$$

where γ is the standard Gaussian measure on the line. Using (1.7) one can easily verify that γ is an invariant probability of the process $\xi^x(t)$.

The generator of the Wiener process on R^n is $\Delta/2$. The generator of the Ornstein–Uhlenbeck process on R^n is $\Delta f(x)/2 - (x, \nabla f(x))/2$. The semigroup T^0_t is called the *Ornstein–Uhlenbeck semigroup.*

The only difference between the semigroups T_t and T^0_t is that the latter has the generator $\Delta - x\nabla$ and corresponds to the process generated by the stochastic differential equation

$$d\xi^x(t) = \sqrt{2}\,dw(t) - \xi^x(t)\,dt.$$

In the sequel we shall use the following remark: let α and β be two positive numbers and let $\zeta^x(t)$ be the process governed by the equation

$$d\zeta^x(t) = \alpha\,dw(t) - \beta\zeta^x(t)\,dt, \qquad \zeta^x(0) = x.$$

Then the invariant probability of $\zeta^x(t)$ is given by the density

$$p_{\alpha,\beta}(t) = \sqrt{\pi\alpha^2/\beta}\,\exp\left(-\frac{\beta}{\alpha^2}t^2\right).$$

Notice that putting $\alpha = 1/\sqrt{n}$ and $\beta = 1/2$ we find that the invariant measure of $\zeta^x(t)$ coincides with the law of the process $dw(t)/\sqrt{n}$ at time 1, which has the density $\sqrt{n/2\pi}\exp(-nt^2/2)$.

The same connection between the Wiener process and the Ornstein–Uhlenbeck process exists in infinite dimensions. Recall that a stochastic process $W(t)$ with values in E^* is called a *Wiener process associated with H* if for any $v \in E$ with $\|v\|_H = 1$ the scalar process $\langle W(t), v\rangle$ is the standard Wiener process (see Bogachev [10]). It is known that in this case there exists a process $\xi^x(t)$ satisfying (1.6). This process is Gaussian and its invariant probability μ coincides with the law of the random element $W(1)$. We shall show that in our setting there is an analogue of the process ξ in S whose invariant probability is the law of the Lévy Brownian motion at time 1.

Denote by E_n the linear span of the vectors $e_{-n}, \ldots, e_n$.

For any natural numbers j and n let $e_j^{(n)} = \sqrt{2n+1}\, e_j$. Note that the vectors $e_1^{(n)}, \ldots, e_n^{(n)}$ form an orthonormal basis in the Hilbert space E_n equipped with the inner product

$$(u, v)_n = \frac{1}{2n+1} \sum_{i=-n}^{n} u_i v_i, \quad \text{where } u = \sum_{i=-n}^{n} u_i e_i,\ v = \sum_{i=-n}^{n} v_i e_i.$$

We shall prove that the normalized sums

$$S_n(t) := \frac{1}{2n+1} \sum_{i=-n}^{n} w_i(t) e_i^{(n)} = \frac{1}{\sqrt{2n+1}} \sum_{i=-n}^{n} w_i(t) e_i \tag{1.8}$$

converge (in a certain sense) to the Lévy Brownian motion introduced by Accardi et al. in [5] and that the associated Ornstein–Uhlenbeck processes $\Xi_n(t)$ on E_n, defined by

$$d\Xi_n(t) = dS_n(t) - \tfrac{1}{2}\Xi_n(t)\,dt, \qquad \Xi_n(0) = 0,$$

converge to a process ξ_t which belongs to a Markov family $\{\xi_t^x\}$ whose generator can be identified with (1.2): this is the *Lévy Ornstein–Uhlenbeck process.*

We consider $S_n(t)$ and $\Xi_n(t)$ as E-valued random processes (and hence as S-valued random processes when E is mapped into S as explained above). Put $\pi_n x = \sum_{i=-n}^{n} x_i e_i$. Setting

$$S_n^x(t) := S_n(t) + x \quad \text{and} \quad \Xi_n^x(t) := \Xi_n(t) + x - \pi_n x + e^{-t/2}\pi_n x,$$

we obtain the processes starting from any point $x \in E$. For every x, these are standard finite-dimensional processes in the plane $x + H_n$. Denote the corresponding semigroups by $\{P_t^{(n)}\}$ and $\{T_t^{(n)}\}$, respectively. As explained above,

$$T_t^{(n)} f(x) = \int_E f(x - \pi_n x + e^{-t/2}\pi_n x + \sqrt{1-e^{-t}}\, y)\, \mu_n(dy), \tag{1.9}$$

where μ_n is the Gaussian measure on E_n obtained as the image of the standard Gaussian measure on R^{2n+1} under the map

$$(t_{-n}, \ldots, t_n) \in R^{2n+1} \mapsto \frac{1}{\sqrt{2n+1}} \sum_{i=-n}^{n} t_i e_i \in E_n. \tag{1.10}$$

The generator $\Delta_n/2$ of the process $S_n^x(t)$ is

$$\frac{1}{2}\Delta_n = \frac{1}{2}\frac{1}{2n+1} \sum_{i=-n}^{n} \partial_{e_i}^2,$$

while the generator of the process $\Xi_n^x(t)$ is

$$L_n = \frac{1}{2}\left[\frac{1}{2n+1} \sum_{i=-n}^{n} \partial_{e_i}^2 - \sum_{i=-n}^{n} x_i \partial_{e_i}\right].$$

Note that if, starting Ξ_n at a point x, we put $\Xi_n(t)+e^{-t/2}x$ instead of the definition above, then we get a similar process, but with the infinite-dimensional drift (so that the last sum will be taken in infinite limits). However, we prefer to start with properly finite-dimensional objects.

For the process $S_n^x(t)$ we get the following representation of the transition semigroup:

$$P_t^{(n)} f(x) = f * \mu_{n,t}(x) = \int_E f(x-y)\, \mu_{n,t}(dy), \tag{1.11}$$

where $\mu_{n,t}$ is the Gaussian measure on E equal the image of the standard Gaussian measure on R^{2n+1} under the mapping

$$(t_{-n}, \ldots, t_n) \mapsto \sqrt{\frac{t}{2n+1}} \sum_{i=-n}^{n} t_i e_i.$$

In accordance with (1.10), in the following we shall use the notation

$$\mu_n := \mu_{n,1}. \tag{1.12}$$

Clearly, its Fourier transform equals

$$\exp\left(-\frac{t}{4n+2} \sum_{i=-n}^{n} x_i^2\right).$$

THEOREM 1.1. (i) *For any fixed $t \geqslant 0$ the operators $T_t^{(n)}\colon \mathscr{F} \to C_b(E)$ and $P_t^{(n)}\colon \mathscr{F} \to C_b(E)$ converge strongly, i.e., pointwise in the sup-norm, to bounded operators $T_t\colon \mathscr{F} \to \mathscr{F}$ and $P_t\colon \mathscr{F} \to \mathscr{F}$, respectively. The families $\{T_t\}_{t\geqslant 0}$ and $\{P_t\}_{t\geqslant 0}$ are strongly continuous Markov semigroups on $\mathscr{F}$, hence they admit unique extensions to strongly continuous semigroups (denoted by the same symbols) on $\mathscr{C} \cong C(S)$.*

(ii) *The semigroups $\{T_t\}$ and $\{P_t\}$ give rise to two Markov families $\{\Xi_{t,x}\}$, $\Xi_{0,x} = x$, and $\{W_{t,x}\}$, $W_{0,x} = x$, with state space S. Moreover, the processes $\Xi_t := \Xi_{t,\theta}$ and $W_t := W_{t,\theta}$ are the limits in distribution of the finite-dimensional processes $\Xi_n(t)$ and $S_n(t)$ (embedded into S), respectively.*

(iii) *The law Λ of W_1 is an invariant Radon probability of the process Ξ.*

(iv) *The process Ξ_t^Λ defined as $\Xi_{t,x}$ with the initial distribution Λ is symmetric.*

Proof. Applying formulas (1.9) and (1.10) to a function $f = \tilde{\nu}$ ($\nu \in \mathscr{M}$) in the space $\mathscr{F}$ we get

$$P_t^{(n)} f(x) = \int_E \tilde{\nu}(x-y)\, \mu_{n,t}(dx), \tag{1.13}$$

$$T_t^{(n)} f(x) = \int_E \tilde{\nu}(y)\, \lambda_{n,t,x}(dy), \tag{1.14}$$

where $\lambda_{n,t,x}$ is the image of the measure μ_n under the map

$$y \mapsto x - \pi_n x + e^{-t/2} \pi_n x + \sqrt{1-e^{-t}}\, y.$$

Now note that, for any measure λ on E and any measure ν on E^*, the following Parseval identity follows from the Fubini–Tonelli theorem:

$$\int_{E^*} \tilde{\lambda}(y)\,\nu(dy) = \int_E \tilde{\nu}(x)\,\lambda(dx). \tag{1.15}$$

Indeed,

$$\begin{aligned}\int_{E^*} \tilde{\lambda}(y)\,\nu(dy) &= \int_{E^*}\int_E \exp(i\langle y, x\rangle)\,\lambda(dx)\,\nu(dy)\\ &= \int_E\int_{E^*} \exp(i\langle y, x\rangle)\,\nu(dy)\,\lambda(dx) = \int_E \tilde{\nu}(x)\,\lambda(dx).\end{aligned}$$

Therefore, (1.14) gives, again for $f = \tilde{\nu}\in\mathscr{F}$,

$$\begin{aligned}T_t^{(n)} f(x) &= \int_E \tilde{\lambda}_{n,t,x}(z)\,\nu(dz)\\ &= \int_{E^*} \exp(i\langle z, x-\pi_n x+e^{-t/2}\pi_n x\rangle)\exp\left(-\frac{1-e^{-t}}{4n+2}\sum_{i=-n}^{n} z_i^2\right)\nu(dz).\end{aligned}$$

Similar calculations give the following expression for $P_t^{(n)}$:

$$P_t^{(n)} f(x) = \int_{E^*} \exp(i\langle z, x\rangle)\exp\left(-\frac{t}{4n+2}\sum_{i=-n}^{n} z_i^2\right)\nu(dz).$$

Since ν-a.e.

$$\|z\|_\nu^2 = g(z) = \lim_{n\to\infty}\frac{1}{2n+1}\sum_{i=-n}^{n} z_i^2,$$

and $\langle z, \pi_n x\rangle \to \langle z, x\rangle$ in ν-measure, by the dominated convergence we get

$$\lim_{n\to\infty} T_t^{(n)} f(x) = \int_{E^*} \exp(i\langle z, e^{-t/2}x\rangle)\exp\left(-\frac{1-e^{-t}}{2}\|z\|_\nu^2\right)\nu(dz), \tag{1.16}$$

$$\lim_{n\to\infty} P_t^{(n)} f(x) = \int_{E^*} \exp(i\langle z, x\rangle)\exp\left(-\frac{t}{2}\|z\|_\nu^2\right)\nu(dz). \tag{1.17}$$

Moreover, again by the dominated convergence it follows that the limits in (1.16) and (1.17) exist uniformly in $x\in E$ and locally uniformly in t. Denoting the right-hand sides of (1.16) and (1.17) by $T_t f$ and $P_t f$, respectively, we get

$$\|T_t^{(n)} f - T_t f\|_{C_b(E)} \to 0, \qquad \|P_t^{(n)} f - P_t f\|_{C_b(E)} \to 0. \tag{1.18}$$

Note that the measures on E^*:

$$\exp\left(-\frac{1-e^{-t}}{2}\|\cdot\|_\nu^2\right)\nu \quad \text{and} \quad \exp\left(-\frac{t}{2}\|\cdot\|_\nu^2\right)\nu$$

are finite and belong to $\mathscr{M}$. Therefore, both semigroups $\{T_t\}$ and $\{P_t\}$ map $\mathscr{F}$ into itself and send real functions from $\mathscr{F}$ to real functions and positive definite functions to positive definite ones. Moreover, from (1.13) and (1.14) it is clear that if $f = \tilde{v}$ is pointwise positive, then both $P_t f$ and $T_t f$ are pointwise positive, a fact that was not evident from the original construction in Accardi et al. [5]. Since S is compact, these semigroups give rise to two homogeneous Markov families $\{W_{t,x}\}$, $W_{0,x} = x$, and $\{\Xi_{t,x}\}$, $\Xi_{0,x} = x$, $x \in S$, with state space S having $\{P_t\}$ and $\{T_t\}$ as the transition semigroups, respectively. This well-known fact follows, e.g., from the Riesz–Markov theorem combined with the Kolmogorov theorem. The proof is similar to that of a metric case (see, e.g., Wentzell [29], Theorem 8.4). The only difference is that the transition probabilities $P(t, x, \cdot)$, which arise in the proof, have the measurability property $x \mapsto P(t, x, B)$ for all sets B from the Baire σ-field $\mathscr{B}a(S)$ which may be smaller than the Borel σ-field $\mathscr{B}(S)$. However, since by the compactness of S every measure on $\mathscr{B}a(S)$ extends uniquely to a Radon measure on $\mathscr{B}(S)$, this is not a significant restriction. In particular, all the transition probabilities $P(t, x, \cdot)$ can be regarded as Radon measures.

By our construction, for any fixed t the laws of $S_n(t)$ and $\Xi_n(t)$ (more precisely, the laws of $\Pi S_n(t)$ and $\Pi \Xi_n(t)$) converge weakly to the laws of W_t and Ξ_t, respectively. Moreover, we have also the weak convergence of the finite-dimensional distributions of these processes. Indeed, let $t_1 < \ldots < t_k$ be fixed moments and let $f \in C(S^k)$. Let us verify that

$$\lim_{n\to\infty} \int_{S^k} f(y_1, \ldots, y_k) P^{(n)}_{t_1,\ldots,t_k}(dy_1 \ldots dy_k) = \int_{S^k} f(y_1, \ldots, y_k) P_{t_1,\ldots,t_k}(dy_1 \ldots dy_k),$$

where $P^{(n)}_{t_1,\ldots,t_k}$ and $P_{t_1,\ldots,t_k}$ denote the measures on S^k induced by $(\Pi\Xi^n(t_1), \ldots, \Pi\Xi^n(t_k))$ and $(\Xi_{t_1}, \ldots, \Xi_{t_k})$, respectively. Clearly, it suffices to check this for f from a subset in $C(S^k)$ with dense linear span. Such a subset is formed by the functions f of the form $f = f_1 \ldots f_k$, where $f_j(y_1, \ldots, y_k) = g_j(y_j)$ with $g \in \mathscr{F}$. To simplify the notation we consider the case $k = 2$. Then the right-hand side of the equality under the question is given by

$$\int_S g_1(y_1)\Big(\int_S g_2(y_2) P(t_2 - t_1, y_1, dy_2)\Big) P_{t_1}(dy_1) = \int_S g_1(y)\, T_{t_2 - t_1} g_2(y) P_{t_1}(dy).$$

An analogous formula holds true for $P^{(n)}_{t_1,\ldots,t_k}$. Therefore, by the strong convergence of the semigroups and the weak convergence of $P^{(n)}_{t_1}$ we obtain the desired equality. For the process W_t the proof is analogous.

Let us check that the law of W_1 (which is a Radon probability measure on the compact S) is an invariant probability for the process Ξ_t. To this end, it suffices to note that the images (under Π) of the invariant probabilities μ_n of the processes $\Xi^x_n(t)$ converge weakly to the measure Λ. Finally, since each $\{T^{(n)}_t\}$ is μ_n-symmetric, $\{T_t\}$ is Λ-symmetric.

Therefore, there exists also a process Ξ^Λ_t with the same transition semigroup $\{T_t\}$, but with the initial distribution Λ which is its invariant measure.

Note that the measure Λ is defined on S. However, by the isomorphism between $C(S)$ and $\mathscr{C}$, it generates a continuous linear functional on $\mathscr{C}$, hence on $\mathscr{F}$. We denote this functional by the same symbol Λ. It follows from the considerations above that, for any $f = \tilde{\nu} \in \mathscr{F}$,

$$\Lambda(f) = \int_{E^*} \exp(-\|z\|_\nu^2/2)\,\nu(dz). \tag{1.19}$$

Formula (1.16) gives the following explicit expression for the action of T_t on $\mathscr{F}$:

$$T_t f(x) = \int_{E^*} \exp(i\langle z, e^{-t}x\rangle)\exp\left(-\frac{1-e^{-t}}{2}\|z\|_\nu^2\right)\nu(dz). \tag{1.20}$$

This action consists of scaling the measure ν and then changing the argument in the Fourier transform. The action of P_t on $\mathscr{M}$ is expressed by the formula

$$\mu \mapsto \exp(-t\,\|\cdot\|_\mu^2/2)\,\mu. \tag{1.21}$$

Let us make one important remark. In Theorem 1.1 we constructed the semigroup $\{T_t\}$ on the Banach space $C(S)$ (with our convention to identify $C(S)$ and $\mathscr{C}$). On the other hand, Λ is an invariant probability for $\{T_t\}$, and, as we already know, $\{T_t\}$ is a Markovian semigroup on $L^2(\Lambda)$. Clearly, $C(X)$ is dense in $L^2(\Lambda)$, since Λ is a Radon measure. However, the sup-norm in $C(S)$ is stronger than the L^2-norm. Therefore, $\{T_t\}$ may have different generators on both spaces. In order to avoid confusion, by generator A_L of $\{T_t\}$ we shall mean its generator in $L^2(\Lambda)$. Domain of A_L in $L^2(\Lambda)$ is denoted by $D(A_L)$.

Another useful observation is that for each element $f \in \mathscr{C}$ the C^*-algebras generated by $\{T_t f,\ t \geqslant 0\}$ and $\{P_t f,\ t \geqslant 0\}$ are separable and stable under the corresponding semigroup. Hence, the results above hold true if we replace S by the spectra of these C^*-algebras which are metrizable compacta. This enables us to use the whole machinary of Fukushima [16] and Ma and Röckner [21]. The same is true for any countable subset of $\mathscr{C}$ instead of f. The properties of the process on these metrizable state spaces will be the subject of a separate paper.

PROPOSITION 1.2. *The generator of the semigroup* $\{T_t\}$ *on* $\mathscr{C}$ *coincides with* (1.2) *on all* $f \in \mathscr{F}$ *from its domain. For any such* $f = \tilde{\nu}$ *the following equality holds*:

$$A_L f(x) = \tfrac{1}{2}\int_{E^*} [-i\langle z, x\rangle - \|z\|_\nu^2]\exp(i\langle z, x\rangle)\,\nu(dz). \tag{1.22}$$

In a similar way, the generator of the semigroup $\{P_t\}$ *on* $\mathscr{C}$ *is given by* (1.1) *on all* $f \in \mathscr{F}$ *from its domain and*

$$\tfrac{1}{2}\Delta_L f(x) = -\tfrac{1}{2}\int_{E^*} \|z\|_\nu^2 \exp(i\langle z, x\rangle)\,\nu(dz). \tag{1.23}$$

The proposition follows from the direct calculations which we omit.

Remark 1.3. It is worth mentioning that the domain of the operator A_L differs from the domain of Δ_L and that for an element $f \in \mathscr{F}$ the function $A_L f$

typically is not the Fourier transform of a measure. To see this, let us note that if $f \in \mathscr{F}$ is in the domain of the first order operator $x\nabla$ and $x\nabla f \in \mathscr{F}$, then

$$(x\nabla) f\colon\ x \mapsto i \int \langle z, x\rangle \exp(i\langle z, x\rangle)\, \nu(dz) \in \mathscr{C}.$$

This condition can be written as

$$\frac{d}{dt} f(e^t \cdot)|_{t=0} \in \mathscr{C}.$$

EXAMPLE 1.4. *Let ν be the Gaussian measure on E^* with Fourier transform $f(x) = \exp(-(x, x)_H/2)$. Then f is in the domain of the generators of both semigroups $\{T_t\}$ and $\{P_t\}$ on $\mathscr{C}$ (hence also on $L^2(\Lambda)$) and*

$$(x\nabla) f(x) = -(x, x) \exp(-(x, x)_H/2)$$

is in $\mathscr{C}$, but is not the Fourier transform of any bounded measure on E^*.

Proof. The function $(x, x)_H \exp(-(x, x)_H/2)$ belongs to the uniform closure of $\mathscr{F}$, since it is the uniform limit of the functions

$$-\frac{T_t f - f}{t} + \frac{P_t f - f}{t}$$

which are in $\mathscr{F}$. Obviously, by (1.21), f is in the domain of the generator of $\{P_t\}$ on $\mathscr{C}$. The fact that f is in the domain of the generator of the semigroup $\{T_t\}$ on $\mathscr{C}$ can be verified directly. To this end, notice that since ν is ergodic under the e-shift, by the ergodic theorem (cf. Shiryaev [27] or Parthasarathy [23]) the function g equals 1 ν-a.e. Hence

$$T_t f(x) = \exp(-(1-e^{-t})/2) \exp(-e^{-t}(x, x)_H/2).$$

Assume now that there is a measure λ on E^* with

$$\tilde{\lambda}(x) = (x, x)_H \exp(-(x, x)_H/2).$$

The finite-dimensional projection λ_n of this measure (under projecting onto E_n) has the Fourier transform

$$\tilde{\lambda}_n(x) = \sum_{i=-n}^{n} x_i^2 \exp\Big(-\frac{1}{2} \sum_{i=-n}^{n} x_i^2\Big).$$

One can easily check that

$$\lambda_n = \Big(\sum_{i=-n}^{n} (1 - x_i^2)\Big) \nu_n,$$

where ν_n is the standard Gaussian measure on E_n (endowed with the inner product from H). Now, to get a contradiction with our assumption, it suffices to prove that the variations of the measures λ_n are not bounded in total. Indeed, if it were the case, the sequence of independent identically distributed random variables $\xi_n(x) = 1 - x_i^2$ on (E^*, ν) would have the following property:

$$\sup_n E\Big|\sum_{i=-n}^{n} \xi_i\Big| < \infty.$$

Note that $E\xi_n = 0$. It then follows from a classical result (see Shiryaev [27] or Vakhania et al. [28], p. 228) that the series $\sum_{i=-\infty}^{\infty} \xi_i$ converges in mean. According to another result from probability theory (see [27], p. 414) we get

$$\sum_{i=-\infty}^{\infty} E\frac{\xi_i^2}{1+|\xi_i|} < \infty,$$

which is impossible since the ξ_n's have identical distributions.

Note that the operator $x\nabla_L$ acts on measures in the following way:

$$(x\nabla_L)\mu = \sum_{n=-\infty}^{\infty} -i\langle\cdot, e_n\rangle d_{e_n}\mu,$$

where $d_e\mu$ stands for the Fomin derivative of the measure μ along the vector e (see Bogachev and Smolyanov [13] or Daletskii and Fomin [15] for more information about differentiable measures). Denoting by β_i the logarithmic derivative of μ along e_i (the Radon–Nikodym density of $d_{e_i}\mu$ with respect to μ) we get

$$(x\nabla_L)\mu = \sum_{n=-\infty}^{\infty} -i\langle\cdot, e_n\rangle \beta_n\mu.$$

Using Ψ-invariance of μ one can check that $\beta_i \circ \Psi = \beta_{i+1}$. Thus, the functions $\langle\cdot, e_n\rangle \beta_n$ are equally distributed, and the series of these functions cannot converge in mean. In particular, if these functions are independent as random variables on (E^*, μ), then $(x\nabla_L)\mu$ is not a finite measure, however, it can happen, as we have seen above, that its Fourier transform belongs to $\mathscr{C}$ (which is the case if $\tilde{\mu}$ is in the domain of the semigroup $\{T_t\}$ on $\mathscr{C}$).

Remark 1.5. (i) As could be noticed from the reasoning above, the same results hold true for some other choices of our initial functional space $\mathscr{F}$. For example, it is possible to impose a stronger restriction $x_i^{2p} \in L^1(\mu)$ for an integer $p \geqslant 1$, which leads to a smaller space of measures $\mathscr{M}_{2p}$ (or even take $\mathscr{M}_\infty = \bigcap_{p\geqslant 1} \mathscr{M}_{2p}$).

(ii) It has been proved by Accardi and Bogachev [2] (with the class $\mathscr{M}_4$ replacing $\mathscr{M}$) that the processes W_t and Ξ_t have the following analogue of the path continuity property: for each function $f \in C(S)$ the scalar processes $f(\Xi_t)$ and $f(W_t)$ have continuous modifications. Moreover, for any $f \in \mathscr{F}_4$ (Fourier transforms of measures from $\mathscr{M}_4$) such scalar processes satisfy the estimate

$$E|f(\Xi_t) - f(\Xi_s)|^4 \leqslant C(f)|t-s|^2,$$

and hence have α-Hölder continuous modifications with any $\alpha \in (0, 1/2)$. As noted by M. Röckner, the same is true for the symmetric process Ξ_t^Λ with initial distribution Λ (for any space $\mathscr{F}_{2p}$ or $\mathscr{F}_\infty$ instead of $\mathscr{M}$). This can be deduced from the proof of Proposition 2.12 in Albeverio and Kusuoka [7]. This problem will be discussed in more detail elsewhere.

2. The Lévy gradient and the associated Dirichlet form. We shall use the term a "pre-Dirichlet form $\mathscr{E}$" for a closable form $\mathscr{E}$ whose closure is a Dirichlet form in the usual sense (see Ma and Röckner [21]). In such a case the closure will be denoted by the same symbol $\mathscr{E}$. To simplify formulas, we shall consider *real* functions when dealing with Dirichlet forms.

Notice that for any n the process $\Xi_n(t)$ with initial distribution μ_n is the diffusion on E corresponding to the pre-Dirichlet form

$$\mathscr{E}_n(f, f) = -(L_n f, f)_{L^2(\mu_n)} = (2n+1)^{-1} \int \sum_{i=-n}^{n} \left(\partial_{e_i} f(x)\right)^2 \mu_n(dx), \tag{2.1}$$

and that (2.1) can be rewritten as

$$\mathscr{E}_n(f, f) = \int (\nabla f(x), \nabla f(x))_n \mu_n(dx).$$

Lemma 2.1. *Let μ and ν be two Ψ-invariant measures on E^* such that $\langle \cdot, e_i \rangle \in L^1(\mu)$ and $\langle \cdot, e_i \rangle \in L^1(\nu)$. Put $\mu_i := \langle \cdot, e_i \rangle \mu$ and $\nu_i := \langle \cdot, e_i \rangle \nu$. Then the measure $\mu_i * \nu_i$ is absolutely continuous with respect to $\mu * \nu$ and its Radon–Nikodym derivative ϱ_i satisfies the equality*

$$\varrho_i(\Psi x) = \varrho_{i+1}(x) \quad \mu * \nu\text{-a.e.}$$

Proof. The first assertion is trivial. To prove the second one let us note that for any bounded Borel function f on E^* there is a Borel function g such that $f = g \circ \Psi$. Therefore, from the Ψ-invariance of μ, ν and $\mu * \nu$ we get

$$\begin{aligned}
&\int f(z) \varrho_{i+1}(z) \mu * \nu(dz) = \int f(z) \mu_{i+1} * \nu_{i+1}(dz) \\
&= \iint f(x+y) \langle x, e_{i+1} \rangle \langle y, e_{i+1} \rangle \mu(dx) \nu(dy) \\
&= \iint g(\Psi x + \Psi y) \langle \Psi x, e_i \rangle \langle \Psi y, e_i \rangle \mu(dx) \nu(dy) \\
&= \iint g(x+y) \langle x, e_i \rangle \langle y, e_i \rangle \mu(dx) \nu(dy) = \int g(z) \mu_i * \nu_i(dz) \\
&= \int g(z) \varrho_i(z) \mu * \nu(dz) = \int g(\Psi z) \varrho_i(\Psi z) \mu * \nu(dz) = \int f(z) \varrho_i(\Psi z) \mu * \nu(dz).
\end{aligned}$$

Lemma 2.2. *The sequence $(\mathscr{E}_n)$ of pre-Dirichlet forms converges pointwise on $\mathscr{F}$. We denote its limit by $\mathscr{E}_L$.*

Proof. Let $f = \tilde{\nu}$. Then, by a property of the Fourier transform,

$$(\partial_{e_i} f)^2 = (-\nu_i * \nu_i)^\sim,$$

where $\nu_i = \langle\cdot, e_i\rangle \nu$. According to Lemma 2.1, the measure $\nu_i * \nu_i$ is absolutely continuous with respect to $\nu * \nu$ and for the corresponding Radon–Nikodym density ϱ_i we have $\varrho_{i+1} = \varrho_i \circ \Psi$. Therefore, from (2.1) and (1.15) we get

$$(2.2)\quad \mathscr{E}_n(f,f) = \int_{E^*} \frac{1}{2n+1} \exp\left(-\frac{1}{4n+2}\sum_{i=-n}^{n} \langle z, e_i\rangle^2\right) \sum_{i=-n}^{n} \varrho_i(z)\, \nu * \nu(dz).$$

Applying again the ergodic theorem we get the convergence of the integrals in (2.2).

Since the operator A_L is the generator of a symmetric Markovian semigroup, it is self-adjoint and nonpositive (see Ma and Röckner [21]). It is natural to ask about the relation of its domain with our initial functional class $\mathscr{F}$. This class has a simple description, so it would be useful to know whether it uniquely determines our main objects. This is indeed so in the following sense.

PROPOSITION 2.3. (i) *A_L is essentially self-adjoint on $\mathscr{F}_0 = \mathscr{F} \cap D(A_L)$.*

(ii) *The quadratic form $\mathscr{E}_L$ obtained in Lemma 2.2 is the pre-Dirichlet form corresponding to the operator A_L on $\mathscr{F}_0$, and $\mathscr{F}_0$ is a core of the Dirichlet form obtained as the closure of $\mathscr{E}_L$ (this closure will be denoted also by $\mathscr{E}_L$).*

Proof. (i) It is not hard to prove that $\mathscr{F}_0$ is dense in $L^2(\Lambda)$. Indeed, for each n the mappings

$$G_n f \mapsto \int_0^\infty n e^{-ns} T_s f\, ds$$

send $\mathscr{F}$ into $\mathscr{F}$. Since $G_n f \to f$ in $L^2(\Lambda)$ and $G_n f \in D(A_L)$ (see Yosida [30], Theorem 1, Chapter IX, § 3), we infer that $\mathscr{F}_0$ is dense in $\mathscr{F}$, hence also in $L^2(\Lambda)$. It is worth noting that the same argument shows that the class $\mathscr{F}_{00}$ defined as the intersection of $\mathscr{F}$ with the domain of the generator of $\{T_t\}$ on $\mathscr{C} \cong C(S)$ is dense in $C(S)$, hence also in $L^2(\Lambda)$. Since the inclusion $T_t(D(A_L)) \subset D(A_L)$ always holds (see Ma and Röckner [21]), we obtain $T_t\mathscr{F}_0 \subset \mathscr{F}_0$ and $T_t\mathscr{F}_{00} \subset \mathscr{F}_{00}$. According to a standard result (see, e.g., Theorem X.49 in Reed and Simon [24]) this implies the essential self-adjointness of A_L on $\mathscr{F}_0$ (and also on $\mathscr{F}_{00}$). It follows then that $\mathscr{F}_0$ and $\mathscr{F}_{00}$ are cores for the Dirichlet form $\mathscr{E}_L$.

(ii) Let $f \in \mathscr{F}_{00}$ be real. We have to prove the equality

$$\mathscr{E}_L(f,f) = -\Lambda(f A_L f).$$

By definition,

$$\mathscr{E}_L(f,f) = \lim_{n\to\infty} \int_{E^*} \frac{1}{2n+1} \sum_{i=-n}^{n} \varrho_i(z) \exp\left(-\frac{1}{4n+2}\sum_{i=-n}^{n} \langle z, e_i\rangle^2\right) \nu * \nu(dz).$$

On the other hand, since $A_L f \in \mathscr{C}$, we have

$$\Lambda(f A_L f) = \lim_{n\to\infty} \int f(x)\, A_L f(x)\, \mu_n(dx).$$

Note that

$$\begin{aligned}\int f(x) A_L f(x) \mu_n(dx) &= \int_{E_n} f(x)\Big[\Delta_L f(x) - \sum_{i=-\infty}^{\infty} x_i \partial_{e_i} f(x)\Big] \mu_n(dx)\\ &= \int_{E_n} f(x)\Big[\Delta_L f(x) - \sum_{i=-n}^{n} x_i \partial_{e_i} f(x)\Big] \mu_n(dx)\\ &= \int_{E_n} f(x)\Big[\Delta_L^{(n)} f(x) - \sum_{i=-n}^{n} x_i \partial_{e_i} f(x)\Big] \mu_n(dx)\\ &\quad + \int_{E_n} f(x)\big[\Delta_L f(x) - \Delta_L^{(n)} f(x)\big] \mu_n(dx).\end{aligned}$$

The first term on the right-hand side of the expression above coincides with (2.2). So it suffices to check that

$$\lim_{n\to\infty} \int f(x)\big[\Delta_L f(x) - \Delta_L^{(n)} f(x)\big] \mu_n(dx) = 0.$$

By the Parseval equality this reduces to the following:

$$\begin{aligned}\lim_{n\to\infty} \int \exp\Big(\frac{1}{4n+2} - \sum_{i=-n}^{n} z_i^2\Big) \nu * (g\nu)(dz)&\\ - \int \exp\Big(-\frac{1}{4n+2}\sum_{i=-n}^{n} z_i^2\Big) \nu * (g_n \nu)(dz) &= 0,\end{aligned}$$

where

$$g_n(z) = \frac{1}{2n+1}\sum_{i=-n}^{n} z_i^2$$

and g is given by (1.5). The integral on the left-hand side of the expression above can be estimated by $\|\nu\|\,\|g\nu - g_n\nu\|$, which converges to zero by the ergodic theorem.

Now we shall construct a gradient (which will be called the *Lévy gradient*) in such a way that the Dirichlet form $\mathscr{E}_L$ will have the form

$$\mathscr{E}_L(f, f) = \Lambda\big((\nabla_L f, \nabla_L f)_{\mathscr{H}}\big), \tag{2.3}$$

where $\mathscr{H}$ is the Hilbert space defined below.

Let X be the space of all real sequences (x_n) satisfying the condition

$$\limsup_n \frac{1}{n}\sum_{i=-n}^{n} x_i^2 < \infty. \tag{2.4}$$

As shown in Accardi and Obata [4], there exists a pre-scalar product $(\cdot, \cdot)_0$ on X with the following property: if the limit

$$\lim_{n\to\infty} \frac{1}{2n+1}\sum_{i=-n}^{n} h_i k_i$$

exists for two elements $h = (h_n)$ and $k = (k_n)$ in X, then this limit coincides with $(h, k)_0$. The completion of the quotient space $X/(\cdot, \cdot)_0$ with respect to this inner product is denoted by $\mathscr{K}$, and its complexified space by $\mathscr{K}^C$. Note that $\mathscr{K}$ is a nonseparable Hilbert space.

For any $f = \tilde{\nu} \in \mathscr{F}$ we define its $\mathscr{K}$-gradient as follows. The $\mathscr{K}^C$-valued function $\nabla_L f$ is defined by

$$\nabla_L f(x) = \Big(i \int_{E^*} \exp(i\langle z, x\rangle)\langle z, e_n\rangle\, \nu(dz)\Big)_{n=-\infty}^{\infty}.$$

The estimate

$$\frac{1}{2n+1}\sum_{i=-n}^{n}\Big(\int_{E^*} |\langle z, e_i\rangle|\,|\nu|(dz)\Big)^2 \leqslant \frac{1}{2n+1}\int \sum_{i=-n}^{n} \langle z, e_i\rangle^2\, |\mu|(dz)$$

and the ergodic theorem show that, for any $x \in E$, $\nabla_L f(x)$ is an element of $\mathscr{K}^C$ (more precisely, it defines an equivalence class in this space). Note that if both ν and $f = \tilde{\nu}$ are real, then $\nabla_L f$ takes values in $\mathscr{K}$.

The gradient constructed above on $\mathscr{F}$ induces the gradient for the corresponding elements in $C(S)$. Note that this gradient is the limit in the sense of Accardi et al. [3] of the finite-dimensional E_n-gradients, i.e.

$$\big(\nabla_L f(x), \nabla_L f(x)\big)_{\mathscr{K}} = \lim_{n\to\infty}\big(\nabla_{E_n} f(x), \nabla_{E_n} f(x)\big)_n.$$

THEOREM 2.4. *For any real $f \in \mathscr{F}$, equality* (2.3) *holds. In addition, the Dirichlet form $\mathscr{E}_L$ is a diffusion Dirichlet form, that is: there exists an algebra $\mathscr{A}$ of bounded real functions in the domain of A_L, dense in the real $L^2(\Lambda)$ and stable under the action of C_b^∞-functions, and for any $f \in \mathscr{A}$, $\phi \in C_b^\infty$ the following equality holds*:

$$A_L \phi(f) = \phi'(f) A_L f + \phi''(f)\Gamma(f, f),$$

where $\Gamma(f,f) = [A_L(f^2) - 2f A_L f]/2$.

Proof. Let $f = \tilde{\nu} \in \mathscr{F}$ be real. Then

$$\mathscr{E}_L(f, f) = \lim_{n\to\infty}(2n+1)^{-1}\int \sum_{i=-n}^{n}\big(\partial_{e_i} f(x)\big)^2 \mu_n(dx)$$

and

$$\Lambda\big((\nabla_L f, \nabla_L f)_{\mathscr{K}}\big) = \lim_{n\to\infty}(2n+1)^{-1}\Lambda\Big(\sum_{i=-n}^{n}\big(\partial_{e_i} f(x)\big)^2\Big).$$

Using the notation of Lemma 2.1 and identity (1.15), the right-hand sides of the relations above can be written as

$$\lim_{n\to\infty}\int_{E^*}\frac{1}{2n+1}\sum_{i=-n}^{n}\varrho_i(z)\exp\Big(-\frac{1}{4n+2}\sum_{i=-n}^{n}\langle z, e_i\rangle^2\Big)\nu * \nu(dz)$$

and

$$\lim_{n\to\infty} \int_{E^*} \frac{1}{2n+1} \sum_{i=-n}^{n} \varrho_i(z) \exp\left(-\frac{1}{2}\|z\|_{\nu*\nu}^2\right) \nu*\nu(dz).$$

Applying once again the ergodic theorem to the measure $\nu*\nu$ we get (2.3).

To prove the second statement, let us take for $\mathscr{A}$ the space of all functions $g = \phi \circ f$, where $\phi \in C_b^\infty(R^1)$ and a real function f is in the class $\mathscr{F}_0 = \mathscr{F} \cap D(A_L)$ used in Proposition 2.3. Clearly, $\mathscr{A}$ is stable under compositions with C_b^∞-functions and is contained in $D(A_L)$, since the latter is also stable under such compositions (by the property of Markov semigroups; see Ma and Röckner [21]). By virtue of Proposition 2.3, A_L is essentially self-adjoint on $\mathscr{A}$. It remains to check the formula for $A_L(\phi \circ f)$. If ϕ is a polynomial, this formula can be easily verified directly. It should be noted that, although polynomials are not in C_b^∞, for any fixed $f \in \mathscr{A}$, the polynomial p of f coincides with $\phi(f)$ for a function $\phi \in C_b^\infty(R^1)$ which agrees with p on a segment that contains the values of f (recall that $\mathscr{A}$ consists of bounded functions). Let now ϕ and f be fixed. $I = [-C, C]$, where $C > \sup|f|$. Taking a sequence of polynomials p_n which converge to f uniformly on I so that $\{p_n'\}$ and $\{p_n''\}$ converge uniformly to the corresponding derivatives of f, we obtain the desired identity.

3. Hypercontractivity. Recall that the semigroup $\{T_t\}$ associated with an invariant probability measure μ and a Dirichlet form $\mathscr{E}$ is called *hypercontractive* if the estimate

$$\|T_t f\|_q \leqslant \|f\|_p \tag{3.1}$$

holds for any $t > 0$, $p > 1$, $q > 1$, such that

$$e^t \geqslant \left(\frac{q-1}{p-1}\right)^{1/2}.$$

It is known (see Gross [17], Bakry and Emery [9]) that the hypercontractivity is equivalent to the following logarithmic Sobolev inequality:

$$\int f(x)^2 \ln|f(x)|\, \mu(dx) \leqslant \mathscr{E}(f,f) + \|f\|_2^2 \ln\|f\|_2. \tag{3.2}$$

An important example of a hypercontractive diffusion semigroup is the Ornstein–Uhlenbeck semigroup (see [17]). In this particular case (3.1) reads as follows:

$$\int f(x)^2 \ln|f(x)|\, \mu(dx) \leqslant \int (\nabla f(x), \nabla f(x))\, \mu + \|f\|_2^2 \ln\|f\|_2. \tag{3.3}$$

PROPOSITION 3.1. *The Ornstein–Uhlenbeck semigroup $\{T_t\}$ associated with the Lévy Laplacian is hypercontractive. In addition, the following logarithmic Sobolev inequality holds*:

$$\int f(x)^2 \ln|f(x)|\, \Lambda(dx) \leqslant \int (\nabla_L f(x), \nabla_L f(x))_{\mathscr{H}}\, \Lambda(dx) + \|f\|_2^2 \ln\|f\|_2. \tag{3.4}$$

Proof. The hypercontractivity, i.e. the estimate (3.1), follows from the convergence of the semigroups $\{T_t^{(n)}\}$ to $\{T_t\}$ combined with the fact that these semigroups are hypercontractive by [17].

4. Comparison with the standard Ornstein–Uhlenbeck process and concluding remarks. It is very interesting to compare the process Ξ_t constructed above with the standard (infinite-dimensional) Ornstein–Uhlenbeck process ξ_t in E^* associated with H. Using the basis $\{e_n\}$ we can define this process just as

$$\xi_t = \sum_{n=-\infty}^{\infty} \xi_n(t)\, e_n,$$

where $\{\xi_n(t)\}$ is a sequence of independent standard real Ornstein–Uhlenbeck processes. Denoting by $W(t)$ the Wiener process in E^* constructed in a similar way by means of a sequence of independent real Wiener processes $w_n(t)$ we get

$$d\xi_t = dW(t) - \tfrac{1}{2}\xi_t\, dt.$$

What happens if we try to obtain this process ξ_t by the method applied above for constructing Ξ_t? Keeping the notation of the previous sections we have the finite-dimensional processes

$$\eta_n^x(t) = \sum_{i=-n}^{n} \xi_i(t)\, e_i + x - \pi_n x + e^{-t/2}\pi_n x$$

and the corresponding semigroups $T_t^{\eta,n}$ given by

$$T_t^{\eta,n} f(x) = \int f(x - \pi_n x + e^{-t/2}\pi_n x + \sqrt{1-e^{-t}}\, y)\, \gamma_n(dy),$$

where γ_n is the Gaussian measure on E_n with the Fourier transform $\exp\left(-\sum_{i=-n}^{n} x_i^2/2\right)$. By the Parseval formula (with $f = \tilde{\nu}$) we have

$$T_t^{\eta,n} f(x) = \int \exp(i\langle z, x - \pi_n + e^{-t/2}\pi_n x\rangle) \exp\left(-\frac{1-e^{-t}}{2}\sum_{i=-n}^{n} z_i^2\right)\nu(dz).$$

Thus, if ν is the Dirac measure δ, the limit equals 1, otherwise it is zero. However, in this situation we get a nontrivial limit semigroup if we choose a bigger functional space. For example, let $\mathscr{F}_0$ be the collection of the Fourier transforms of measures of the form $\mu = \nu + \lambda$, where $\nu \in \mathscr{M}$ and λ is a finite Borel measure on some of the subspaces E_n. This new class $\mathscr{M}_0$ of measures can be written as an algebraic sum $\mathscr{M} + \mathscr{M}_{\text{fin}}$, where $\mathscr{M}_{\text{fin}}$ stands for the class of all measures concentrated on the finite-dimensional subspaces E_n. It follows from the S-invariance of all measures in $\mathscr{M}$ that the intersection $\mathscr{M} \cap \mathscr{M}_{\text{fin}}$ contains only measures concentrated at the origin, that is, only measures proportional to the Dirac measure δ. The class of the Fourier transforms of measures in $\mathscr{M}_{\text{fin}}$ will be denoted by $\mathscr{F}_{\text{fin}}$. For any $\lambda \in \mathscr{M}_{\text{fin}}$ the function $\tilde{\lambda}$ depends only on a finite number of variables x_i. Hence, this function can be viewed as a function on E^*. On this bigger functional space, the limit semigroup (which is the transition semigroup $\{T_t^\xi\}$ of the process $\xi_{t,x}$ on E^*) acts nontrivially by the formula

$$T_t^\xi f(x) = \int_{E^*} f(e^{-t/2} x + \sqrt{1-e^{-t}}\, y)\, \gamma(dy),$$

where γ is the Gaussian measure on E^* with Fourier transform $\exp(-(x, x)_H/2)$ (this is the invariant measure of the Ornstein–Uhlenbeck process ξ). In particular, our new space contains all trigonometric functions of the form

$$\exp\big(i \sum_{j=-n}^{n} c_j x_j\big),$$

since these are the Fourier transforms of the atomic measures at $c_{-n} e_{-n} + \ldots + c_n e_n$. It is known that the linear span of such functions is a core of the Dirichlet form corresponding to the process ξ_t^γ (which is the Ornstein–Uhlenbeck process with initial distribution γ) and defined by the formula

$$\mathscr{E}^\xi(f, f) = \int_{E^*} (\nabla_H f(z), \nabla_H f(z))_H\, \gamma(dz).$$

Here $\nabla_H f(z)$ is defined from the relation

$$(\nabla_H f(z), h)_H = \partial_h f(z) \quad \text{for all } h \in H.$$

However, at this point the parallel between the processes Ξ_t and ξ_t is broken. Though the operators A_L and ∇_L are well defined on $\mathscr{F}_{\text{fin}}$ (A_L is vanishing on $\mathscr{F}_{\text{fin}}$), this subspace is far from being a core for the Dirichlet form $\mathscr{E}_L$. This shows the essentially infinite-dimensional character of the objects constructed above. Certainly, since we have a process corresponding to the semigroup $\{T_t\}$, and thus to the Dirichlet form $\mathscr{E}_L$, our situation agrees with the general theory developed in Fukushima [16] and Ma and Röckner [21]. A novelty is that it does not fit the framework of the usual infinite-dimensional Dirichlet forms considered in Albeverio and Röckner [8], Ma and Röckner [21]. Thus, we get an interesting example of a Dirichlet form of the "gradient type" which is essentially infinite dimensional. It is worth mentioning that the process Ξ_t may have some common features with the processes constructed and studied in Bogachev et al. [12]. It is possible to construct Sobolev classes and capacities over Λ in the spirit of Bouleau and Hirsch [14] (see Accardi and Bogachev [2]).

In conclusion, note that the methods of this paper apply to more general differential and pseudodifferential operators connected with the Lévy Laplacian. Analogous constructions have sense in the quantum case.

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