AN APPLICATION OF WAVELET ANALYSIS TO PRICING AND HEDGING DERIVATIVE SECURITIES

BY

JURI HINZ (TÜRINGEN)

Abstract. This work provides an application of wavelet analysis to pricing and hedging path-dependent contingent claims within the framework of the Black–Scholes model.

1. Introduction. A European contingent claim written on an asset is a financial contract which gives its owner the right to receive a payoff at the expiration date \( T \). This payoff depends on the market behaviour (the path) of the underlying asset during the time \([0, T]\) as determined in the contract. Two problems arise naturally when dealing with contingent claims. The problem of pricing: Calculate the fair price of the contingent claim at each time \( 0 < t < T \) using the behaviour of the underlying asset during the time \([0, t]\). The problem of hedging: Having sold the contingent claim, how can the seller insure against the upcoming random loss at the time \( T \)? In some cases of interest, in particular in the case of the Black–Scholes market, we find the solution of these problems in terms of the so-called arbitrage-free pricing.

In the Black–Scholes model we specify three financial assets traded continuously during the time \([0, T]\). The corresponding prices are modelled by adapted stochastic processes \((\tilde{S}_t)_{t \in [0,T]}, (\tilde{B}_t)_{t \in [0,T]}, \) and \((\tilde{Y}_t)_{t \in [0,T]}\) on the filtered probability space \((\Omega, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\), where \((\mathcal{F}_t)_{t \in [0,T]}\) denotes the natural filtration generated by some Brownian motion \((\tilde{W}_t)_{t \in [0,T]}\). The stock process \((\tilde{S}_t)_{t \in [0,T]}\) with initial value \( S_0 > 0 \) describes the price of a risky asset. It is given by

\[
\tilde{S}_t := e^{\sigma W_t + (\mu - \frac{1}{2} \sigma^2) t} S_0 \quad \text{for all } t \in [0, T].
\]

The constants \( \mu > 0 \) and \( \sigma > 0 \) are known as the appreciation rate and the stochastic volatility of the stock. The bond process \((\tilde{B}_t)_{t \in [0,T]}\) represents a risk-free security, assumed to continuously compound in value at the fixed interest rate \( r > 0 \), meaning \( \tilde{B}_t := e^{rt} \) for all \( t \in [0, T] \). The terminal payoff of the contingent claim is given by an \( \mathcal{F}_T \)-measurable random variable \( \tilde{Y}_T \). The process \((\tilde{Y}_t)_{t \in [0,T]}\), which is to be determined, corresponds to the behaviour of the market price of the contingent claim during \([0, T]\). The main idea of the
arbitrage-free pricing is to introduce the discounted processes as $(S_t := e^{-rt} \tilde{S}_t)_{t \in [0,T]}$ and $(Y_t := e^{-rt} \tilde{Y}_t)_{t \in [0,T]}$. They describe the prices of the stock and of the claim if the risk-free security is chosen as the numeraire asset. Supposing the absence of arbitrage and following standard arguments (see, for example, [10], [5], [12]), we are led to the following statement: There exists a probability measure $P$, which is equivalent to $\tilde{P}$ such that the discounted processes $(S_t)_{t \in [0,T]}$ and $(Y_t)_{t \in [0,T]}$ are martingales under $P$. In our setting the measure $P$ is unique and the Radon-Nikodym derivative $dP/d\tilde{P}$ is explicitly obtained. Furthermore, from the theorem of Girsanov it follows that $(S_t)_{t \in [0,T]}$ satisfies

$$S_t = \exp \left\{ \sigma W_t - \frac{\sigma^2}{2} t \right\} S_0 \quad \text{for all } t \in [0, T],$$

where $(W_t)_{t \in [0,T]}$ is a Brownian motion with respect to $P$ and $S_0 > 0$. In the following let us suppose that $Y_T \in L^2(\Omega, \mathcal{F}_T, P)$. In this situation the problem of pricing is easily solved: The fair (discounted) price of the contingent claim at the time $t \in [0, T]$ is found by calculating the conditional expectation of $Y_T$ under $P$: $(Y_t := E_P(Y_T | \mathcal{F}_t))_{t \in [0, T]}$. The problem of hedging admits the following treatment: Representing the square-integrable martingale $(Y_t)_{t \in [0,T]}$ as the stochastic integral (this is possible in our setting)

$$Y_t = Y_0 + \int_0^t \gamma_s dW_s \quad \text{for all } t \in [0, T],$$

we obtain $(\gamma_t)_{t \in [0,T]}$ and determine the process $(\eta, \theta) = ((\eta_t, \theta_t))_{t \in [0,T]}$ (the trading strategy) as $\eta_t = Y_t - \gamma_t, \sigma^{-1}$ and $\theta_t = \gamma_t (\sigma \tilde{S}_t)^{-1}$ for all $t \in [0, T]$. In such a strategy, $\theta_t$ describes the number of units of risky asset held at the time $t$, and $\eta_t$ describes the amount invested in the riskless asset at the time $t$. The strategy $(\eta, \theta)$ satisfies

(1) \quad $\tilde{Y}_t = Y_0 + \int_0^t \eta_s dB_s + \int_0^t \theta_s d\tilde{S}_s \quad \text{for all } t \in [0, T],$

(2) \quad $\tilde{\gamma}_t = \theta_t \tilde{S}_t + \eta_t \tilde{B}_t \quad \text{for all } t \in [0, T].$

From the equation (1) it follows that, starting with initial investment $Y_0$, the trading strategy $(\eta, \theta)$ replicates the payoff $\tilde{Y}_T$ of the contingent claim. The equation (2) means that in order to attain $\tilde{Y}_t$ in this way only the investment $Y_0$ at the time $t = 0$ is needed. Such a strategy is called self-financing. (For additional information we refer the reader to [12], [5], [15].) The seller of contingent claim may apply this trading strategy to avoid the risk completely. However, since closed-form expressions of $Y_t$ and $\gamma_t$ are not always available, it is important to study numerical methods (see, for example, [5], [6], [1]).

In this work we apply some results from the wavelet theory in order to obtain square-mean approximations of $(\gamma_t)_{t \in [0,T]}$ and of $(\eta_t)_{t \in [0,T]}$. Let us explain which $L^2$-approximations are meant. We choose the orthonormal basis $(\varphi^k)_{k \in \mathbb{N}}$
of \( L^2[0, T] \) by putting \( \varphi^0_s := \sqrt{1/T} \),

\[
\varphi^{2k}_s := \frac{2}{T} \cos \left( \frac{2\pi ks}{T} \right),
\]

\[
\varphi^{2k+1}_s := \frac{2}{T} \sin \left( \frac{2\pi ks}{T} \right)
\] for all \( k \geq 1, s \in [0, T] \)

and define the family \( (E^n)_{n \in \mathbb{N}} \) of linear subspaces of \( L^2[0, T] \), where \( E^n \) is spanned by \( (\varphi^k_s)_{k=0} \). For each \( n \in \mathbb{N} \) let us also introduce the \( \sigma \)-algebra \( \sigma_n \), generated by \( \{ \int_0^t z_s dW_s : z \in E^n \} \), the random variable \( Y^n_T := E_P(Y_T | \sigma_n) \), the martingale \( (Y^n_t := E_P(Y^n_T | \mathcal{F}_t))_{t \in [0,T]} \), and the predictable process \( (y^n_t)_{t \in [0,T]} \) defined by

\[
Y^n_T = E_P(Y^n_T) + \int_0^T y^n_s dW_s.
\]

Note that, in view of the properties of the Hermite polynomials,

\[
\{ X : X \text{ is a polynomial in } \int_0^T \varphi^0_s dW_s, \ldots, \int_0^T \varphi^s_s dW_s \}
\]

is a dense subspace of \( L^2(\Omega, \sigma_n, P) \). Moreover, it is well known (see [11], Chapter 4.2) that

\[
\bigcup_{n=0}^\infty \{ X : X \text{ is a polynomial in } \int_0^T \varphi^0_s dW_s, \ldots, \int_0^T \varphi^s_s dW_s \}
\]

is a dense subspace of \( L^2(\Omega, \mathcal{F}_T, P) \). Hence \( \bigcup_{n=0}^\infty L^2(\Omega, \sigma_n, P) \) is dense in \( L^2(\Omega, \mathcal{F}_T, P) \). This implies that the sequence \( (Y^n_T)_{n \in \mathbb{N}} \) converges to \( Y_T \) in the square mean. Doob's maximal inequality and the isometry of stochastic integral yield:

\[
\lim_{n \to \infty} E_P\left( \sup_{t \in [0,T]} |Y^n_t - Y_t|^2 \right) = 0, \quad \lim_{n \to \infty} E_P\left( \int_0^T |y^n_t - y_t|^2 dt \right) = 0.
\]

Let \( n \in \mathbb{N} \) be sufficiently large. Then, in view of (3), the processes \( (Y^n_t)_{t \in [0,T]} \) and \( (y^n_t)_{t \in [0,T]} \) are seen to be \( L^2 \)-approximations of \( (Y_t)_{t \in [0,T]} \) and \( (y_t)_{t \in [0,T]} \), respectively.

Throughout this paper let \( n \in \mathbb{N} \) be fixed. For each \( t \in [0, T] \) we introduce \( \mathbb{R}^{n+1} \)-valued Gaussian random variable

\[
\Phi_t = (\Phi^k_t)_{k=0} = (\int_0^t \varphi^k_s dW^k_s)_{k=0}.
\]

Its covariance matrix \( G_t \) satisfies

\[
G_t = (\int_0^t \varphi^k_s \varphi^l_s ds)_{k,l=0}.
\]

Note that \( 1 - G_t \) is positive definite for all \( t \in [0, T] \). We shall denote the transpose (of a matrix) by \( * \) and write \( \varphi_t \) for the (column) vector.
(\varphi_t^0, \ldots, \varphi_t^n) \in \mathbb{R}^{n+1}. For all \( t \in [0, T] \) and \( \omega \in \Omega \) we define the random variables \( K_{t,\omega}, K'_{t,\omega} \in L^2(\Omega, \mathcal{F}_T, P) \) as
\[
K_{t,\omega} := |\det(1 - G)|^{-1/2} \times \exp \left\{ -\frac{1}{2} (\Phi_T - \Phi_t(\omega))^* (1 - G)^{-1} (\Phi_T - \Phi_t(\omega)) \right\} \exp \left\{ \frac{1}{2} \Phi_T^* \Phi_T \right\},
\]
\[
K'_{t,\omega} := (\Phi_T - \Phi_t(\omega))^* (1 - G)^{-1} \varphi_t K_{t,\omega}.
\]

Using the notation above, we can formulate the main result of this work as follows:

**Proposition 1.** For all \( t \in [0, T] \) and \( \omega \in \Omega \):
\[
Y_t^n(\omega) = E_P(K_{t,\omega} Y_T),
\]
\[
y_t^n(\omega)_\xi = E_P(K'_{t,\omega} Y_T).
\]

Note that since \( \Phi_t \) is \( \mathcal{F}_t \)-measurable, the value \( \Phi_t(\omega) \in \mathbb{R}^{n+1} \) is observed at the time \( t \). For this reason a calculation of (4) and of (5) involves only the evaluation of the mean value of the random variables \( Y_T K_{t,\omega} \) and \( Y_T K'_{t,\omega} \). This may be done numerically.

2. The mathematical background. Hilbert spaces used in this work are separable and inner products are linear on the right. The linear and the closed linear space spanned by a set \( M \) are denoted by \( \text{lin} M \) and by \( \overline{\text{lin}} M \), respectively. All integrals of Hilbert space-valued functions are understood in the weak sense. The group of unitary operators on the Hilbert space \( \mathcal{H} \) is denoted by \( \mathcal{U}(\mathcal{H}) \). The \( \sigma \)-algebra generated by a set \( M \) of random variables is denoted by \( \sigma(M) \). The space of continuous functions on a topological space \( X \) is denoted by \( C(X) \).

The wavelet analysis was introduced by Grossmann et al. in [7] and [8], and was motivated by applications in the signal processing. We recall some recent results from this theory.

Let \( G \) be a locally compact group equipped with a left Haar measure \( \mu \). A strongly continuous irreducible unitary representation \( U \) of \( G \) on the Hilbert space \( \mathcal{H} \) is called *square integrable* if there exists a vector \( v \in \mathcal{H} \) satisfying
\[
v \neq 0 \quad \text{and} \quad \int_G |\langle U(g) v, v \rangle|^2 \mu(\mathrm{d}g) < \infty.
\]

Such a vector \( v \) is called a *wavelet*. Given \( G, U, \mathcal{H} \), and \( v \in \mathcal{H} \) as above we introduce \( V : \mathcal{H} \rightarrow C(G) \), \( h \mapsto V h \) by putting \( V h(g) := \langle U(g) v, h \rangle \) for all \( g \in G \) and \( h \in \mathcal{H} \). The mapping \( V \) is called the *wavelet transform*. In [7] it is shown that the wavelet transform is, up to a positive constant, an isometric operator from \( \mathcal{H} \) into \( L^2(G, \mu) \). Let us choose the left Haar measure \( \mu \) such that \( V \) becomes isometric. The adjoint \( V^* \) is given by
\[
V^* \xi = \int_G \xi(g) U(g) v \mu(\mathrm{d}g) \quad \text{for all} \ \xi \in L^2(G, \mu).
\]
From \( V^* V = 1 \) we infer the inversion formula of the wavelet transform:

\[
(7) \quad h = \int \frac{Vh(g)U(g)\nu(g)}{dg} \quad \text{for all } h \in \mathcal{H}.
\]

In the following we also need some special constructions of Hilbert spaces; for proofs and details we refer the reader to [14], p. 92. Let \( M \) be any set. The map \( k: M \times M \to \mathbb{C} \) is called a positive definite kernel on \( M \) if the matrix \( (k(x_i, x_j))_{i,j=1}^n \) is positive semidefinite for all \( x_1, \ldots, x_n \in M \) and \( n \in \mathbb{N} \). Let \( k \) be a positive definite kernel on \( M \). Then there exists a Hilbert space \( K \) and a map \( e: M \to K \) such that \( \langle e(x), e(y) \rangle = k(x, y) \) for all \( x, y \in M \) and the linear space \( \operatorname{lin} \{ e(x): x \in M \} \) is dense in \( K \). The pair \((e, K)\) is called the Kolmogoroff decomposition of the positive definite kernel \( k \). Let \((e_1, K_1)\) and \((e_2, K_2)\) be Kolmogoroff decompositions of \( k \). Then there exists a unitary operator \( q: K_1 \to K_2 \) such that \( q e_1(x) = e_2(x) \) holds for all \( x \in M \). In this sense the Kolmogoroff decomposition is unique. The Kolmogoroff decomposition is useful in constructing new Hilbert spaces; for example, we obtain the direct sum \( H_1 \oplus H_2 \) of Hilbert spaces \( H_1 \) and \( H_2 \) by decomposing the kernel \( k \) on \( H_1 \times H_2 \), given by

\[
k: \langle (h_1, h_2), (h_1', h_2') \rangle \mapsto \langle h_1, h_1' \rangle + \langle h_2, h_2' \rangle.
\]

In this case, \( e \) is given as

\[
e: (h_1, h_2) \mapsto h_1 \oplus h_2.
\]

Similarly, the symmetric Fock space \( \Gamma(H) \) over the Hilbert space \( H \) is defined by the decomposition \((e, \Gamma(H))\) of the following kernel \( k \) on \( H \):

\[
k(h, h') := e^{\langle h, h' \rangle} \quad \text{for all } h, h' \in H.
\]

The vector \( e(x) \in \Gamma(H) \) is called the exponential vector corresponding to \( x \in H \). The exponential vectors are linearly independent and the map \( e: H \to \Gamma(H) \) is continuous.

Remark 1. We consider one concrete realization of the symmetric Fock space over \( L^2[0, T] \). For each \( z \in L^2[0, T] \) and \( t \in [0, T] \) we define the random variable \( \delta_t(z) \) on our probability space as

\[
\delta_t(z) := \exp \left\{ \int_0^t z_s dW_s - \frac{1}{2} \int_0^t z_s^2 ds \right\}.
\]

A calculation shows that \( E_p(\delta_T(z)\delta_T(z')) = e^{\langle z, z' \rangle} \) holds for all \( z, z' \in L^2[0, T] \). Using the fact that \( \operatorname{lin} \{ \delta_T(z): z \in L^2[0, T] \} \) forms a dense subspace of \( L^2(\Omega, \mathcal{F}_T, P) \) (see the proof of Lemma 5.36 of [9]), we conclude that \((\delta_T(\cdot), L^2(\Omega, \mathcal{F}_T, P))\) defines a decomposition of the kernel \((z, z') \mapsto e^{\langle z, z' \rangle} \) on \( L^2[0, T] \), and therefore \( L^2(\Omega, \mathcal{F}_T, P) \) can be regarded as the symmetric Fock space over \( L^2[0, T] \).

Let \( K \) be a Hilbert space and \( T := \{ \tau \in \mathbb{C}: |\tau| = 1 \} \) be the one-dimensional torus. We endow the set \( G_K := T \times K \) with the multiplication \( \circ \) as
follows:

\[(\tau, z) \circ (\tau', z') := (\tau \tau' \exp \{-i \Im \langle z, z' \rangle\}, z + z') \quad \text{for all } (\tau, z), (\tau', z') \in G_K,\]

and we obtain a non-abelian topological group \((G_K, \circ)\). In the case where \(K\) is a finite-dimensional Hilbert space, \((G_K, \circ)\) is a locally compact unimodular group, which is called the Weyl–Heisenberg group. The Haar measure \(\omega_{G_K}\) of the Weyl–Heisenberg group \(G_K\) is given by \(\omega_{G_K} := \omega_{T} \otimes \omega_{K}\), where \(\omega_{T}\) and \(\omega_{K}\) denote the Haar measures of \(T\) and \(K\), respectively.

Let \(\Gamma(K)\) be the Fock space over \(K\). For any \(z \in K\) the Fock–Weyl operator \(\mathcal{W}_K(z)\) well defined by its action on all exponential vectors as

\[\mathcal{W}_K(z)e(z') = \exp \{-\frac{1}{2}||z||^2 - \langle z, z' \rangle\} e(z + z') \quad \text{for all } z, z' \in K.\]

It can be shown (see [14], p. 135) that each \(\mathcal{W}_K(z)\) is a unitarity on \(\Gamma(K)\) and that the Fock–Weyl operators obey the relation

\[\mathcal{W}_K(z) \mathcal{W}_K(z') = \exp \{-i \Im \langle z, z' \rangle\} \mathcal{W}_K(z + z') \quad \text{for all } z, z' \in K,\]

from which it follows that the map

\[U_K: G_K \to \mathcal{U}(\Gamma(K)), \quad (\tau, z) \mapsto \tau \mathcal{W}_K(z),\]

defines a unitary representation (the Fock representation) of \(G_K\). This representation is strongly continuous and irreducible (see [14], p. 142).

Remark 2. In the case where \(K\) is finite dimensional the Fock representation \(U_K\) of the Weyl–Heisenberg group \(G_K\) is square integrable and \(e(0)\) can be chosen as a wavelet (compare with (6)), since

\[
\int_{G_K} |\langle U_E(g)e(0), e(0) \rangle|^2 \omega_{G_K}(dg)
= \int_{K} \int_{T} |\langle U_E((\tau, z))e(0), e(0) \rangle|^2 \omega_{T}(d\tau) \omega_{K}(dz)
= \int_{K} \int_{T} |\tau \exp \{-||z||^2/2\} e(z), e(0) \rangle|^2 \omega_{T}(d\tau) \omega_{K}(dz)
= \omega_{T}(T) \int_{K} \exp \{-||z||^2\} \omega_{K}(dz) < \infty.
\]

Consider a finite-dimensional subspace \(E\) of a given Hilbert space \(H\). Let \((e, \Gamma(H))\) be the decomposition defining the symmetric Fock space over \(H\). We denote by \(Y(E) \subset \Gamma(H)\) the space \(\lim \{e(z): z \in E\} \subset \Gamma(H)\) and by \(P_{Y(E)}\) the orthogonal projector onto \(Y(E)\). Let us also introduce the following transform:

\[\mathcal{L}: \Gamma(H) \to C(E), \quad \mathcal{L}h(z) := \langle e(z), h \rangle \text{ for all } z \in E.\]

From this definition we obtain \(\mathcal{L} = \mathcal{L}P_{Y(E)}\). Let \(\omega_E\) be the Haar measure of \(E\) normalized as

\[
\int_{E} \exp \{-||z||^2\} \omega_E(dz) = 1.
\]
and $\gamma_E$ be the probability measure
\[
\gamma_E(dz) := \exp \{-||z||^2\} \omega_E(dz).
\]

**Proposition 2.** Using the above notation the following holds:
(i) $\mathcal{L}$ is a bounded operator mapping from $\Gamma(H)$ into $L^2(E, \gamma_E)$. Moreover, $\mathcal{L}^* \mathcal{L} = P_{\gamma(E)}$.
(ii) The adjoint $\mathcal{L}^*: L^2(E, \gamma_E) \to \Gamma(H)$ is given by
\[
\mathcal{L}^* \xi = \int_E \xi(z) e(z) \gamma_E(dz) \text{ for all } \xi \in L^2(E, \gamma_E).
\]

**Proof.** (i) It is obvious that the decomposition $(e|E, \gamma(E))$ corresponds to the symmetric Fock space over $E$, and therefore we may identify $\gamma(E)$ with $\Gamma(E)$. The appropriate Fock–Weyl operators $\{\mathcal{W}_E(z); z \in E\} \subset \mathcal{U}(\gamma(E))$ are given by
\[
\mathcal{W}_E(z) := P_{\gamma(E)} \mathcal{W}_H(z)|_{\gamma(E)} \text{ for all } z \in E.
\]

From the second remark it follows that
\[
\hat{U}_E: G_E \to \mathcal{U}(\gamma(E)), \quad (\tau, z) \mapsto \tau \mathcal{W}_E(z),
\]
defines a representation of $G_E$ which is unitarily equivalent to the Fock representation $U_E$. For this reason, $\hat{U}_E$ is square integrable, and we may define the wavelet transform according to $\hat{U}_E$ choosing $e(0)$ as the wavelet. This transform becomes isometric if the Haar measure $\omega_G = \omega_T \otimes \omega_E$ is normalized as
\[
\omega_T(T) = \int_E \exp \{-||z||^2\} \omega_E(dz) = 1,
\]
which implies for all $h \in \gamma(E)$
\[
||h||^2 = \int_{G_E} |\langle \hat{U}_E(g) e(0), h \rangle|^2 \omega_{G_E}(dg)
= \int_T \int_E |\tau \exp \{-||z||^2/2\} e(z), h\rangle|^2 \omega_{E}(dz) \omega_{T}(d\tau)
= \int_E ||e(z), h\rangle \gamma_E(dz) = ||\mathcal{L}h||^2.
\]
Hence $||P_{\gamma(E)} h||^2 = ||\mathcal{L} P_{\gamma(E)} h||^2 = ||\mathcal{L}h||^2$ for all $h \in \Gamma(H)$. Finally, we are led to $P_{\gamma(E)} = \mathcal{L}^* \mathcal{L}$ by polarizing $||P_{\gamma(E)} h||^2 = ||\mathcal{L}h||^2$ for all $h \in \Gamma(H)$.

(ii) is straightforward. ∎

3. **Pricing contingent claims.** Let $\hat{Y}_T \in L^2(\Omega, \mathcal{F}_T, P)$ be a contingent claim and $n \in N$. For the remainder of this work we identify $L^2(\Omega, \mathcal{F}_T, P)$ with the symmetric Fock space over $L^2[0, T]$ as described in the first remark. We also choose the finite-dimensional subspace
\[
E^n := \text{lin}\{\varphi^0, \ldots, \varphi^n\} \subset L^2[0, T]
\]
and consider the transform $\mathcal{L}: L^2(\Omega, \mathcal{F}_T, P) \to L^2(\mathbb{R}^n, \gamma_{E^n})$ which corresponds to $E^n$ in this context.

Since pointwise arguments are important in this section, let us agree that for each $c = (c_0, \ldots, c_n) \in \mathbb{C}^{n+1}$ and $t \in [0, T]$ the vector $\mathcal{E}_t\left(\sum_{k=0}^n c_k \varphi^k \right) \in L^2(\Omega, \mathcal{F}_T, P)$ is represented by the function

$$
\Omega \to \mathbb{C}, \quad \omega \mapsto \exp \left\{ \sum_{k=0}^n c_k \Phi_k^t(\omega) - \frac{1}{2} \sum_{k,j=0}^n c_k c_j \int_0^t \varphi^k \varphi^j_s ds \right\}
$$

$$
= \exp \{ c^* \Phi_t(\omega) - \frac{1}{2} c^* G_t c \}.
$$

**Proposition 3.** For all $t \in [0, T]$, $Y^n_t = \int_{\mathbb{R}^n} Y_T(z) \mathcal{E}_t(z) \gamma_{E^n}(dz)$ holds in the weak sense and for all $t \in [0, T]$ pointwise.

**Proof.** The identity $Y(E^n) := \lim \{ \mathcal{E}_t(z) : z \in \mathbb{R}^n \}$ holds by definition. Applying Lemma 5.36 of [9] we conclude that $Y(E^n) = L^2(\Omega, \sigma_n, P)$, which implies that the conditional expectation $Y^n_t = E^P(Y_t | \mathcal{F}_t)$ coincides with the projection $P_{Y(E^n)} Y_T$. From Proposition 2 it follows that

$$
Y^n_t = P_{Y(E^n)} Y_T = \mathcal{L}^* \mathcal{L} Y_T = \int_{\mathbb{R}^n} \mathcal{L} Y_T(z) \mathcal{E}_t(z) \gamma_{E^n}(dz)
$$

holds in the weak sense for all $t \in [0, T]$. To show the pointwise representation it suffices to prove that for $(\omega, t) \in \Omega \times [0, T]$ the function $f_{t,\omega} : \mathbb{R}^n \to \mathbb{C}$, $z \mapsto \mathcal{E}_t(z)(\omega)$ is contained in $L^2(\mathbb{R}^n, \gamma_{E^n})$.

$$
\int_{\mathbb{R}^n} |f_{t,\omega}(z)|^2 \gamma_{E^n}(dz) = \int_{\mathbb{R}^n} |\mathcal{E}_t(z)(\omega)|^2 \exp \{ -\|z\|^2 \} \omega_{E^n}(dz)
$$

$$
= \int_{\mathbb{C}^{n+1}} |\exp \{ c^* \Phi_t(\omega) - \frac{1}{2} c^* G_t c \}|^2 \exp \{ -\bar{c}^* c \} \frac{dc}{\pi^{n+1}}
$$

$$
= \int_{\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}} \exp \{ 2u^* \Phi_t(\omega) - u^* G_t u + v^* G_t v \} \exp \{ -u^* u - v^* v \} \frac{du dv}{\pi^{n+1}}
$$

$$
= \int_{\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}} \exp \{ 2u^* \Phi_t(\omega) - u^*(G_t + 1) u - v^*(1 - G_t) v \} \frac{du dv}{\pi^{n+1}} < \infty.
$$

The last inequality holds since the matrix $(1 - G_t)$ is positive definite for all $t \in [0, T]$. □

We are now able to prove (4). By Proposition 3, for all $(\omega, t) \in \Omega \times [0, T]$ we obtain

$$
Y^n_t(\omega) = \int_{\mathbb{R}^n} \mathcal{L} Y_T(z) f_{t,\omega}(z) \gamma_{E^n}(dz) = \langle \mathcal{L}^* f_{t,\omega}, \mathcal{L} Y_T \rangle_{L^2(\mathbb{R}^n, \gamma_{E^n})}
$$

$$
= \langle \mathcal{L}^* f_{t,\omega}, Y_T \rangle_{L^2(\Omega, P)} = E^P(\mathcal{K}_{t,\omega} Y_T),
$$
where $K_{t,\omega} = \mathcal{L}^* f_{t,\omega}$. This random variable is calculated explicitly as:

$$K_{t,\omega} = \int_{\mathcal{F}_n} f_{t,\omega}(z) \mathcal{E}_T(z) \gamma_{E^n}(dz)$$

$$= \int_{\mathcal{F}_n} \exp \{ \varepsilon^* \Phi_1(\omega) - \frac{1}{2} \varepsilon^* \mathcal{G}_T \varepsilon \} \exp \{ \varepsilon^* \Phi_T - \frac{1}{2} \varepsilon^* \varepsilon \} \frac{dc}{\pi^{n+1}}$$

$$= |\det (1 - \mathcal{G}_T)^{-1/2}| \exp \left\{ -\frac{1}{2} \left[ \Phi_T - \Phi_1(\omega) \right](1 - \mathcal{G}_T)^{-1} \left( \Phi_T - \Phi_1(\omega) \right) \right\} \exp \left\{ \frac{1}{2} \Phi^* \Phi_T \right\},$$

which proves the statement (4).

4. Hedging contingent claims. To apply tools from the wavelet analysis to the hedging problem we consider the Hilbert space of predictable processes: $L^2(\Omega \times [0, T], \mathcal{F}, \mathbb{P} \otimes \lambda)$. Here $\mathcal{F}$ denotes the $\sigma$-algebra of predictable sets and $P \otimes \lambda$ denotes the restriction to $\mathcal{F}$ of the measure-theoretic product of $P$ with the Lebesgue measure $\lambda$ on $[0, T]$. Then, by its construction, the stochastic integral

$$I: L^2(\Omega \times [0, T], \mathcal{F}, \mathbb{P} \otimes \lambda) \to L^2(\Omega, \mathbb{F}_T, \mathbb{P}), \quad h = (h_{t})_{t \in [0, T]} \mapsto \int_{0}^{T} h_t dW_t,$$

is an isometric operator. Each exponential vector $\mathcal{E}_T(z)$ is the terminal value of the martingale

$$\mathcal{E}_T(z) = (\mathcal{E}_t(z))_{t \in [0, T]} = \left( \exp \left\{ \int_{0}^{t} z_s dW_s - \frac{1}{2} \int_{0}^{t} z_s^2 ds \right\} \right)_{t \in [0, T]}.$$
in the weak sense. Using $\delta_T(z) - \delta_0(z) = Ie(z)$, we are led to

$$Y_T^n - Y_0^n = \int_{E^n} \mathcal{L} Y_T(z) Ie(z) \gamma_E^n(dz) = I \int_{E^n} \mathcal{L} Y_T(z) e(z) \gamma_E^n(dz)$$

since $I$ is bounded. This proves the weak integral representation of $y^n$. To prove the second assertion it suffices to show that for each $(\omega, t) \in \Omega \times [0, T[$ the function

$$f_{t,\omega} : E^n \to C, \quad z \mapsto e_t(z)(\omega),$$

is contained in $L^2(E^n, \gamma_{E^n})$. This is true since $(1 - G_t)$ is positive definite for $t \in [0, T[$:

$$\int_{E^n} |f_{t,\omega}|^2 \gamma_{E^n}(dz) = \int_{E^n} |\varphi|^2 |\delta_t(z)(\omega)|^2 \gamma_{E^n}(dz)$$

$$= \int_{\mathbb{R}^{n+1}} |\varphi|^2 \exp \left\{ \frac{c^* \Phi_t(\omega) - \frac{1}{2} c^* G_t c}{\gamma_{E^n}} \right\} \frac{dc}{\pi^{n+1}}$$

$$= \int_{\mathbb{R}^{n+1}} |\Phi_t(u + iv)|^2 \exp \left\{ \frac{2u^* \Phi_t(\omega)}{\gamma_{E^n}} \right\} - u^* (G_t + 1) u - v^* (1 - G_t) v \frac{du dv}{\pi^{n+1}} < \infty. \quad \blacksquare$$

Now we are able to show (5) by evaluating

$$\int_{E^n} \mathcal{L} Y_T(z) e_t(z)(\omega) \gamma_{E^n}(dz), \quad \text{where} \ (\omega, t) \in \Omega \times [0, T[.$$

We get

$$y^n_t(\omega) = \int_{E^n} \mathcal{L} Y_T(z) f_{t,\omega}(z) \gamma_{E^n}(dz) = \langle \mathcal{L} Y_T, \mathcal{L} Y_T \rangle_{L^2(E^n, \gamma_{E^n})}$$

$$= \langle \mathcal{L}^* f_{t,\omega}, Y_T \rangle_{L^2(\Omega, P)} = E_p(K_{t,\omega} Y_T^n),$$

where $K_{t,\omega} = \mathcal{L} f_{t,\omega}$. This random variable is calculated explicitly as:

$$K_{t,\omega} = \int_{E^n} f_{t,\omega}(z) \delta_T(z) \gamma_{E^n}(dz)$$

$$= \int_{\mathbb{R}^{n+1}} \varphi \ exp \left\{ \frac{c^* \Phi_t(\omega) - \frac{1}{2} c^* G_t c}{\gamma_{E^n}} \right\} \frac{dc}{\pi^{n+1}}$$

$$\times \exp \left\{ \frac{c^* \Phi_T - \frac{1}{2} c^* c}{\gamma_{E^n}} \right\} \frac{dc}{\pi^{n+1}}$$

$$= (\Phi_T - \Phi_t(\omega))^* (1 - G_t)^{-1} \varphi_t |\det (1 - G_t)^{-1/2}|$$

$$\times \exp \left\{ - \frac{1}{2} (\Phi_T - \Phi_t(\omega))^* (1 - G_t)^{-1} (\Phi_T - \Phi_t(\omega)) \right\} \exp \left\{ \frac{1}{2} \Phi^* \Phi_T \right\},$$

which proves (5).
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Mathematisches Institut
Universität Tübingen
72076 Tübingen, Germany
email: juri.hinz@uni-tuebingen.de

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