ON CERTAIN SUBCLASSES OF THE CLASSES $L_c$

BY

T. RAJBÄ* (WROCLAW)

Abstract. Loève in [5] introduced the classes $L_c$ associated with number $c$, $c \in \mathbb{R}$, as the classes of probability measures satisfying the condition (1). Many authors investigated those classes ([2], [5]–[9], [20], [21]). In this paper we consider certain subclasses $L_{c_1,...,c_k}$ of the classes $L_c$. We prove that they coincide with the classes of distributions of series of some random variables and with the classes of limit distributions of some normed sums. We give a characterization of certain classes $D_{c_1,...,c_k}$ associated with $L_{c_1,...,c_k}$.

Urbanik in [18] introduced the concept of the decomposability semigroup associated with probability measure $P$, as the set of all numbers $c$, such that $P \in L_c$ ([11]–[14]). The class $L$ of self-decomposable distributions coincides with the class of probability measures $P$ such that $D(P) \supset [0, 1]$. The class $L_m$, $m \geq 1$, of multiply self-decomposable distributions may be described as the class of probability measures $P$ such that $P \in L_{c_1,...,c_m}$, for every $c_1, \ldots, c_m \in [0, 1]$, or in terms of multiply decomposability semigroups it is equivalent to the inclusion $D_m(P) \supset [0, 1]^m$, where $D_m(P)$ is the multiply decomposability semigroup defined by the formula $D_m(P) = \{(c_1, \ldots, c_m); P \in L_{c_1,...,c_m}\}$ ([3], [4], [10], [15]–[17], [19]).

Let $\varphi$ be the characteristic function of a probability measure on the real line $\mathbb{R}$. We say ([2], [3]) that $\varphi$ is $c$-decomposable, $c \in \mathbb{R}$, if

$$\varphi(t) = \varphi(ct) \varphi_c(t), \quad t \in \mathbb{R},$$

for some characteristic function $\varphi_c$. $L_c$ is the family of all $c$-decomposable laws. $L_0$ and $L_1$ are the families of all laws. Every $L_c$ is closed under compositions and passages to the limit.

Let $X$ be a random variable with the characteristic function $\varphi$. The probability distribution of the random variable $X$ will be denoted by $\mathcal{L}(X)$. Rewriting (1) in terms of random variables we obtain $\varphi \in L_c$ if and only if

$$\mathcal{L}(X) = \mathcal{L}(cX + X_c)$$

* Institute of Mathematics, Wroclaw University.
for some random variable \( X \) with the characteristic function \( \varphi_e \), such that \( X \) and \( X_e \) are independent.

For nondegenerate and \( c \)-decomposable laws, the inequality \( |c| \leq 1 \) is satisfied. Further, if \( \varphi \) is nondegenerate and \( c \)-decomposable with \( 0 < |c| < 1 \), then \( \varphi \) is the characteristic function of a continuous distribution [21]. In the sequel we consider only nondegenerate laws and the numbers \( c \) such that \( 0 < |c| < 1 \).

For nondegenerate \( \varphi \), \( \varphi \in L_c \), \( 0 < |c| < 1 \), if and only if it is a characteristic function of

\[
X(c) = \sum_{k=0}^{\infty} c^k Z_k,
\]

where \( Z_k \), \( k = 0, 1, 2, \ldots \), are independent and identically distributed random variables. Then the series converges a.s. (almost surely) and \( \varphi_{Z_k} = \varphi_e \), where \( \varphi_{Z_k} \) means the characteristic function of \( Z_k \) (see [5] and [6]).

Rewriting (3) in terms of characteristic functions, we obtain \( \varphi \in L_c \) if and only if

\[
\varphi(t) = \prod_{k=0}^{\infty} \varphi_e(c^k t)
\]

for some characteristic function \( \varphi_e \).

Further, \( \varphi \in L_c \) if and only if it is the limit of a sequence of characteristic functions of normed sums \( S_n/B_n \) of independent random variables with \( B_n/B_{n+1} \to c \):

\[
\varphi_{S_n/B_n}(t) \to \varphi(t),
\]

where \( S_n = Y_1 + \ldots + Y_n \), and \( Y_1, Y_2, \ldots \) are independent random variables (see [2]).

Now we define certain subclasses of the classes \( L_c \).

Let \( c_1, c_2, \ldots, c_k \in \mathbb{R} \), \( k \geq 1 \). We say that \( \varphi \) belongs to \( L_{c_1,c_2,\ldots,c_k} \) if and only if

\[
\varphi(t) = \varphi(c_1 t) \varphi_{c_1}(t), \quad \varphi_{c_1}(t) = \varphi_{c_1}(c_2 t) \varphi_{c_1,c_2}(t), \ldots
\]

\[
\ldots, \varphi_{c_1,\ldots,c_{k-1}}(t) = \varphi_{c_1,\ldots,c_{k-1}}(c_k t) \varphi_{c_1,\ldots,c_k}(t)
\]

for some characteristic functions \( \varphi_{c_1}, \varphi_{c_1,c_2}, \varphi_{c_1,\ldots,c_k} \).

We note that (6) is equivalent to the following statement:

\[
\varphi \in L_{c_1}, \varphi_{c_1} \in L_{c_2}, \ldots, \varphi_{c_1,\ldots,c_{k-1}} \in L_{c_k}.
\]

Obviously, if \( \varphi \in L_{c_1,\ldots,c_k} \), then

\[
\varphi(t) = \varphi(c_1 t) \varphi_{c_1}(c_2 t) \ldots \varphi_{c_1,\ldots,c_{k-1}}(c_k t) \varphi_{c_1,\ldots,c_k}(t),
\]

\[
\mathcal{L}(X) = \mathcal{L}(c_1 X + c_2 X_{c_1} + c_3 X_{c_1,c_2} + \ldots + c_k X_{c_1,\ldots,c_{k-1}} + X_{c_1,\ldots,c_k})
\]

for some independent random variables \( X, X_{c_1}, X_{c_1,c_2}, \ldots, X_{c_1,\ldots,c_k} \) with characteristic functions \( \varphi, \varphi_{c_1}, \varphi_{c_1,c_2}, \ldots, \varphi_{c_1,\ldots,c_k} \), respectively.
We say that $\psi$ belongs to $D_{c_1,\ldots,c_k}$ if and only if there exist characteristic functions $\varphi, \varphi_{c_1}, \ldots, \varphi_{c_1,\ldots,c_k}$ satisfying (6) such that $\varphi_{c_1,\ldots,c_k} = \psi$. For $c_1 = c_2 = \ldots = c_k = c$, instead of $L_{c_1,\ldots,c_k}, \varphi_{c_1,\ldots,c_k}, X_{c_1,\ldots,c_k}, D_{c_1,\ldots,c_k}$ we will write $L_{c,(k)}, \varphi_{c,(k)}, X_{c,(k)}, D_{c,(k)}$, respectively.

We note that

(10) \[ L_{c_1,\ldots,c_k-1,c_k} \subset L_{c_1,\ldots,c_k-1}, \quad L_{c,(k)} \subset L_{c,(k-1)}. \]

Let $Z$ be a random variable with the characteristic function $\psi$. We say that $\psi$ belongs to $D_{(k)}$ if and only if

(11) \[ E(\ln(|Z|+1)) < \infty. \]

The following theorem is a generalization of the theorem proved by Zaksilos in [20] for $k = 1$.

**Theorem 1.** For each $k$, $k \geq 1$, the classes $D_{c,(k)}$, $0 < |c| < 1$, are independent of $c$ and coincide with the class $D_{(k)}$.

**Proof.** Given $k \geq 1$ and $0 < |c| < 1$. Let $\{Z_{j_1,\ldots,j_k}\}_{j_1,\ldots,j_k=0}^{\infty}$, and let $Z$ be independent and identically distributed random variables with an arbitrary common characteristic function $\psi$.

Let $N = \{0, 1, 2, \ldots\}$ and $N^k = \{(j_1, \ldots, j_k), j_1, \ldots, j_k \in N\}$. Consider a one-to-one and onto mapping $x: N \to N^k$. For elements of $N^k$ we put

(12) \[ x(n) = ((x(n))_1, \ldots, (x(n))_k), \quad |x(n)| = \sum_{j=1}^{k} (x(n))_j, \quad n = 0, 1, 2, \ldots \]

We are going to investigate the convergence of the series

(13) \[ \sum_{n=0}^{\infty} c^{(|x(n)|)} Z_{x(n)}, \]

say to a random variable $X_k(c)$.

As in [20] (see also [1]), we consider two series:

(i) \[ \sum_{n=0}^{\infty} P(|c^{(|x(n)|)} Z_{x(n)}| > 1)), \]

(ii) \[ \sum_{n=0}^{\infty} E(|c^{(|x(n)|)} Z_{x(n)}|; |c^{(|x(n)|)} Z_{x(n)}| < 1)). \]

We observe that the convergence of the series (i) and (ii) is equivalent to the convergence of the following series (i') and (ii') and, consequently, (i'') and (ii''):

(i') \[ \sum_{j_1=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} P(|c^{j_1} \ldots c^{j_k} Z_{j_1,\ldots,j_k}| > 1)), \]

(ii') \[ \sum_{j_1=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} E(|c^{j_1} \ldots c^{j_k} Z_{j_1,\ldots,j_k}|; |c^{j_1} \ldots c^{j_k} Z_{j_1,\ldots,j_k}| < 1)). \]
Taking into account the equality $\mathcal{L}(Z_{j_1,...,j_k}) = \mathcal{L}(Z)$ we can write the series (i'') in the form

$$\sum_{n=0}^{\infty} \left( \sum_{j_1+...+j_k=n} P(\{|c^{j_1}...c^{j_k}Z_{j_1,...,j_k}| > 1\}) \right)$$

The convergence of the above series is equivalent to the convergence of the series

$$\sum_{n=0}^{\infty} \binom{n+k-1}{k-1} P(|Z| > |c^{-1}|^n)$$

The above series is convergent if and only if the series

$$\sum_{n=0}^{\infty} (n+1)^k P(\{|\ln|Z| > n \ln |c^{-1}|\})$$

is convergent, which is equivalent to satisfying the condition (11).

The series (ii'') equals

$$\sum_{n=0}^{\infty} \left( \frac{n+k-1}{k-1} \right) |c^n| E(|Z|; |Z| < |c^{-1}|^n)$$

The convergence of the above series is equivalent to the convergence of the series

$$\sum_{n=0}^{\infty} (n+1)^k |c^n| E(|Z|; |Z| < |c^{-1}|^n)$$

$$= \sum_{n=0}^{\infty} (n+1)^k |c^n| E(|Z|; |Z| < 1)$$

$$+ \sum_{n=0}^{\infty} (n+1)^k |c^n| \sum_{j=0}^{n-1} E(|Z|; |c^{-1}|^j \leq |Z| < |c^{-1}|^{j+1})$$

$$\leq E(|Z|; |Z| < 1) \sum_{n=0}^{\infty} (n+1)^k |c|^n$$

$$+ \sum_{n=0}^{\infty} (n+1)^k |c^n| \sum_{j=0}^{n-1} |c^{-1}|^{j+1} P(|c^{-1}|^j < |Z| \leq |c^{-1}|^{j+1})$$

$$\leq \frac{(k-1)!}{(1-|c|)^k} + \sum_{j=0}^{\infty} P(|c^{-1}|^j \leq |Z| < |c^{-1}|^{j+1}) \sum_{n=j+1}^{\infty} (n+1)^k |c|^{n-j-1}.$$
Since
\[
\sum_{n=j+1}^{\infty} (n+1)^{k-1} |c|^{n-j-1} = \sum_{i=0}^{\infty} (i+j+2)^{k-1} |c|^i
\]
\[
= \sum_{i=0}^{\infty} \left( \sum_{m=0}^{k-1} \binom{k-1}{m} (j+1)^m (i+1)^{k-1-m} \right) |c|^i
\]
\[
= \sum_{m=0}^{k-1} \binom{k-1}{m} (j+1)^m \sum_{i=0}^{\infty} (i+1)^{k-1-m} |c|^i
\]
\[
\leq \sum_{m=0}^{k-1} \binom{k-1}{m} (j+1)^m \frac{(k-1-m)!}{1-|c|} \left( \frac{1}{1-|c|} \right)^{k-1-m}
\]
\[
\leq \frac{(k-1)!}{1-|c|} \left( \frac{1}{1-|c|} + j+1 \right)^{k-1},
\]
for the series in expressions (14) we have the inequality
\[
\sum_{j=0}^{\infty} P\left(\{ |c^{-1}|^j \leq |Z| < |c^{-1}|^{j+1} \} \right) \sum_{n=j+1}^{\infty} (n+1)^{k-1} |c|^{n-j-1}
\]
\[
\leq \frac{(k-1)!}{1-|c|} \sum_{j=0}^{\infty} P\left(\{ |c^{-1}|^j \leq |Z| < |c^{-1}|^{j+1} \} \right) \left( \frac{1}{1-|c|} + j+1 \right)^{k-1}.
\]
Since the series in the above expression is convergent if and only if \( E(\ln^{k-1}(|Z|+1)) < \infty \), this completes the proof that the convergence of two series (i) and (ii) is equivalent to satisfying the condition (11). We note that condition (11) is independent of \( c \). Hence we conclude that the series (13) is convergent if and only if the condition (11) holds; moreover, the convergence of the series (13) is independent of the choice of the mapping \( x \). Thus we can write \( X_k(c) \) in the form
\[
X_k(c) = \sum_{j_1=0}^{\infty} \cdots \sum_{j_k=0}^{\infty} c^{j_1} \cdots c^{j_k} Z_{j_1,\ldots,j_k}.
\]
Putting
\[
\varphi_{c,(k)}(t) = \psi(t), \quad \varphi_{c,(k-1)}(t) = \prod_{n=0}^{\infty} \varphi_{c,n}(c^n t), \ldots
\]
\[
\ldots, \quad \varphi_{c,(j-1)} = \prod_{n=0}^{\infty} \varphi_{c,(j)}(c^n t), \quad 1 \leq j \leq k,
\]
\[
\varphi(t) = \varphi_{c,(0)}(t),
\]
we complete the proof of the theorem.

Let \( k \geq 1 \) and \( 0 < |c_j| < 1 \) for \( 1 \leq j \leq k \). Consider now the series
\[
\sum_{j_1=0}^{\infty} \cdots \sum_{j_k=0}^{\infty} c^{j_1} \cdots c^{j_k} Z_{j_1,\ldots,j_k}
\]
for \( Z_{j_1,\ldots,j_k} \) as in Theorem 1.
Then taking into account the inequality
\[
(\min_{1 \leq j \leq k} |c_j|^{h_1 + \ldots + h_k}) \leq |c_1|^{h_1} \ldots |c_k|^{h_k} \leq (\max_{1 \leq j \leq k} |c_j|)^{h_1 + \ldots + h_k},
\]
as a corollary to Theorem 1 we obtain

**Theorem 2.** Let \( k \geq 1 \). The classes \( D_{c_1, \ldots, c_k}, 0 < |c_j| < 1, 1 \leq j \leq k, \) coincide with the class \( D_{(k)} \).

From the proof of the above two theorems we obtain immediately the following two theorems:

**Theorem 3.** Let \( k \geq 1 \) and \( 0 < |c_j| < 1 \) for \( 1 \leq j \leq k \). Let \( \varphi \) be a characteristic function. Then the following conditions are equivalent:

(a) \( \varphi \in L_{c_1, \ldots, c_k} \).

(b) \( \varphi \) is the characteristic function of

\[
X(c_1, \ldots, c_k) = \sum_{j_1 = 0}^{\infty} \ldots \sum_{j_k = 0}^{\infty} c_1^{j_1} \ldots c_k^{j_k} Z_{j_1, \ldots, j_k},
\]

where \( \{Z_{j_1, \ldots, j_k} \}_{j_1, \ldots, j_k = 0}^{\infty} \) are independent identically distributed random variables with the same characteristic function \( \psi \). Then the series converges a.s.

(c) \( \varphi \) is the characteristic function of the form

\[
\varphi(t) = \prod_{j_1 = 0}^{\infty} \ldots \prod_{j_k = 0}^{\infty} \psi(c_1^{j_1} \ldots c_k^{j_k})
\]

for some characteristic function \( \psi \).

**Theorem 4.** Let \( k \geq 1 \) and \( 0 < |c| < 1 \). Let \( \varphi \) be a characteristic function. Then the following conditions are equivalent:

(a) \( \varphi \in L_{c, (k)} \).

(b) \( \varphi \) is the characteristic function of

\[
X_k(c) = \sum_{n=0}^{\infty} c^n \sum_{j=1}^{\binom{n+k-1}{k-1}} Z_{n,j},
\]

where \( \{Z_{n,j} \}_{n,j} \), \( n = 0, 1, 2, \ldots, j = 1, 2, \ldots, \binom{n+k-1}{k-1} \), are independent identically distributed random variables with the same characteristic function \( \psi = \varphi_{c,(k)} \). Then the series converges a.s.

(c) \( \varphi \) is the characteristic function of the form

\[
\varphi(t) = \prod_{n=0}^{\infty} [\psi(c^n t)]^{\binom{n+k-1}{k-1}}
\]

for some characteristic function \( \psi = \varphi_{c,(k)} \).

In the next theorem we show that the classes \( L_{c,(k)} \) coincide with some limit distributions of normed sums.

**Theorem 5.** Let \( k \geq 1 \), \( 0 < |c| < 1 \), and \( \varphi \) be a characteristic function. Then \( \varphi \in L_{c,(k)} \) if and only if there exists a sequence of positive numbers
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$B_0, B_1, \ldots, B_n/B_{n+1} \to c$ and a sequence of random variables $U_0, U_1, \ldots$ such that, for independent random variables $\{X_{n,j}\}_{n,j}$, $n = 0, 1, 2, \ldots$, $j = 1, 2, \ldots, \binom{n+k-1}{k-1}$,

$$\mathcal{L}(X_{n,1}) = \mathcal{L}(U_n) \quad \text{and} \quad \mathcal{L}(X_{n,j}) = \mathcal{L}(U_i) \quad \text{for} \quad k > 1,$$

(19)

$$\left(\frac{n-i-1+k-1}{k-1}\right) < j \leq \binom{n-i+k-1}{k-1}, \quad 0 \leq i \leq n-1, \quad k > 1,$$

(20)

$$Y_{n,(k)} = \sum_{j=1}^{\binom{n+k-1}{k-1}} X_{n,j},$$

(21)

$$S_{n,(k)} = Y_{1,(k)} + \ldots + Y_{n,(k)}$$

such that the characteristic function of $S_{n,(k)}/B_n$ is convergent to the characteristic function $\varphi$,

(22)

$$\varphi_{S_{n,(k)}/B_n}(t) \to \varphi(t).$$

Proof. Let $B_0, B_1, \ldots, B_n/B_{n+1} \to c$ be a sequence of positive integers and $U_0, U_1, \ldots$ be a sequence of random variables. Suppose that for independent random variables $\{X_{n,j}\}_{n,j}$, $n = 0, 1, 2, \ldots$, $j = 1, 2, \ldots, \binom{n+k-1}{k-1}$, as in (19), and for $Y_{n,(k)}$ and $S_{n,(k)}$, $n = 1, 2, \ldots$, defined by (20) and (21), respectively, the convergence (22) holds. Since

$$\varphi_{S_{n,(k)}/B_n}(t) = \varphi_{S_{n-1,(k)}/B_{n-1}}\left(\frac{B_{n-1}}{B_n} t\right) \varphi_{Y_{n,(k)}/B_n}(t),$$

where $\varphi_{S_{n-1,(k)}/B_{n-1}}\left(\frac{B_{n-1}}{B_n} t\right) \to \varphi(ct)$,

denoting by $\varphi_{c(1)}$ the characteristic function which is the limit of $\varphi_{Y_{n,(k)}/B_n}(t)$ (see [2]; without loss of generality we can assume that $\varphi_{Y_{n,(k)}/B_n}(t)$ is convergent to a characteristic function, passing to a subsequence if necessary), we obtain

(23)

$$\varphi(t) = \varphi(ct) \varphi_{c(1)}(t)$$

and, consequently, $\varphi \in L_c$. This completes the proof of "if" assertion for $k = 1$.

Now suppose that $k > 1$. Then

$$\varphi_{Y_{n,(k)}/B_n}(t) = [\varphi_{U_0/B_n}(t)]^{n+k-2\choose k-2} \ldots [\varphi_{U_n/B_n}(t)]^{0+k-2\choose k-2}$$

$$= \varphi_{Y_{n-1,(k)}/B_{n-1}}\left(\frac{B_{n-1}}{B_n} t\right) [\varphi_{U_0/B_n}(t)]^{n+k-3\choose k-3} \ldots [\varphi_{U_n/B_n}(t)]^{0+k-3\choose k-3},$$

where

$$\varphi_{Y_{n,(k)}/B_n}(t) \to \varphi_{c(1)}(t), \quad \varphi_{Y_{n-1,(k)}/B_{n-1}}\left(\frac{B_{n-1}}{B_n} t\right) \to \varphi_{c(1)}(ct).$$
Hence, as for $\varphi_{c,(1)}$, we can put

$$\varphi_{c,(2)} = \lim_{n \to \infty} \left[ \varphi_{U_0/B_n}(t) \right]^{(n+k-3, k-3)} \ldots \left[ \varphi_{U_n/B_n}(t) \right]^{(0+k-3, k-3)}.$$  

Then we obtain $\varphi_{c,(1)} \in L_c$ and, consequently, $\varphi \in L_{c,(2)}$.

It is not difficult to show (by induction) that for $1 \leq j \leq k-1$

$$(24) \quad \varphi_{c,(j)}(t) = \varphi_{c,(j)(ct)} \varphi_{c,(j+1)}(t),$$

where

$$\varphi_{c,(j)}(t) = \lim_{n \to \infty} \left[ \varphi_{U_0/B_n}(t) \right]^{(n+k-j+1, k-j+1)} \ldots \left[ \varphi_{U_n/B_n}(t) \right]^{(0+k-j+1, k-j+1)}, \quad 1 \leq j \leq k-1,$$

$$\varphi_{c,(k)}(t) = \lim_{n \to \infty} \varphi_{U_n/B_n}(t).$$

By (23) and (24) we obtain $\varphi \in L_{c,(k)}$. Thus, $\varphi \in L_{c,(k)}$ in both the cases, and the "if" assertion is proved.

Now suppose that $\varphi \in L_{c,(k)}$, $k \geq 1$. It follows from Theorem 4 (b) that there exist independent and identically distributed random variables $\{Z_{n,j}\}_{n,j}$, $n = 0, 1, 2, \ldots, j = 1, 2, \ldots, (n+k-1)$, with common characteristic function $\psi$, i.e.

$$\varphi_{Z_{n,j}} = \varphi_Z = \psi, \quad n = 0, 1, 2, \ldots, j = 1, 2, \ldots, (n+k-1),$$

such that $\varphi$ is the characteristic function of

$$X_k(c) = \sum_{n=0}^{\infty} c^n \sum_{j=0}^{n+k-1} Z_{n,j}.$$  

Let $B_0, B_1, \ldots$ be a number sequence such that $B_n/B_{n+1} \to c$. Put $U_n = B_n Z_{n,1}, X_{n,1} = U_n, n = 0, 1, 2, \ldots$ In the case $k > 1$ we put

$$X_{n,j} = B_j Z_{n,j}, \quad \binom{n+i-k-1}{k-1} < j \leq \binom{n+i-k-1}{k-1}, \quad 0 \leq i \leq n-1.$$  

Further, we put

$$Y_{n,(k)} = \sum_{j=1}^{n+k-1} X_{n,j}, \quad S_{n,(k)} = Y_{1,(k)} + \ldots + Y_{n,(k)}, \quad n = 1, 2, \ldots$$

In the case $k \geq 2$ we have

$$\varphi_{S_{n,(k)}}(t) = \left[ \varphi_{B_n/B_n Z_n}(t) \right]^{(0+k-2)} \left[ \varphi_{B_{n-1}/B_n Z_n}(t) \right]^{(0+k-2)} \ldots$$

$$\ldots \left[ \varphi_{B_1/B_n}(t) \right]^{(0+k-2)} \ldots + \left[ \varphi_{B_0/B_n}(t) \right]^{(0+k-2)} \ldots + \left[ \varphi_{B_0/B_n}(t) \right]^{(n+k-2)}.$$
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Taking into account that
\[
\binom{0+k-2}{k-2} + \ldots + \binom{j+k-2}{k-2} = \binom{j+k-1}{k-1}, \quad j = 0, 1, 2, \ldots, n,
\]
we obtain
\[
\varphi_{S_n(k)/B_n}(t) = [\psi(t)]^\alpha \left[ \psi \left( \frac{B_{n-1}}{B_n} \right) \right]^{a+k-1} \left( \frac{B_0}{B_1} \frac{B_1}{B_2} \ldots \frac{B_{n-1}}{B_n} \right)^{(a+k-1)}.
\]
Formula (25) in the case $k = 1$ evidently also holds. From (25) it follows that
\[
\varphi_{S_n(k)/B_n}(t) = \prod_{n=0}^{\infty} [\psi(c^n t)]^{a+k-1},
\]
and this, by Theorem 4 (c), yields $\varphi_{S_n(k)/B_n}(t) \to \varphi(t)$. Thus the assertion “only if” is proved.

REFERENCES


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