ON MARTINGALE MEASURES FOR STOCHASTIC PROCESSES WITH DISCRETE TIME

BY

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Abstract. Let \( (X(t); t \in \mathbb{N}^*) \) be a random sequence adopted to a filtration \((\mathcal{F}_t)\) in \((\Omega, \mathcal{F}, P)\) satisfying some natural assumption. If none of the events \((X(t+1) > X(t)), (X(t+1) < X(t))\) can be predicted, i.e. none contains some \(A \in \mathcal{F}_t, P(A) > 0\), then \((X(t), \mathcal{F}_t)\) is a martingale for some probability \(P^*\) on \(\mathcal{F}\). It is a version of the "fundamental theorem of option pricing".

1. Introduction. Let \(X(t), t \in \mathbb{R}\), be a stochastic process. If \(X(t) = e^{mt+\sigma w(t)}\) with \(w(t)\) being a Wiener process, then \(X(t)\) becomes a martingale with respect to \(P^*\) being a probability equivalent to the original one \(P\). This theory, initiated by Girsanov, has been very tempting and widely researched for the last 30 years (we only mention monographs [4] and [11]–[13]). As one of the most famous applications of the theory one should mention the Black–Scholes model describing a replication strategy for European options (see [1], [8], [10] and [12]).

In the so-called financial mathematics, many efforts were also devoted to the formulation of the so-called "no free lunch" condition which, in more general situations, guarantees the existence of a martingale measure \(P^*\) equivalent to the original probability \(P\). The notion of free lunch is defined (in a non-effective way) by the use of some space of strategies \(\Theta(t)\) being stochastic processes predictable for some filtration \((\mathcal{F}_t)\). The construction of the martingale measure \(P^*\) is obtained by some development of the Banach–Mazur theorem of the separating of convex sets (cf. [3], [7]–[10] and [12]). Free lunch conditions look simpler for processes indexed by discrete finite times (cf. [2] and [6]).

In the paper we use one scalar stochastic process \(X(t)\) which corresponds to the simplest case of one security. The strategy is described by our position \(\Theta(t)\) in the security. We assume that all our outcomes and incomes are cumulated in a riskless bond.

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We propose a simple condition (analogous to that of Dalang–Morton–Willinger with zero interest rate [2]) which assures the existence of a martingale measure. This condition, later referred to as the change of sign property, states that
\[ P\left(\{X(t) > X(s)\} \cap A\right) > 0 \iff P\left(\{X(t) < X(s)\} \cap A\right) > 0 \]
for any \( A \in \mathcal{F}_s \), \( s < t \). Our arguments are rather classical. The required martingale measure \( P^* \) is obtained by the Kolmogorov extension theorem (see [4] and [13]). The main result is contained in Theorem 3.3.

2. Elementary examples. To explain the possibilities and restrictions appearing in constructing a martingale measure, let us consider some elementary examples.

2.1. Example. Suppose we are tossing a symmetric coin. Assume that \( \omega = (\varepsilon_1, \varepsilon_2, \ldots) \) is a sequence of outcomes, \( \varepsilon_i = 0 \) or 1 depending on the result of the \( i \)-th toss. Let \( \mathcal{F}_0 = \{\emptyset, \Omega\} \), \( \mathcal{F}_i = \sigma(\varepsilon_1, \ldots, \varepsilon_i) \) (i.e., a \( \sigma \)-field generated by random variables \( \varepsilon_1, \ldots, \varepsilon_i \)) and \( \mathcal{F} = \sigma(\varepsilon_1, \varepsilon_2, \ldots) \). Let \( X(t) = \sum_{i=1}^{t} (\varepsilon_i - \beta) \) for some \( \beta \in (0, 1) \). Then \( X(t) \) is a martingale with respect to the sequence \( (\mathcal{F}_i) \) for \( \beta = \frac{1}{2} \). For \( \beta \neq \frac{1}{2} \), \( X(t) \) becomes a martingale if the original probability \( P(\varepsilon_i = 0) = P(\varepsilon_i = 1) = \frac{1}{2} \) is replaced by \( P^*(\varepsilon_i = 1) = \beta = 1 - P^*(\varepsilon_i = 0) \), which corresponds to an asymmetric coin. Moreover, \( P^* \) is uniquely determined. Thus each martingale measure \( P^* \) satisfies
\[ P^*\left(\left\{ \omega; \lim_{n \to \infty} \frac{1}{n} (\varepsilon_1 + \ldots + \varepsilon_n) = \beta \right\}\right) = 1 \]
by the strong law of large numbers, while
\[ P\left(\left\{ \omega; \lim_{n \to \infty} \frac{1}{n} (\varepsilon_1 + \ldots + \varepsilon_n) = \frac{1}{2} \right\}\right) = 1. \]
Thus \( P \) and \( P^* \) are singular for \( \beta \neq \frac{1}{2} \).

When \( X(t) \) is indexed by an infinite set of \( t \)'s, it is impossible to obtain a martingale measure \( P^* \) equivalent to \( P \).

2.2. Example. As previously, we toss a coin obtaining outcomes \( \omega = (\varepsilon_1, \varepsilon_2, \ldots) \). Let us put
\[ \Omega^0 = \left\{ \omega; \lim_{n \to \infty} \frac{1}{n} (\varepsilon_1 + \ldots + \varepsilon_n) = \frac{1}{2} \right\} \quad \text{(then } P(\Omega^0) = 1), \]
\[ \mathcal{F}^0 = \{A \cap \Omega^0; A \in \mathcal{F} = \sigma(\varepsilon_1, \varepsilon_2, \ldots)\} \quad \text{and } X^0(t) = X(t)|_{\Omega^0}. \]
Since \( P(\Omega^0) = 1 \), the finite-dimensional distributions of the processes \( X^0(t) \) and \( X(t) \) are identical.

Suppose that there exists a martingale measure \( P_0^* \) on \( (\Omega^0, \mathcal{F}^0) \) for the process \( X^0(t) \). Then \( P_0^*(\varepsilon_i = 1) = \beta = 1 - P_0^*(\varepsilon_i = 0) \) and, by the strong law of large numbers,
\[ P_0^*(\Omega^0) = P_0^*\left(\left\{ \omega; \lim_{n \to \infty} \frac{1}{n} (\varepsilon_1 + \ldots + \varepsilon_n) = \frac{1}{2} \right\}\right) = 0, \]
which is a contradiction.
It is worth noting that $\Omega^0$ is not a closed set in the Tikhonov topology in $\Omega = \{0, 1\}^{N^+}$ (namely, $\overline{\Omega^0} = \Omega$). We shall show that the closure of the set of trajectories of the process is a natural support of a martingale measure $P^*$.

3. Main results. Let $Y(t), t \in N^+$, be a stochastic process on a probability space $(\Omega, \mathcal{F}, P)$. By $Y(t)$ we also denote its canonical representation on the space $(R^{N^+}, \sigma(\mathcal{F}), P_Y)$. Thus

1. $Y(t)(\omega) = \varepsilon_i$ for $\omega = (\varepsilon_1, \varepsilon_2, \ldots) \in R^{N^+}$;
2. $\mathcal{G} = \bigcup_{n \in N^+} \mathcal{G}_n$;
3. $\mathcal{G}_n = \{\mathcal{G}_n(A^n); (A^n) \in B_{R^n}\}$;
4. $\mathcal{G}_n(A^n) = \{(\varepsilon_1, \varepsilon_2, \ldots) \in R^{N^+}; (\varepsilon_1, \ldots, \varepsilon_n) \in A^n, A^n \in B_{R^n}$ (i.e., $\sigma$-fields of Borel sets in $R^n$);
5. $P_Y(\mathcal{G}_n(A^n)) = P_n(A^n)$ for a finite-dimensional distribution $P_n(A^n) = P((Y(1), \ldots, Y(n)) \in A^n)$ for $n \in N^+$. Obviously, the image $Y[\Omega]$ can be treated as a subspace of $R^{N^+}$ (proper, in general).

We need some modification of the classical Kolmogorov theorem. To explain new elements precisely, we decided to formulate two self-interesting lemmas. The following exercise will be used. For any set $T \subset X$ and any family $\mathcal{R} \subset 2^X$, we have

$$\sigma(\mathcal{R} \cap T) = \sigma(\mathcal{R}) \cap T.$$ 

Obviously, $\mathcal{R} \cap T$ means $\{R \cap T; R \in \mathcal{R}\}$.

3.1. Lemma. If, in the introduced notation $2^0-4^0$, $T$ is any set closed in $R^{N^+}$ in the Tikhonov topology, $P_n$ is a probability distribution on $\mathcal{G}_n \cap T$, $n \in N^+$, satisfying

$$P_{n+1}(\mathcal{G}_{n+1}(A_n \times R) \cap T) = P_n(\mathcal{G}_n(A_n) \cap T),$$

then there exists a uniquely defined probability measure $P$ on $\sigma(\mathcal{G} \cap T) = T \cap \sigma(\mathcal{G})$ such that

$$P_n = P|_{\mathcal{G}_n \cap T}.$$ 

Proof. Let $\mathcal{T}_n = T \cap \mathcal{G}_n$, $\mathcal{T} = \bigcup (T \cap \mathcal{G}_n) = T \cap \mathcal{G}$. For $B \in \mathcal{T}$, taking any representation $B$ in the form $B = \bigcap T \cap \mathcal{G}_n(A_n)$, we can uniquely define the function

$$Q(B) = P_n(T \cap \mathcal{G}_n(A_n))$$

which is finitely-additive and normed on $\mathcal{T}$. It remains only to prove 'continuity'. Let $B_1 \supset B_2 \supset \ldots$, $Q(B_i) \geq \varepsilon > 0$. To use the classical Kolmogorov construction (see [4] and [13]), one has to show that $\bigcap B_i \neq \emptyset$. We consider

$$\tilde{P}_n(C_n) = P_n(T \cap C_n) \quad \text{for} \quad C_n \in \mathcal{G}_n,$$

obtaining a consistent system of distributions on $\mathcal{G}_n$.\"
From the Kolmogorov lemma we infer that if $C_1 \supseteq C_2 \supseteq C_3 \supseteq \ldots$ and

$$\bar{p}_n(C_n) \geq \varepsilon > 0,$$

then there exists $\omega \in \bigcap_i C_i$.

We put $C_n = \mathcal{C}_n(A_n \cap T_i)$ for projections

$$T_n = \{(e_1, e_2, \ldots, e_n); (e_1, e_2, \ldots, e_n, \eta_{n+1}, \eta_{n+2}, \ldots) \in T\}
\quad \text{for some } \eta_{n+1}, \eta_{n+2}, \ldots,$$

assuming that $B_n = \mathcal{C}_n(A_n)$.

Let $\omega$ be as in the Kolmogorov lemma. Since $T$ is closed, we have $\omega \in T$ and

$$\omega \in T \cap \mathcal{C}_n(A_n) = \bigcap_n B_n.$$

The lemma is proved.

3.2. Lemma. If, in the introduced notation, $\mathcal{F} = \sigma(Y(1), Y(2), \ldots)$ and $Y[\Omega]$ is closed in $R^{N^*}$ in the Tikhonov topology, and if $P^*_Y|\mathcal{W}_n \sim P_Y|\mathcal{W}_n$ for some probability measure $P^*_Y$ on $\sigma(\mathcal{F})$, then there exists a uniquely defined probability measure $P^*$ on $\mathcal{F}$ satisfying

$$P^*_Y(A) = P^*(Y^{-1}[A]), \quad A \in \sigma(\mathcal{F}).$$

Proof. We put $T = Y[\Omega]$ and define $P_n(C_n \cap T) = P_n(C_n)$ for $C_n \in \mathcal{C}_n$. Distributions $P_n$ are well defined: if $C_n \cap T = C_n' \cap T$, then

$$(C_n \Delta C_n') \cap T = \emptyset;$$

it follows that $P_Y(C_n \Delta C_n') = 0$, so $P^*_n(C_n \Delta C_n') = 0$.

The condition of consistency (c) in Lemma 3.1 is obvious from the definition of $P_n$'s. The probability measure $P$ on $T \cap \sigma(\mathcal{F})$ exists by Lemma 3.1, and $P_n = P|\mathcal{W}_n \cap T$.

The measure $P^*$ that is being looked for can be defined by the formula

$$P^*(Y^{-1}(A)) = P(A \cap Y[\Omega]) \quad \text{for } A \in \sigma(\mathcal{F}).$$

The measure $P$ on $\sigma(\mathcal{F}) \cap T$ corresponds to a measure $P^*_Y(A) = P(A \cap T)$ on $\sigma(\mathcal{F})$. But from $P_n = P|\mathcal{W}_n \cap T$ we get $P^*_n(C_n) = P^*_Y(C_n)$ for $C_n \in \mathcal{F}$. The uniqueness of the extension of a countable additive function completes the proof (cf. [5]).

Remark. Obviously, to prove Lemma 3.2, it is enough to show that $P^*_Y = 0$ for any $A \in \sigma(\mathcal{F})$ disjoint from $Y[\Omega]$, or that $A \cap Y[\Omega] \neq \emptyset$ when $P^*_Y(A) = \varepsilon > 0$. It seems natural to repeat Kolmogorov's arguments for decreasing cylinders $C_1 \supseteq C_2 \supseteq \ldots$ defined by projections of $A$,

$$C_n = \{(e_1, e_2, \ldots); (e_1, \ldots, e_n, \eta_{n+1}, \ldots) \in A \text{ for some } \eta_{n+1}, \eta_{n+2}, \ldots\}.$$

An element $\omega \in \bigcap_n C_n$ satisfies $\omega \in Y[\Omega]$ but it may happen that $\omega \notin A$. Lemma 3.2 cannot be obtained in such a way.
Martingale measures

For a sequence of bounded random variables \((X(t), t \in \mathbb{N}^+)\) on a probability space \((\Omega, \mathcal{F}, P)\), let

\[ \mathcal{F} = \sigma(X(0), X(1), \ldots), \quad X(0) = 0, \]

(2) \[ X(\Omega) = \{X(t)(\omega); \omega \in \Omega\} \text{ is a closed set in the Tikhonov topology in } R^{N^+}. \]

Let us write \(\mathcal{F}_t = \sigma(X(0), \ldots, X(t)), t \in \mathbb{N}\).

3.3. Theorem. Under assumption (2) the following conditions are equivalent:

(i) \(P\left( A \cap (X(t+1) > X(t)) \right) > 0 \Leftrightarrow P\left( A \cap (X(t+1) < X(t)) \right) > 0 \) for any \( t \in \mathbb{N}, A \in \mathcal{F}_t \) (the change of sign property);

(ii) there exists a measure \(P^*\) on \(\mathcal{F}\) for which \((X(t))\) is a martingale with respect to \((\mathcal{F}_t), P^*|\mathcal{F}_t \sim P|\mathcal{F}_t, t \in \mathbb{N}\).

Proof. It is enough to prove that (i) implies (ii). Let us put \(Y(t) = X(t) - X(t-1), t = 1, 2, \ldots\) To use Lemma 3.1, we discuss, at first, a canonical representation \((R^N, \sigma(\mathcal{G}), P_Y)\) for the process \(Y(t)\). There exists a measure \(P^*_Y\) on \(\sigma(\mathcal{G})\) (cf. notation 3o at the beginning of Section 3) satisfying

(3) \[ E_{P^*_Y} Y(t+1) = 0 \]

(for conditional expectation with respect to a \(\sigma\)-field \(\mathcal{G}\), and a probability \(P^*_Y\)),

(4) \[ P^*_Y|\mathcal{G}_t \sim P_Y|\mathcal{G}_t, \quad t \in \mathbb{N}. \]

To show this, we define by induction a sequence of probabilities \(P(t)\) on \(\sigma(\mathcal{G})\) satisfying

(5) \[ P(t+1)|\mathcal{G}_t = P(t)|\mathcal{G}_t, \]

(6) \[ E_{P(t)} Y(t+1) = 0, \quad t \in \mathbb{N}. \]

Let \(P(0) = P_Y\). Define \(\varphi_1(\omega) \equiv 1\) if \(P_Y(Y(1) > 0) = 0\); otherwise

\[ \varphi_1(\omega) = \begin{cases} x(\omega) & \text{for } Y(1)(\omega) > 0, \\ 1 & \text{for } Y(1)(\omega) = 0, \\ y(\omega) & \text{for } Y(1)(\omega) < 0 \end{cases} \]

with \(x, y\) uniquely determined by

\[ x(\omega) E_{P_Y}^{\mathcal{G}_0} Y(1)^+ - y(\omega) E_{P_Y}^{\mathcal{G}_0} Y(1)^- = 0, \]

\[ x(\omega) E_{P_Y}^{\mathcal{G}_0} 1_{Y(1) > 0} + y(\omega) E_{P_Y}^{\mathcal{G}_0} 1_{Y(1) < 0} = E_{P_Y}^{\mathcal{G}_0} 1_{\{Y(1) \neq 0\}} \]

with \(\mathcal{G}_0 = \{\emptyset, R^{N^+}\}\), and \(1_{\{Y(1) > 0\}}(\omega) = 0\) or \(1\) when \(\omega \notin \{Y(1) > 0\}\) or \(\omega \in \{Y(1) > 0\}\). Then

\[ P(0)|\mathcal{G}_0 = P(1)|\mathcal{G}_0, \]

\[ E_{P(1)} Y(1) = 0 \quad \text{for } dP(1)/dP_Y = \varphi_1. \]
Assume that $P(0), \ldots, P(n)$ are defined so that (5) and (6) are satisfied for $t = 0, \ldots, n-1$. Let $\varphi_{n+1}(\omega) = 1$ if $P(n)(Y(n+1) > 0) = 0$; otherwise

$$
\varphi_{n+1}(\omega) = \begin{cases} 
  x(\omega) & \text{for } Y(n+1)(\omega) > 0, \\
  1 & \text{for } Y(n+1)(\omega) = 0, \\
  y(\omega) & \text{for } Y(n+1)(\omega) < 0,
\end{cases}
$$

where $x(\omega)$ and $y(\omega)$ are uniquely determined almost everywhere on a set $(Y(n+1) \neq 0)$ by

$$
x(\omega) E_{\mathcal{F}_n}^P Y(n+1)^+ - y(\omega) E_{\mathcal{F}_n}^P Y(n+1)^- = 0,
$$

$$
x(\omega) E_{\mathcal{F}_0}^P 1_{(Y(n+1) > 0)} + y(\omega) E_{\mathcal{F}_0}^P 1_{(Y(n+1) < 0)} = E_{\mathcal{F}_0}^P 1_{(Y(n+1) \neq 0)}. 
$$

Then we obtain (5) and (6) with $t = n$ for $P(n+1)$ defined by

$$
dP(n+1)/dP_Y = \varphi_{n+1}. 
$$

By the Kolmogorov extension theorem, the measure $P^\mathcal{F}_0, P^\mathcal{F}_0|_{\mathcal{F}_0}$ is uniquely defined on $\sigma(\mathcal{G})$, and conditions (3) and (4) are satisfied.

Let us return to the space $\Omega$. For bounded random variables $Y(t)$, assumption (2) implies that $Y[\mathcal{Q}]$ is closed in $\mathbb{R}^N$. Then formula (1) defines a probability $P^*$ on $\mathcal{F} = \sigma(Y(1), Y(2), \ldots)$ by virtue of Lemma 3.2. Equivalence (4) implies $P^*|_{\mathcal{F}_t} \sim P|_{\mathcal{F}_t}$ for $\mathcal{F}_t = \sigma(Y(0), \ldots, Y(t))$ as $\mathcal{F}_t = Y^{-1}(\mathcal{G}_t)$. The equality $E_{\mathcal{F}_n}^P Y(t+1) = 0$ is a consequence of (3) by elementary changes of variables in integrals.

Obviously, $(X(t)) = (X(0)+Y(1)+\ldots+Y(t))$ is a martingale with respect to $P^*$, and $\sigma$-fields $\sigma(X(0), \ldots, X(t)) = \sigma(Y(1), \ldots, Y(t))$.

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Received on 16.2.1998;
revised version on 25.5.1998