A NOTE ON THE RATE OF CONVERGENCE FOR P-CONVOLUTIONS ON R^d

BY

ANNA K. PANORSKA* (CHATTANOOGA, TENNESSEE)

Abstract. This paper focuses on p-convolutions, a class of generalized convolutions of random vectors. It establishes the rate of convergence in uniform (Kolmogorov) metric for normalized n-fold p-convolution of random vectors to a generalized stable law. The method of proof relies on the application of probability metrics. Applications of generalized stable laws to financial data are also mentioned.

1. Introduction. The theory of generalized convolutions represents a unifying approach to many binary operations on probability measures. Urbanik [11]–[13] developed a theory of generalized convolutions of probability measures on the positive half-line (Urbanik convolutions). Urbanik's ideas inspired a series of results: a characterization of the domain of attraction of stable measures in Urbanik's sense (Bingham [1] and [2]), normed rings given by Urbanik convolutions (Volkovich [15], [16]), rate of convergence in the Central Limit Theorem for Urbanik convolutions (Panorska [6]). All of the aforementioned works dealt with generalized convolutions of probability measures on R^+. In this paper, for the first time, we consider an example of generalized convolution in R^d, p-convolution, and study the rate of convergence in the generalized Central Limit Theorem. We state our main result, Theorem 2.1, in Section 2, after preliminary definitions and notation. In Section 3, we include some auxiliary lemmas and prove Theorem 2.1 using probability metrics techniques (see Zolotarev [17] and Rachev [8] for details on probability metrics).

2. Notation and statement of the main result. Let X and Y be i.i.d. random vectors in R^d, and \|·\| be the Euclidean norm in R^d. Vatan [14] first pointed out the importance of studying the generalized sum

\[ X \circ Y = (X^{(p)} + Y^{(p)})^{1/p}, \]

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where \( X^{(p)} = X \|X\|^{-p-1} \) with \( p > 0 \). The generalized sum (1) defines a commutative and associative operation on random vectors called \( p \)-convolution. In what follows, we always assume that the operation on random vectors is fixed, i.e. that \( p \) is fixed. Since \( p \)-convolution generalizes summation of random vectors, we can define an analog of stable measure with respect to it. We say that a vector \( Y_a \) is symmetric \((\alpha, p)\)-stable if

\[
Y_a \overset{d}{=} n^{-\alpha} (Y_{a}^{(1)} \circ \ldots \circ Y_{a}^{(n)})
\]

holds for any \( n \geq 1 \), where \( Y_{a}^{(i)} \), \( i = 1, \ldots, n \), are i.i.d. copies of \( Y_a \), and \( \circ \) is defined in (1).

Remark 1. For any \( X \in \mathbb{R}^d \), \((X^{(p)})^{(1/p)} = X\).

Remark 2. It follows that if \( Y_a \) is \((\alpha, p)\)-stable, then

\[
Y_{a}^{(p)} \overset{d}{=} n^{-p/\alpha} [(Y_{a}^{(1)})^{(p)} + \ldots + (Y_{a}^{(n)})^{(p)}]
\]

is a symmetric (strictly) Paretian \((\alpha/p)\)-stable (for a complete treatment of Paretian stable random vectors see Janicki and Weron [3] or Samorodnitsky and Taqqu [9]). Thus, for any \((\alpha, p)\)-stable vector, we must have \( 0 < \alpha \leq 2p \).

Remark 3. Any \((\alpha, p)\)-stable \( Y_a \) belongs to the domain of attraction of a Paretian \( \alpha \)-stable law, and is heavy-tailed: \( P(|Y_a| > x) \sim ax^{-\alpha} \) as \( x \to \infty \), for some positive constant \( a \).

The following characterization of \( p \)-stability is frequently used in our proofs:

**Lemma 2.1.** For any \((\alpha, p)\)-stable vector \( Y_a \) and \( \lambda, \lambda_1, \lambda_2 \geq 0 \) such that \( \lambda^a = \lambda_1^a + \lambda_2^a \), we have \( \lambda_1 Y_{a}^{(1)} \circ \lambda_2 Y_{a}^{(2)} \overset{d}{=} \lambda Y_a \), where \( Y_{a}^{(1)}, Y_{a}^{(2)} \) are i.i.d. copies of \( Y_a \).

**Proof.** Since \( Y_{a}^{(p)} \) is Paretian \((\alpha/p)\)-stable, by Theorem 2.1.2 in [9] we have

\[
\lambda_1^a (Y_{a}^{(1)})^{(p)} + \lambda_2^a (Y_{a}^{(2)})^{(p)} \overset{d}{=} \lambda^a (Y_{a}^{(p)}).
\]

Taking \((1/p)\) “power” of both sides of the above equation and using Remark 1 we obtain

\[
\lambda_1 Y_{a}^{(1)} \circ \lambda_2 Y_{a}^{(2)} \overset{d}{=} \lambda Y_a \quad \blacksquare
\]

It is easy to see that \((\alpha, p)\)-stable random vectors are the only distributional limits of (scaled) \( n \)-fold \( p \)-convolutions of symmetric random vectors in \( \mathbb{R}^d \). Random vector \( X \) is said to belong to the domain of attraction of an \((\alpha, p)\)-stable vector \( Y_a \) if

\[
c_n (X_1 \circ \ldots \circ X_n) \overset{d}{=} Y_a \quad \text{as} \quad n \to \infty,
\]

where \( X_1, X_2, \ldots \) are i.i.d. copies of \( X \). We shall study the rate of convergence in (3) when \( X_1 \) belongs to the normal domain of attraction of \( Y_a \), that is when
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$c_n = n^{-1/a}$ in (3). Our main result, stated in Theorem 2.1 below, gives the rate of convergence in (3) in terms of the uniform (Kolmogorov) metric $q$ in $\mathcal{F}(R^d)$ — the space of $R^d$-valued random variables. By definition

$$q(X, Y) = \sup_{C \in \mathcal{C}} |P(X \in C) - P(Y \in C)|,$$

where $\mathcal{C}$ is the family of all convex Borel sets in $R^d$. In addition, we shall utilize the total variation metric $\text{Var}$,

$$\text{Var}(X, Y) = 2 \sup_{A \in \mathcal{B}(R^d)} |P(X \in A) - P(Y \in A)|,$$

and its smoothed version $\tilde{\varphi}_{r,a}$,

$$\tilde{\varphi}_{r,a}(X, Y) = \sup_{h \in R} |h|^r \text{Var}(X \circ h Y_a, Y \circ h Y_a),$$

where $X, Y \in \mathcal{F}(R^d)$, and $Y_a$ is a symmetric $(\alpha, p)$-stable random vector independent of $X$ and $Y$. For clarity, we shall use the following notation:

$$X^{\circ n} = X_1 \circ \ldots \circ X_n \quad \text{and} \quad Y_a^{\circ n} = Y_a^{(1)} \circ \ldots \circ Y_a^{(n)},$$

where $X_1, \ldots, X_n$ and $Y_a^{(1)}, \ldots, Y_a^{(n)}$ are i.i.d. copies of $X$ and $Y_a$, respectively. Finally, we define

$$\tilde{\tau}_{r,p,a} := \max \{q(X, Y_a), \tilde{\varphi}_{r,a}(X, Y_a), [\tilde{\varphi}_{r,a}(X, Y_a)]^{1/(r-a/p)}\}.$$

We shall state the main result now.

**Theorem 2.1.** If $rp > \alpha$, then the following estimate holds:

$$q(n^{-1/a} X^{\circ n}, Y) \leq C \tilde{\tau}_{r,p,a}(X, Y_a) n^{1-rp/\alpha} + C \tilde{\tau}_{r,p,a} n^{-p/\alpha}$$

for some constant $C$.

**3. Proof of the main result.** We shall need the following four lemmas before we can prove Theorem 2.1.

**Lemma 3.1.** $\tilde{\varphi}_{r,a}$ is an ideal metric of order $r$ with respect to operation $\circ$, that is $\tilde{\varphi}_{r,a}$ satisfies the following two conditions:

(i) Regularity: $\tilde{\varphi}_{r,a}(X \circ Z, Y \circ Z) \leq \tilde{\varphi}_{r,a}(X, Y)$.
(ii) Homogeneity: $\tilde{\varphi}_{r,a}(cX, cY) = c^r \tilde{\varphi}_{r,a}(X, Y)$, $c > 0$.

**Note.** $\text{Var}$ is an ideal metric (of order 0) for operation $\circ$.

**Proof.** Regularity of $\tilde{\varphi}_{r,a}$ follows from regularity of $\text{Var}$. To show homogeneity of order $r$, write

$$\tilde{\varphi}_{r,a}(cX, cY) = \sup_{h \in R} |h|^r \text{Var}[c(X \circ h/c Y_a), c(Y \circ h/c Y_a)]$$

$$= \sup_{h > 0} h^r \text{Var}(X \circ h/c Y_a, Y \circ h/c Y_a) = c^r \tilde{\varphi}_{r,a}(X, Y).$$
Lemma 3.2. Let \( X, Y \in \mathcal{X}(R^d) \), and \( \delta > 0 \). Then for some positive constant \( C \) the following smoothing inequality holds:

\[
q(X, Y) \leq Cq(X \circ \delta^{1/p} Y_a, Y \circ \delta^{1/p} Y_b) + C\delta.
\]

Proof. First, we show that \( q(X, Y) = q(X^{(p)}, Y^{(p)}) \). By Remark 1, for any Borel set \( C \in \mathscr{G} \), we have \( \{X^{(p)} \in C\} \leftrightarrow \{X \in C^{(1/p)}\} \), where \( C^{(1/p)} = \{c^{(1/p)} : c \in C\} \). Consequently,

\[
q(X^{(p)}, Y^{(p)}) = \sup_{C \in \mathcal{G}} |P(X^{(p)} \in C) - P(Y^{(p)} \in C)|
\]

\[
= \sup_{C \in \mathcal{G}} |P(X \in C^{(1/p)}) - P(Y \in C^{(1/p)})| = q(X, Y).
\]

Then, recall the classical smoothing inequality for summation of random vectors (see Zolotarev [17] and Rachev [8]):

\[
q(X, Y) \leq Cq(X + \delta \Theta, Y + \delta \Theta) + C\delta,
\]

where \( X, Y \in \mathcal{X}(R^d) \) and \( \Theta \) is a symmetric (Paretian) \( \alpha \)-stable random vector, and write

\[
q(X, Y) = q(X^{(p)}, Y^{(p)})
\]

\[
\leq Cq(X^{(p)} + (\delta^{1/p} Y_a)^{(p)}, Y^{(p)} + (\delta^{1/p} Y_b)^{(p)}) + C\delta
\]

\[
= Cq(X \circ \delta^{1/p} Y_a, Y \circ \delta^{1/p} Y_b) + C\delta. \quad \square
\]

Lemma 3.3. For any \( X, Y, Z, R \in \mathcal{X}(R^d) \), \( \lambda > 0 \), and symmetric \( \alpha \)-stable \( Y_a \), the following two inequalities hold:

(i) \[
q(\lambda Y \circ X \circ Y, \lambda Y \circ X \circ Z) \leq q(X, R) \text{Var}(\lambda Y \circ Y, \lambda Y \circ Z)
\]

\[
+ q(\lambda Y \circ R \circ Y, \lambda Y \circ R \circ Z),
\]

(ii) \[
q(\lambda Y \circ X \circ Y, \lambda Y \circ R \circ Y) \leq q(X, R) \text{Var}(\lambda Y \circ Y, \lambda Y \circ Z)
\]

\[
+ q(\lambda Y \circ X \circ Z, \lambda Y \circ R \circ Z).
\]

Proof. We will prove part (i), and skip the details of the analogous proof of (ii). We have

\[
q(\lambda Y \circ X \circ Y, \lambda Y \circ X \circ Z)
\]

\[
= q((\lambda Y)^{(p)} + X^{(p)} + Y^{(p)}, (\lambda Y)^{(p)} + X^{(p)} + Z^{(p)})
\]

\[
= \sup_{\lambda \in \mathcal{G}} |P((\lambda Y)^{(p)} + X^{(p)} + Y^{(p)} \in A) - P((\lambda Y)^{(p)} + X^{(p)} + Z^{(p)} \in A)|
\]

\[
= \sup_{\lambda \in \mathcal{G}} |\int P(X^{(p)} \in A - a) d(F_{(\lambda Y)^{(p)} + Y^{(p)}} - F_{(\lambda Y)^{(p)} + Z^{(p)}})(a)| = L,
\]

where \( F_X \) denotes the distribution function of a random vector \( X \). Let us put

\[
F_{(\lambda Y)^{(p)} + Y^{(p)}} - F_{(\lambda Y)^{(p)} + Z^{(p)}} = F_{(X,Y)} - F_{(X,Z)},
\]
Then $L$ can be bounded as follows:

$$\begin{align*}
L &\leq \sup_{A \in \mathcal{E}} \left| \left( P(X^{(p)} \in A - x) - P(R^{(p)} \in A - x) \right) d(F(x,y) - F(x,z)) (y) \right| \\
&\quad + \sup_{A \in \mathcal{E}} \left| P(R^{(p)} \in A - y) d(F(x,y) - F(x,z)) (y) \right| \\
&\leq q(X^{(p)}, R^{(p)}) \left| d(F(x,y) - F(x,z)) \right| \\
&\quad + q((\lambda Y_x^{(p)} + R^{(p)} + Y^{(p)} + (\lambda Y_x^{(p)} + R^{(p)} + Z^{(p)})) \\
&= q(X, R) \text{Var}(\lambda Y_x^O, \lambda Y_z^O) + q(\lambda Y_x \circ R \circ Y, \lambda Y_z \circ R \circ Z). \quad \blacksquare
\end{align*}$$

The following estimate is an analog of a classical result (Senatov [10]), and will be used several times in the proof of the main theorem.

**Lemma 3.4.** For $X, Y \in \mathcal{X}(R^d)$ and $\lambda, \lambda_1, \lambda_2 \geq 0$ such that $\lambda^2 = \lambda_1^2 + \lambda_2^2$, we have

$$\text{Var}(\lambda_1 Y_x^{(1)} \circ X \circ \lambda_2 Y_x^{(2)}, \lambda_1 Y_x^{(1)} \circ Y \circ \lambda_2 Y_x^{(2)}) \leq \lambda^{-rp} \bar{\gamma}_{r,\alpha}(X, Y),$$

where $Y_x^{(1)}, Y_x^{(2)}$ are i.i.d. copies of $Y_x$.

**Proof.** By Lemma 2.1, we have $\lambda_1 Y_x^{(1)} \circ \lambda_2 Y_x^{(2)} \overset{d}{=} \lambda Y_x$, and thus

$$\text{Var}(\lambda_1 Y_x^{(1)} \circ X \circ \lambda_2 Y_x^{(2)}, \lambda_1 Y_x^{(1)} \circ Y \circ \lambda_2 Y_x^{(2)}) = \text{Var}(X \circ \lambda Y_x, Y \circ \lambda Y_x) \leq \lambda^{-rp} \sup_{\lambda > 0} \lambda^r \text{Var}(X \circ \lambda Y_x, Y \circ \lambda Y_x) = \lambda^{-rp} \bar{\gamma}_{r,\alpha}(X, Y). \quad \blacksquare$$

Now we prove the main result.

**Proof of Theorem 2.1.** Note that $C$ depends on $r$ and $\alpha$, but it does not depend on the distributions of $X$ and $Y_x$. We will follow the main idea of Senatov [10] and Rachev [8], pp. 264–281. We will proceed by induction. For $n = 1, 2$, the assertion of the theorem is easy to see. Let us assume that, for all $j < n$,

$$q(j^{1/\alpha} X^{\circ j}, Y_x) \leq C\bar{\gamma}_{r,\alpha}(X, Y_x)j^{-rp/\alpha} + C\bar{\gamma}_{r,\alpha}j^{-p/\alpha}.$$  

(5)

Set $\delta := A\bar{\gamma}_{r,\alpha} n^{-p/\alpha}$ and $m = \lceil n/2 \rceil$, where $\lceil . \rceil$ denotes the greatest integer function. Using the stability property of $Y_x$, Lemma 3.4, and the triangle inequality ($m$ times), we obtain

$$\begin{align*}
q(n^{-1/\alpha} X^{\circ n}, Y_x) &\leq Cq(\delta^{1/p} Y_x \circ n^{-1/\alpha} X^{\circ n}, \delta^{1/p} Y_x \circ n^{-1/\alpha} Y_x^{\circ n}) + C\delta \\
&\leq q(\delta^{1/p} Y_x \circ n^{-1/\alpha} X^{\circ n}, \delta^{1/p} Y_x \circ n^{-1/\alpha} X^{\circ (n-1)} \circ n^{-1/\alpha} Y_x^{\circ n}) \\
&\quad + \sum_{j=1}^{m} q(\delta^{1/p} Y_x \circ n^{-1/\alpha} X^{\circ (n-j)}, \delta^{1/p} Y_x \circ n^{-1/\alpha} X^{\circ (n-j-1)} \circ n^{-1/\alpha} Y_x^{(j)} + n^{-1/\alpha} Y_x^{(j+1)}) \\
&\quad + q(\delta^{1/p} Y_x \circ n^{-1/\alpha} X^{\circ (n-m-1)} \circ n^{-1/\alpha} Y_x^{(m+1)}, \delta^{1/p} Y_x \circ n^{-1/\alpha} Y_x^{\circ n}) + C\delta \\
&= J_1 + J_2 + J_3 + C\delta.
\end{align*}$$


We estimate $J_1$ using Lemma 3.3 (i) with

$$X = n^{-1/\alpha} X^{O(n^{-1})}, \quad Y = n^{-1/\alpha} X_n, \quad Z = n^{-1/\alpha} Y_a^n, \quad \text{and} \quad R = n^{-1/\alpha} Y_a^{(n-1)}.$$

Then we get

$$J_1 \leq \varrho(n^{-1/\alpha} X^{O(n^{-1})}, n^{-1/\alpha} Y_a^{O(n^{-1})}) \times \text{Var}(\delta^{1/p} Y_a \circ n^{-1/\alpha} X_n, \delta^{1/p} Y_a \circ n^{-1/\alpha} Y_a^n)
+ \varrho(\delta^{1/p} Y_a \circ n^{-1/\alpha} Y_a^{(n-1)} \circ n^{-1/\alpha} X_n, \delta^{1/p} Y_a \circ Y_a^{O(n)}) = I_1 + I_2.$$

To estimate $J_2$ we use Lemma 3.3 (ii) with

$$X = n^{-1/\alpha} X^{O(n^{-j-1})}, \quad R = n^{-1/\alpha} Y_a^{O(n^{-j-1})}, \quad Y = n^{-1/\alpha} X_{n-j} \circ n^{-1/\alpha} Y_a^{O(j)}$$

and

$$Z = n^{-1/\alpha} Y_a^{(n-j)} \circ n^{-1/\alpha} Y_a^{O(j)}.$$

Then we obtain

$$J_2 \leq \sum_{j=1}^m \varrho(n^{-1/\alpha} X^{O(n^{-j-1})}, n^{-1/\alpha} Y_a^{O(n^{-j-1})})
\times \text{Var}(\delta^{1/p} Y_a \circ n^{-1/\alpha} X_{n-j} \circ n^{-1/\alpha} Y_a^{O(j)}, \delta^{1/p} Y_a \circ n^{-1/\alpha} Y_a^{O(j+1)})
+ \sum_{j=1}^m \varrho(\delta^{1/p} Y_a \circ n^{-1/\alpha} Y_a^{(n-j-1)} \circ n^{-1/\alpha} X_{n-j} \circ n^{-1/\alpha} Y_a^{O(j)},
\delta^{1/p} Y_a \circ n^{-1/\alpha} Y_a^{O(n)})
= I_3 + I_4.$$

We estimate $J_3$ using Lemma 3.4 with $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda = 1$, the fact that $\varrho \leq \frac{1}{2} \text{Var}$, and the ideality properties of $\tilde{V}_{r,p,a}$:

$$J_3 \leq \varrho(m^{-1/\alpha} X^{O(n-m-1)} \circ m^{-1/\alpha} Y_a^{O(m+1)}, m^{-1/\alpha} Y_a^{O(n-m-1)} \circ m^{-1/\alpha} Y_a^{O(m+1)})
\leq \text{Var}(m^{-1/\alpha} X^{O(n-m-1)} \circ m^{-1/\alpha} Y_a^{O(m)}, m^{-1/\alpha} Y_a^{O(n-m-1)} \circ m^{-1/\alpha} Y_a^{O(m)})
\leq \text{Var}(m^{-1/\alpha} X^{(n-m-1)} \circ Y_a, m^{-1/\alpha} Y_a^{O(n-m-1)} \circ Y_a)
\leq m^{-rp/\alpha} (n-m-1) \tilde{V}_{r,p,a}(X, Y_a) \leq D_1 n^{1-\alpha p/\alpha} \tilde{V}_{r,p,a}(X, Y_a)$$

for $n$ large enough and some positive $D_1$. We use Lemma 3.4 with $\lambda_1 = \delta^{1/p}$, $\lambda_2 = (j/n)^p$, and the ideality properties of $\tilde{V}_{r,p,a}$ to estimate the term involving $\text{Var}$ in $I_3$:

$$\text{Var}(\delta^{1/p} Y_a \circ n^{-1/\alpha} X_{n-j} \circ Y_a^{O(j)}, \delta^{1/p} Y_a \circ n^{-1/\alpha} Y_a^{(n-j)} \circ n^{-1/\alpha} Y_a^{O(j)})
\leq 1/(\delta^{1/p} + i/n)^{rp/\alpha} \tilde{V}_{r,p,a}(n^{-1/\alpha} X, n^{-1/\alpha} Y_a) = n^{-rp/\alpha} (\delta^{1/p} + i/n)^{rp/\alpha} \tilde{V}_{r,p,a}(X, Y_a).$$
Also, by inductive hypothesis, we estimate the term involving \( q \) in \( I_3 \):

\[
q\left(n^{-1/\alpha}X^{(n-j-1)}, n^{-1/\alpha} Y_a^{(n-j-1)}\right)
\]
\[
= q\left((n-j-1)^{-1/\alpha} X^{(n-j-1)}, (n-j-1)^{-1/\alpha} Y_a^{(n-j-1)}\right).
\]
\[
= q\left((n-j-1)^{-1/\alpha} X^{(n-j-1)}, Y_a\right)
\]
\[
\leq B(\tilde{v}_{rp,a}(X, Y_a)(n-j-1)\frac{1-r/p}{\alpha} + \tilde{\tau}_{rp,a}(n-j-1)^{-p/\alpha})
\]
\[
\leq B(\tilde{v}_{rp,a}(X, Y_a)(n-m-1)^{1-r/p/a} + \tilde{\tau}_{rp,a}(n-m-1)^{-p/\alpha})
\]
\[
\leq BD_2(\tilde{v}_{rp,a}(X, Y_a)n^{1-r/p/a} + \tilde{\tau}_{rp,a} n^{-p/\alpha})
\]

for sufficiently large \( n \) and some positive \( D_2 \). Note that \( \delta \leq A\tilde{\tau}_{rp,a} n^{-p/\alpha} \), so that

\[
I_3 \leq B(\tilde{v}_{rp,a}(X, Y_a)n^{1-r/p/a} + \tilde{\tau}_{rp,a} n^{-p/\alpha})
\]
\[
\times \sum_{j=1}^{\infty} \tilde{v}_{rp,a}(X, Y_a)/(A^{a/p}(\tilde{v}_{rp,a}(X, Y_a))^{a/p}(r-a/p) + j)^{-r/p/a}
\]
\[
\leq D_3 BA^{a/p-r}(\tilde{v}_{rp,a}(X, Y_a)n^{1-r/p/a} + \tilde{\tau}_{rp,a} n^{-p/\alpha})
\]

for some positive \( D_3 \) and large \( n \). Now we will estimate \( I_2 + I_4 \). We use Lemma 3.4 with \( \lambda_1 = ((n-j-1)/(n-1))^{1/\alpha}, \lambda_2 = (j/(n-1))^{1/\alpha}, \lambda = 1, 2q \leq \text{Var}, \) and the ideality of \( \tilde{v}_{rp,a} \):

\[
I_2 + I_4 = \sum_{j=0}^{m} q(\delta^{1/p} Y_a o n^{-1/\alpha} Y_a^{(n-j-1)} o n^{-1/\alpha} X_{n-j} o n^{-1/\alpha} Y_a^{(j)}, \delta^{1/p} Y_a o n^{-1/\alpha} Y_a^{(0)})
\]
\[
= \sum_{j=0}^{m} q((n-1)^{-1/\alpha} Y_a^{(n-j-1)} o (n-1)^{-1/\alpha} X_{n-j} o (n-1)^{-1/\alpha} Y_a^{(j)}),
\]
\[
(n-1)^{-1/\alpha} Y_a^{(n-j-1)} o (n-1)^{-1/\alpha} Y_a^{(n-j)} o (n-1)^{-1/\alpha} Y_a^{(0)}
\]
\[
\leq \frac{1}{2} \sum_{j=0}^{m} \text{Var} \left( \left(\frac{n-j-1}{n-1}\right)^{1/\alpha} Y_a (n-1)^{1/\alpha} X_{n-j} \left(\frac{j}{n-1}\right)^{1/\alpha} Y_a \right)
\]
\[
\leq \frac{1}{2} \sum_{j=0}^{m} \tilde{v}_{rp,a}(n-1)^{-1/\alpha} X_{n-j}, (n-1)^{-1/\alpha} Y_a^{(n-j)}
\]
\[
\leq m(n-1)^{-r/p/a} \tilde{v}_{rp,a}(X, Y_a) \leq D_4 n^{1-r/p/a} \tilde{v}_{rp,a}(X, Y_a)
\]

for some constant \( D_4 \) and large \( n \). To estimate \( I_1 \), we use Lemma 3.4 with \( \lambda_1 = \lambda = \delta^{i/p}, \lambda = 0 \), the inductive hypothesis (5), and the ideality of \( \tilde{v}_{rp,a} \):

\[
q\left(n^{-1/\alpha} X^{(n-1)}, n^{-1/\alpha} Y_a^{(n-1)}\right) \leq B(\tilde{v}_{rp,a}(X, Y_a)(n-1)^{1-r/p/a} + \tilde{\tau}_{rp,a}(n-1)^{-p/\alpha})
\]
\[
\leq BD_5(\tilde{v}_{rp,a}(X, Y_a)n^{1-r/p/a} + \tilde{\tau}_{rp,a} n^{-p/\alpha})
\]
for some constant $D_5$ and large $n$. Further,

$$
\text{Var}(\delta^{1/p} Y_a \circ n^{-1/\alpha} X_n, \delta^{1/p} Y_a \circ \delta^{1/p} Y^{(n)}) \leq 1/\delta \tilde{v}_{\alpha,p}(X, Y_a) n^{-p/\alpha}.
$$

Since $\tilde{v}_{\alpha,p}(X, Y_a) n^{-p/\alpha} < \delta/A$, we obtain

$$
I_1 \leq BD_5/A (\tilde{v}_{\alpha,p}(X, Y_a) n^{1-r/p} + \tilde{r}_{\alpha,p} n^{-p/\alpha}).
$$

Combining all the above estimates, we get

$$
q(n^{-1/\alpha} X^{on}, Y_a) \leq I_1 + I_2 + I_3 + I_4
\leq CB(D_3/A + D_3 A^{a/p - r})(\tilde{v}_{\alpha,p}(X, Y_a) n^{1-r/p} + \tilde{r}_{\alpha,p} n^{-p/\alpha})
+ C(D_1 + D_4) n^{1-r/p} \tilde{r}_{\alpha,p}(X, Y_a) + CA \tilde{r}_{\alpha,p} n^{-p/\alpha}.
$$

It is clear that we can choose a new constant $C$, so that

$$
q(n^{-1/\alpha} X^{on}, Y_a) \leq C(\tilde{v}_{\alpha,p}(X, Y_a) n^{1-r/p} + \tilde{r}_{\alpha,p} n^{-p/\alpha}).
$$

4. A note on applications. Interesting applications of this theory come from the area of “heavy tail” modeling, in particular in mathematical finance, as discussed by Mittnik and Rachev [5]. The problem of finding the probability distribution of financial asset returns has occupied a prominent place in statistical and financial literature (for a comprehensive review of the literature see Mittnik and Rachev [5] and the references therein). The $(\alpha, p)$-stable laws were successfully used in modeling the distribution of financial data. Panorska [7] fitted $(\alpha, p)$-stable models to foreign exchange rates and interest rates, and showed that they fit the data better than Paretian stable and normal models.

The empirical distribution of financial data is very often heavy-tailed and peaky at the center (i.e. leptokurtic), and thus to model it successfully we need distributions with similar properties. It turns out that variables stable with respect to $p$-convolutions have both of the above properties (for details on modeling and properties of $(\alpha, p)$-stable models see Panorska [7]). Apart from that, they are stable, and thus have domains of attraction, which makes them even more attractive models. The $(\alpha, p)$-stable models have several advantages over the Paretian stable ones. It occurs that their densities can be unimodal or bimodal, depending on $\alpha$ and $p$. Thus, we can fix the tail order $\alpha$, and by varying $p$ we obtain a unimodal or bimodal model. Another advantage of the $(\alpha, p)$-stable laws is that by choosing appropriate $p$ we can get a model with much steeper center peak than that allowed by Paretian stable distributions (financial data is often very peaky at the center). Finally, since $\alpha$ is not bounded above by 2, we can model data with tails lighter than those of Paretian stable distributions (see Loretan and Phillips [4]).

The $(\alpha, p)$-stable models successfully compete with Paretian stable ones in modeling one-dimensional financial asset returns. Extensions to multivariate portfolios' modeling would be a natural next step in this line of research.
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Department of Mathematics
The University of Tennessee at Chattanooga
615 McCallie Ave
Chattanooga, TN 37403-2598, U.S.A.
e-mail: panorska@utkux.utcc.utk.edu

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