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MARTINGALE CHARACTERIZATIONS OF STOCHASTIC PROCESSES ON COMPACT GROUPS

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Abstract. By a classical result of P. Lévy, the Brownian motion $(B_t)_{t\geqslant 0}$ on $\boldsymbol{R}$ may be characterized as a continuous process on $\boldsymbol{R}$ such that $(B_t)_{t\geqslant 0}$ and $(B_t^2-t)_{t\geqslant 0}$ are martingales. Generalizations of this result are usually obtained in the setting of the so-called martingale problem. This paper contains a variant of the martingale problem for stochastic processes on locally compact groups with independent stationary increments that is based on irreducible unitary representations. In particular, for Gaussian processes on compact Lie groups, analogues of the Lévy-characterization above are obtained. It turns out that for certain compact Lie groups even the continuity assumption in this characterization can be dropped.

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1. INTRODUCTION

A classical result of Lévy [11] states that an a.s. continuous process $(B_t)_{t\geqslant 0}$ on $\boldsymbol{R}$ is a Brownian motion if and only if the two processes $(B_t)_{t\geqslant 0}$ and $(B_t^2-t)_{t\geqslant 0}$ are martingales. This type of characterization carries over to very general diffusions or Markov processes in terms of the martingale problem of Stroock and Varadhan; see, for instance, [5], [18], and [20].

The main result of this paper is a Lévy-characterization for Gaussian processes on compact Lie groups in terms of irreducible unitary representations. As a preparation, Section 2 will be devoted to a variant of the martingale problem for stochastic processes on locally compact groups with independent stationary increments. In that section we present generalizations of the well-known fact that a process $(X_t)_{t\geqslant 0}$ on $\boldsymbol{R}$ with independent stationary increments associated with a given convolution semigroup $(\mu_t)_{t\geqslant 0}$ can be characterized by the fact that for all $y\in\boldsymbol{R}$ the processes

$$\left(\hat{\mu}_t(y)^{-1}\cdot\exp(-iyX_t)\right)_{t\geqslant 0}$$

are martingales (where $\hat{\mu}_t(y):=\int_{\boldsymbol{R}}e^{-ixy}\,d\mu_t(x)$ is the Fourier transform of μ_t).

We shall also prove in Section 2 that our "representation-theoretic" version of the martingale problem is equivalent to the usual martingale problem (up to technical details like the underlying spaces of test functions).

In Section 3 we restrict our attention to Gaussian processes on compact Lie groups and prove the following Lévy-characterization:

1.1. THEOREM. *Let ϱ be a faithful finite-dimensional unitary representation of a compact Lie group G. Let $(\mu_t)_{t\geq 0}$ be a Gaussian convolution semigroup and $(X_t)_{t\geq 0}$ be a continuous stochastic process on G. Then $(X_t)_{t\geq 0}$ is a Gaussian process associated with $(\mu_t)_{t\geq 0}$ if and only if for $\pi\in\{\varrho, \varrho\otimes\varrho\}$ the processes*

$$\big(\tilde{\pi}(\mu_t)^{-1}\,\pi(X_t)\big)_{t\geq 0} \quad \text{with } \tilde{\pi}(\mu_t) := \int_G \pi(g)\,d\mu_t(g)$$

are matrix-valued martingales.

Moreover, the continuity assumption for the process in this theorem may be replaced by a stronger martingale condition. More precisely, we prove:

1.2. THEOREM. *Let G, ϱ, and $(\mu_t)_{t\geq 0}$ be given as in Theorem 1.1. Then a stochastic process $(X_t)_{t\geq 0}$ on G is a Gaussian process associated with $(\mu_t)_{t\geq 0}$ if and only if $\big(\tilde{\pi}(\mu_t)^{-1}\,\pi(X_t)\big)_{t\geq 0}$ is a matrix-valued martingale for $\pi\in\{\varrho, \varrho\otimes\varrho, \varrho\otimes\bar{\varrho}\}$ (where $\bar{\varrho}$ is the contragredient representation).*

We give some examples: If G is either the torus $\{z\in\mathbf{C}\colon |z|=1\}$, $SU(2)$, or $SO(n)$ $(n\geq 3)$ with the usual canonical faithful representation ϱ, then either $\varrho\otimes\bar{\varrho}$ is the trivial representation or $\bar{\varrho}=\varrho$ holds. Hence, in all cases, the martingale conditions in Theorems 1.1 and 1.2 are equivalent. We finally mention that, as a by-product, the proofs of the results above lead to new proofs of some well-known facts on Gaussian convolution semigroups; we refer to the monograph [8] for the background.

2. THE MARTINGALE PROBLEM FOR STOCHASTIC PROCESSES ON LOCALLY COMPACT GROUPS

This section is devoted to a variant of the martingale problem for stochastic processes on compact groups with independent stationary increments. This martingale problem will be stated in terms of group representations; it will turn out that it is essentially equivalent to its usual form.

We recapitulate some notation. If not specified otherwise, G will be a locally compact *separable* group. By $M_b(G)$ and $M^1(G)$ we denote the Banach space of all signed regular Borel measures on G and the subspace of all probability measures, respectively.

2.1. Unitary representations of locally compact groups. (1) A unitary representation π of G on some Hilbert space H is a group homomorphism $\pi\colon G\to U(H)$ from G into the space $U(H)$ of all unitary operators such that, for all $a, b\in H$, the coefficients $g\mapsto\langle\pi(g)a, b\rangle$ from G into $\mathbf{C}$ are continuous. The set of all (equivalence classes of) irreducible unitary representations of G will be denoted by $\hat{G}$.

(2) We need the following fact on representations from Section 22 of [7]: Each unitary representation π of G on H can be extended to a strongly continuous Banach algebra homomorphism $\tilde{\pi}$: $M_b(G) \to B(H)$ into the space of all bounded linear operators $B(H)$ on H; for $\mu \in M_b(G)$, the operator $\tilde{\pi}(\mu)$ is characterized by

$$\langle \tilde{\pi}(\mu) a, b \rangle = \int_G \langle \pi(g) a, b \rangle \, d\mu(g) \qquad (a, b \in H). \tag{2.1}$$

2.2. Convolution semigroups. (1) A family $(\mu_t)_{t \geq 0} \subset M^1(G)$ is called a *convolution semigroup* if $\mu_s * \mu_t = \mu_{s+t}$ for $s, t \geq 0$, $\mu_0 = \delta_e$, and the mapping $[0, \infty[\to M^1(G)$, $t \mapsto \mu_t$, is weakly continuous.

(2) Let $(\mu_t)_{t \geq 0} \subset M^1(G)$ be a convolution semigroup on G. A stochastic process $(X_t)_{t \geq 0}$ with values in G is called a *process with independent stationary increments* related to $(\mu_t)_{t \geq 0}$ if for all $n \in N$, $0 \leq t_0 \leq t_1 \leq \ldots \leq t_n$, the increments $X_{t_1} X_{t_0}^{-1}, X_{t_2} X_{t_1}^{-1}, \ldots, X_{t_n} X_{t_{n-1}}^{-1}$ are independent, and if for $s, t \geq 0$ the random variables $X_{s+t} X_s^{-1}$ are μ_t-distributed.

A process $(X_t)_{t \geq 0}$ with independent stationary increments related to $(\mu_t)_{t \geq 0}$ can be also characterized as a Markov process on G whose transition probabilities satisfy

$$P(X_{s+t} \in A \mid X_s = x) = (\mu_t * \delta_x)(A)$$

for $s, t \geq 0$, $x \in G$, and $A \subset G$ a Borel set.

2.3. Remark. Let π be a unitary representation of G on some Hilbert space H, and let $(\mu_t)_{t \geq 0}$ be a convolution semigroup on G. We shall need below that the operators $\tilde{\pi}(\mu_t)$ are invertible in $B(H)$ for all $t \geq 0$. This holds in the following two important cases:

(1) If π is finite-dimensional, then this property holds for all convolution semigroups. In fact, in this case the operators $\tilde{\pi}(\mu_t)$ tend for $t \to 0$ to the identity in the norm-topology. Hence $\tilde{\pi}(\mu_t)$ is invertible for small $t \geq 0$. The general case follows from the semigroup property.

(2) Let $\mu \in M_b^+(G)$ be a bounded, positive regular Borel measure on G, and

$$(\mu_t := e^{-t\|\mu\|} \cdot e^{t\mu})_{t \geq 0} \in M^1(G)$$

the associated compound Poisson convolution semigroup. Then, for each representation π of G, $\tilde{\pi}(\mu_t) = e^{-t\|\mu\|} \cdot e^{t\tilde{\pi}(\mu)}$; hence $\tilde{\pi}(\mu_t)^{-1} = e^{t\|\mu\|} \cdot e^{-t\tilde{\pi}(\mu)} \in B(H)$ exists for all $t \geq 0$. We remark that each convolution semigroup on a discrete group is compound Poisson; see Theorem 4.1.5 of [8].

The following simple notion of operator-valued martingales will be useful in this paper (for a general discussion of Banach space-valued martingales see [12]):

2.4. Operator-valued martingales. Let H be a Hilbert space, and $(Z_t)_{t\geq 0}$ be a $B(H)$-valued stochastic process with filtration $(\mathscr{F}_t)_{t\geq 0}$ (where $B(H)$ carries the weak operator topology). Then $(Z_t)_{t\geq 0}$ is called a *$B(H)$-valued martingale* (with respect to $(\mathscr{F}_t)_{t\geq 0}$) if for all $a, b\in H$ the processes $(\langle Z_t a, b\rangle)_{t\geq 0}$ are martingales (with respect to $(\mathscr{F}_t)_{t\geq 0}$). A similar notion works for local L^2-martingales.

2.5. Lemma. *Let π be a unitary representation of G on some Hilbert space H. Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on G such that the operators $\tilde{\pi}(\mu_t)$ are invertible for all $t \geq 0$. Finally, let $(X_t)_{t\geq 0}$ be a stochastic process on G with filtration $(\mathscr{F}_t)_{t\geq 0}$. Then:*

(1) *If $(X_t)_{t\geq 0}$ has independent stationary increments associated with $(\mu_t)_{t\geq 0}$, then the process $\left(\tilde{\pi}(\mu_t)^{-1}\pi(X_t)\right)_{t\geq 0}$ is a $B(H)$-valued martingale with respect to $(\mathscr{F}_t)_{t\geq 0}$.*

(2) *If $\left(\tilde{\pi}(\mu_t)^{-1}\pi(X_t)\right)_{t\geq 0}$ is a $B(H)$-valued martingale with respect to $(\mathscr{F}_t)_{t\geq 0}$, then*

$$E\left(\langle \pi(X_{s+t})a, b\rangle \mid \mathscr{F}_s\right) = \langle \tilde{\pi}(\mu_t)\pi(X_s)a, b\rangle \quad \textit{for } s, t \geq 0,\ a, b\in H.$$

Proof. (1) Denote the adjoint of $A\in B(H)$ by A^*. Then, for all $s, t \geq 0$, $a, b\in H$,

$$\begin{aligned} E\left(\langle \tilde{\pi}(\mu_{s+t})^{-1}\pi(X_{s+t})a, b\rangle \mid \mathscr{F}_s\right) &= E\left(\langle \pi(X_{s+t})a, \left(\tilde{\pi}(\mu_{s+t})^{-1}\right)^* b\rangle \mid X_s\right) \\ &= \langle \tilde{\pi}(\mu_t * \delta_{X_s})a, \left(\tilde{\pi}(\mu_{s+t})^{-1}\right)^* b\rangle \\ &= \langle \tilde{\pi}(\mu_{s+t})^{-1}\tilde{\pi}(\mu_t)\tilde{\pi}(\delta_{X_s})a, b\rangle = \langle \tilde{\pi}(\mu_s)^{-1}\pi(X_s)a, b\rangle. \end{aligned}$$

(2) The assumptions yield that, for all $s, t \geq 0$, $a, b\in H$,

$$\begin{aligned} E\left(\langle \pi(X_{s+t})a, b\rangle \mid \mathscr{F}_s\right) &= E\left(\langle \tilde{\pi}(\mu_{s+t})^{-1}\pi(X_{s+t})a, \tilde{\pi}(\mu_{s+t})^* b\rangle \mid \mathscr{F}_s\right) \\ &= \langle \tilde{\pi}(\mu_s)^{-1}\pi(X_s)a, \tilde{\pi}(\mu_{s+t})^* b\rangle = \langle \tilde{\pi}(\mu_t)\pi(X_s)a, b\rangle. \quad ∎ \end{aligned}$$

By Lemma 2.5 (1), one part of the following theorem is clear.

2.6. Theorem. *Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on G. Assume that there exists a set S of (equivalence classes of) unitary representations of G with the following properties:*

(a) *For each $\pi\in S$ and $t \geq 0$, the operator $\tilde{\pi}(\mu_t)^{-1}\in B(H)$ exists.*

(b) *Injectivity of the Fourier–Stieltjes transform: If $\mu\in M_b(G)$ satisfies $\tilde{\pi}(\mu) = 0$ for all $\pi\in S$, then $\mu = 0$ holds.*

Then the following statements are equivalent for a stochastic process $(X_t)_{t\geq 0}$ on G:

(1) *$(X_t)_{t\geq 0}$ is a process with independent stationary increments related with $(\mu_t)_{t\geq 0}$.*

(2) *For each $\pi\in S$, the process $\left(\tilde{\pi}(\mu_t)^{-1}\pi(X_t)\right)_{t\geq 0}$ is an operator-valued martingale with respect to the canonical filtration of $(X_t)_{t\geq 0}$.*

Proof. For the proof of (2) ⇒ (1), assume that $(X_t)_{t\geqslant 0}$ is defined on the probability space $(\Omega, \mathscr{A}, P)$. Denote the canonical filtration of $(X_t)_{t\geqslant 0}$ by $(\mathscr{F}_t)_{t\geqslant 0}$. Now fix $s, t \geqslant 0$, $\pi\in S$, and $a, b\in H$. Then, by Lemma 2.5 (2),

$$E\big(\langle \pi(X_{s+t})a, b\rangle \mid \mathscr{F}_s\big) = \langle \tilde{\pi}(\mu_t)\pi(X_s)a, b\rangle \text{ a.s.}$$

Now take $F\in\mathscr{F}_s$ with $P(F) > 0$. Define a probability measure P_F on $(\Omega, \mathscr{A})$ by

$$P_F(A) := \frac{P(A\cap F)}{P(F)} \quad \text{for } A\in\mathscr{A}.$$

Then the distributions $\mu_s^F, \mu_{s+t}^F\in M^1(G)$ of X_s and X_{s+t}, respectively with respect to P_F satisfy

$$\begin{aligned}\langle\tilde{\pi}(\mu_{s+t}^F)a, b\rangle &= \frac{1}{P(F)}\int_F \langle \pi(X_{s+t}(\omega))a, b\rangle\, dP(\omega)\\ &= \frac{1}{P(F)}\int_F E\big(\langle\pi(X_{s+t})a, b\rangle \mid \mathscr{F}_s\big)(\omega)\, dP(\omega)\\ &= \frac{1}{P(F)}\int_F \langle\tilde{\pi}(\mu_t)\pi(X_s(\omega))a, b\rangle\, dP(\omega)\\ &= \langle\tilde{\pi}(\mu_t)\tilde{\pi}(\mu_s^F)a, b\rangle = \langle\tilde{\pi}(\mu_t * \mu_s^F)a, b\rangle.\end{aligned}$$

As this holds for all $a, b\in H$ and all $\pi\in S$, condition (b) implies that $\mu_t * \mu_s^F = \mu_{s+t}^F$ for all $F\in\mathscr{F}_s$, $s, t\geqslant 0$. Hence, for all Borel sets $B\subset G$ and all $F\in\mathscr{F}_s$ with $P(F) > 0$,

$$\begin{aligned}\int_F 1_{\{X_{s+t}\in B\}}\, dP &= P(\{X_{s+t}\in B\}\cap F) = P(F)\cdot\mu_{s+t}^F(B)\\ &= P(F)\cdot(\mu_t * \mu_s^F)(B) = \int_F (\mu_t * \delta_{X_s(\omega)})(B)\, dP(\omega).\end{aligned}$$

As the mapping $\omega\mapsto(\mu_t * \delta_{X_s(\omega)})(B)$ is $\sigma(X_s)$-measurable, and as $\mathscr{F}_s\supset\sigma(X_s)$, we obtain

$$P(X_{s+t}\in B \mid \mathscr{F}_s) = P(X_{s+t}\in B \mid X_s) = (\mu_t * \delta_{X_s(\omega)})(B) \text{ a.e.}$$

for all Borel sets $B\subset K$. Hence $(X_t)_{t\geqslant 0}$ is a process with independent stationary increments related to $(\mu_t)_{t\geqslant 0}$ as claimed. ∎

2.7. Examples. (1) If a convolution semigroup $(\mu_t)_{t\geqslant 0}$ on a locally compact group G has the property that $\tilde{\pi}(\mu_t)^{-1}\in B(H)$ exists for all $t\geqslant 0$ and all $\pi\in\hat{G}$, then conditions (a) and (b) of Theorem 2.6 hold for $S := \hat{G}$. In particular, Remark 2.3 (2) ensures that the assertion of Theorem 2.6 holds with $S = \hat{G}$ for any compound Poisson convolution semigroup.

(2) A locally compact group G is called *almost periodic* if the set of all finite-dimensional unitary representations of G separates points in G, i.e., for all $x, y\in G$ with $x\neq y$ there exists a finite-dimensional unitary representation

π and elements a, b on the underlying Hilbert space with $\langle \pi(x) a, b\rangle \neq \langle \pi(y) a, b\rangle$.

It is well-known (see Theorem 1.3.8 of [8]) that for almost periodic groups G the injectivity of the Fourier–Stieltjes transform holds with $S := \{\pi \in \hat{G}: \dim \pi < \infty\}$. Hence conditions (a) and (b) of Theorem 2.6 hold here for this S and for all convolution semigroups.

(3) We give some examples of almost periodic groups: If all irreducible unitary representations of a locally compact group G are finite-dimensional (i.e., if G is a "Moore group"), then G is almost periodic by the Gelfand–Raikov theorem. In particular, compact groups and locally compact Abelian groups are Moore groups, and hence almost periodic. Moreover, each discrete free group is almost periodic; see Section 22.22 (d) of [7]. For a further discussion of almost periodic groups from a stochastic point of view we refer to [8].

We next compare the martingale characterization 2.6 with the martingale problem due to Stroock and Varadhan [18]. For a discussion of the martingale problem for Markov processes see Sections 4.1 and 4.3 of [5]. Recapitulate the notion that a stochastic process $(X_t)_{t\geq 0}$ has the càdlàg property if almost all paths are right continuous and admit limits from the left.

For simplicity we now restrict our attention to finite-dimensional representations.

2.8. PROPOSITION. *Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on a locally compact group G. Let π be a finite-dimensional unitary representation of G on some Hilbert space H. Then the one-parameter semigroup $(\tilde{\pi}(\mu_t))_{t\geq 0} \subset B(H)$ admits a generator*

$$F := \lim_{t\to 0} \frac{1}{t}\left(\tilde{\pi}(\mu_t) - \mathrm{Id}\right) \in B(H).$$

Moreover, for each càdlàg-process $(X_t)_{t\geq 0}$ on G, the process $\left(\tilde{\pi}(\mu_t)^{-1}\pi(X_t)\right)_{t\geq 0}$ is a $B(H)$-valued martingale if and only if so is $\left(\pi(X_t) - F\cdot\int_0^t \pi(X_s)\,ds\right)_{t\geq 0}$.

Proof. The existence of F is clear. For the main part of the proof identify H with C^n, i.e., elements of $B(H)$ are matrices $(A^{ij})_{i,j=1,\ldots,n}$. Notice that all coefficients of $\left(\tilde{\pi}(\mu_t)^{-1}\pi(X_t)\right)_{t\geq 0}$ and $\left(X_t^\pi := \pi(X_t) - F\cdot\int_0^t \pi(X_s)\,ds\right)_{t\geq 0}$ are uniformly and L^2-bounded on compact time intervals. Hence, these processes are martingales if and only if they are local L^2-martingales (see 4.2.3 of [20]).

Assume now that $\left(\left(\tilde{\pi}(\mu_t)^{-1}\pi(X_t)\right)^{ij}\right)_{t\geq 0}$ is a local L^2-martingale for each i, j. Then all $\left(\pi(X_t)^{ij}\right)_{t\geq 0}$ are semimartingales. Stochastic integration by parts together with $[(e^{-tF})^{kl}, \pi(X_t)^{ij}] = 0$ for the mutual variations (see 7.3.1 and 7.3.13 of [20]) implies

$$(2.2)\qquad d\left(e^{-tF}\pi(X_t)\right)^{ij} = \sum_{k=1}^{n} d\left((e^{-tF})^{ik}\pi(X_t)^{kj}\right) = \sum_{k=1}^{n}\left((e^{-tF})^{ik}\,d\pi(X_t)^{kj} + \left(\frac{d}{dt}e^{-tF}\right)^{ik}\pi(X_t)^{kj}\,dt\right)$$

$$= \sum_{k=1}^{n} \left((e^{-tF})^{ik} \, d\pi(X_t)^{kj} - (e^{-tF} \cdot F)^{ik} \, \pi(X_t)^{kj} \, dt\right)$$

$$= \sum_{k=1}^{n} (e^{-tF})^{ik} \left(d\pi(X_t)^{kj} - (F\pi(X_t))^{kj} \, dt\right),$$

and hence

$$d\pi(X_t)^{ij} - (F\pi(X_t))^{ij} \, dt = \sum_{k=1}^{n} (e^{tF})^{ik} \, d\left(e^{-tF} \pi(X_t)\right)^{kj}.$$

This yields that all components of $(X_t^\pi)_{t\geqslant 0}$ are local L^2-martingales.

The converse conclusion is similar. In fact, if all components of $(X_t^\pi)_{t\geqslant 0}$ are local L^2-martingales, then all $(\pi(X_t)^{ij})_{t\geqslant 0}$ are semimartingales for which (2.2) holds. ∎

Theorem 2.6 and Proposition 2.8 yield that the martingale problem in the sense of [18] is well-posed for processes with independent stationary increments on almost periodic groups in the following way:

2.9. Corollary. *Let $(\mu_t)_{t\geqslant 0}$ be a convolution semigroup on an almost periodic group G. Let π be a finite-dimensional unitary representation of G on some Hilbert space H and $a, b \in H$. Then for all $a, b \in H$ and all "coefficients" $f(x) := \langle \pi(x)a, b\rangle$ the limit*

$$Lf := \lim_{t\to 0} \frac{1}{t} (\mu_t^- * f - f) \in C_b(G)$$

exists. Moreover, a càdlàg-process $(X_t)_{t\geqslant 0}$ on G is a process with independent stationary increments associated with $(\mu_t)_{t\geqslant 0}$ if and only if, for each coefficient f as above, the process $\left(f(X_t) - \int_0^t Lf(X_s)\, ds\right)_{t\geqslant 0}$ is a martingale.

Proof. If f if given by $f(x) = \langle \pi(x)a, b\rangle$ and if F is as in Proposition 2.8, then

$$Lf(x) = \lim_{t\to 0} \left\langle \frac{1}{t} \left(\tilde{\pi}(\mu_t) - \mathrm{Id}\right) \pi(x)a, b \right\rangle = \langle F \cdot \pi(x)a, b\rangle.$$

Hence, in the matrix notation, $L\pi = F \cdot \pi$, and the corollary follows from 2.6 and 2.8. ∎

We finally compare Corollary 2.9 with known results for the martingale problem on a Lie group G. The well-posedness of the martingale problem for Lie groups has been derived by Feinsilver [6] and can be done so also from the discussion of the general martingale problem for Markov processes in [5] (use Proposition 4.1.7 and Theorem 4.4.1 of [5]). To describe the result, denote the space of all continuous functions on G vanishing at infinity by $C_0(G)$. Moreover, let $C_0^2(G)$ be the space of all twice (from the left) differentiable $C_0(G)$-functions. Then, for each convolution semigroup $(\mu_t)_{t\geqslant 0}$ on G, the space $C_0^2(G)$ is contained in the domain of the associated generator

(2.3) $$Lf := \lim_{t\to 0} \frac{1}{t} (\mu_t^- * f - f) \qquad (f \in C_0(G));$$

see Section 4.1 of [8]. Then, according to [6] or [5], the following holds:

2.10. Proposition. *Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on a Lie group G with generator L. Let $(X_t)_{t\geq 0}$ be a càdlàg process on G. Then $(X_t)_{t\geq 0}$ is a process with independent stationary increments associated with $(\mu_t)_{t\geq 0}$ if and only if for each $f\in C_0^2(G)$ the process $(f(X_t)-\int_0^t Lf(X_s)\,ds)_{t\geq 0}$ is a martingale.*

2.11. Remarks. (1) All results above can be generalized to non-unitary representations. However, one has to assume here that the convolution semigroup $(\mu_t)_{t\geq 0}$ on G and the representation(s) π of G have the property that the semigroup $(\tilde{\pi}(\mu_t))_{t\geq 0}\in B(H)$ as defined in Section 2.1 (2) exists as a weakly continuous semigroup. This may be seen as a generalization of classical moment conditions for $(\mu_t)_{t\geq 0}$.

(2) All results of this section can be also extended to stochastic processes on hypergroups. For the theory of hypergroups we refer to the monograph [1]. In particular, an analogue of Theorem 2.6 for commutative hypergroups is given in [15]. A further result of this type for Markov processes on $\mathbf{R}^n$ related with Dunkl operators is discussed in [17].

3. GAUSSIAN PROCESSES ON COMPACT LIE GROUPS

In this section we reduce condition (2) of Theorem 2.6 to a subset of representations of G which is as small as possible. This can be achieved via Itô's formula for processes with continuous paths. Hence, from now on we restrict our attention to Gaussian processes.

3.1. Gaussian semigroups and processes. A convolution semigroup $(\mu_t)_{t\geq 0}$ on a locally compact group G is called *Gaussian* if

$$\lim_{t\to 0}\frac{1}{t}\,\mu_t(G\backslash U_e)=0 \qquad \text{for all neighborhoods } U_e \text{ of } e.$$

If $(\mu_t)_{t\geq 0}$ is a Gaussian semigroup, then each associated stochastic process $(X_t)_{t\geq 0}$ on G with independent stationary increments is called a *Gaussian process associated with* $(\mu_t)_{t\geq 0}$.

For an extensive discussion and examples of Gaussian semigroups we refer to [8].

We recapitulate some well-known facts:

3.2. Facts. (1) A convolution semigroup on a locally compact group is Gaussian if and only if each associated process with independent stationary increments admits a version with continuous paths. In particular, the support of a Gaussian convolution semigroup is contained in the connected component of the group.

For the proof of the "only if" part see Theorem 45 in [13] (or I (9.10) in [2]). The "if" part in (1) can be shown in the same way as Exercise I (9.18) in [2] for $G=\mathbf{R}^n$.

(2) Let G be a compact group. If $(\mu_t)_{t\geq 0}$ is a Gaussian convolution semigroup on G, and if ϱ is a finite-dimensional unitary representation of G on C^n, then

$$(3.1)\qquad |\det(\varrho\otimes\varrho(\mu_t))^{\sim}|\cdot|\det(\varrho\otimes\bar{\varrho}(\mu_t))^{\sim}| = |\det\tilde{\varrho}(\mu_t)|^{4\dim\varrho} \qquad (t\geq 0),$$

where $\bar{\varrho}$ and $\dim\varrho$ denote the contragradient representation and dimension of ϱ, respectively. Conversely, if equality (3.1) holds for all irreducible unitary representations or some faithful unitary representation ϱ of G, then $(\mu_t)_{t\geq 0}$ is a Gaussian convolution semigroup. For the proof see Section 6.2 of [8] or Section 4 of [4].

(3) Gaussian processes on Lie groups can be characterized as unique solutions of certain stochastic differential equations in which Brownian motions on the associated Lie algebra appear; see [9] and [16].

We next show that it suffices for compact Lie groups to consider only finitely many representations in condition (2) of Theorem 2.6. For this notice that each compact Lie group admits a faithful finite-dimensional unitary representation; see Theorem III.4.1 in [3].

3.3. THEOREM. *Let $(\mu_t)_{t\geq 0}$ be a Gaussian convolution semigroup on a compact Lie group G, and let ϱ be a faithful finite-dimensional unitary representation of G. Then the following statements are equivalent for a continuous stochastic process $(X_t)_{t\geq 0}$ on G with filtration $(\mathscr{F}_t)_{t\geq 0}$:*

(1) *$(X_t)_{t\geq 0}$ is a Gaussian process associated with $(\mu_t)_{t\geq 0}$.*

(2) *If $\pi\in\{\varrho, \varrho\otimes\varrho\}$, then $(\tilde{\pi}(\mu_t)^{-1}\pi(X_t))_{t\geq 0}$ is a matrix-valued martingale with respect to $(\mathscr{F}_t)_{t\geq 0}$.*

(3) *For each irreducible unitary subrepresentation π of ϱ or $\varrho\otimes\varrho$, the process $(\tilde{\pi}(\mu_t)^{-1}\pi(X_t))_{t\geq 0}$ is a matrix-valued martingale with respect to $(\mathscr{F}_t)_{t\geq 0}$.*

Proof. The conclusion (1) $\Rightarrow$ (2) is clear by Lemma 2.5 (1), and (2) $\Leftrightarrow$ (3) is obvious. To prove (2) $\Rightarrow$ (1), we may assume that G is a compact subgroup of $U(n)$ for some $n\in N$ and that ϱ is the identity. Let $(X_t)_{t\geq 0}$ be a continuous process on G satisfying condition (2). Realize the tensor product representation $\varrho\otimes\varrho$ on R^{n^2} as matrices with $(\varrho\otimes\varrho(x))^{(ik)(jl)} = x^{ij}x^{kl}$. Moreover, denote the generators of the semigroups $(\tilde{\varrho}(\mu_t))_{t\geq 0}$ and $((\varrho\otimes\varrho(\mu_t))^{\sim})_{t\geq 0}$ by F_1 and F_2, respectively. Let $i, j, k, l\in\{1, \ldots, n\}$. As the coordinates $(X_t^{ij})_{t\geq 0}$ are semimartingales (cf. the proof of Proposition 2.8), Itô's formula yields

$$d(X_t^{ij}X_t^{kl}) = X_t^{ij}\,dX_t^{kl}+X_t^{kl}\,dX_t^{ij}+d[X_t^{ij}, X_t^{kl}],$$

where $[\cdot,\cdot]$ denotes the mutual variance. Therefore,

$$(3.2)\qquad d(X_t^{ij}X_t^{kl})-(F_2\cdot(X_t\otimes X_t))^{(ik)(jl)}dt-X_t^{ij}(dX_t^{kl}-(F_1\cdot X_t)^{kl}dt)$$
$$-X_t^{kl}(dX_t^{ij}-(F_1\cdot X_t)^{ij}dt) = H_{ijkl}(X_t)\,dt+d[X_t^{ij}, X_t^{kl}]$$

with

$$H_{ijkl}(x):= x^{ij}(F_1\cdot x)^{kl}+x^{kl}(F_1\cdot x)^{ij}-(F_2\cdot(x\otimes x))^{(ik)(jl)} \qquad (x\in G).$$

The left-hand side of (3.2) is the differential of a local L^2-martingale by condition (2) and Proposition 2.8. Moreover, as a.s. locally finite variation is preserved under stochastic integration (see 5.3.5 of [20]), the right-hand side of (3.2) has paths with a.s. locally finite variation. It follows from the Doob–Meyer decomposition that (3.2) is a.s. equal to zero; see 5.3.2 of [20]. Hence

$$d[X_t^{ij}, X_t^{kl}] = H_{ijkl}(X_t)\,dt. \tag{3.3}$$

Now consider a "monomial" h on G of the type

$$h(x) = x_1^{i_1} \dots x_n^{i_n} \cdot \overline{x_1}^{j_1} \dots \overline{x_n}^{j_n}, \qquad i_1, \dots, i_n, j_1, \dots, j_n \geqslant 0.$$

As h is holomorphic on G, the complex Itô formula applies. Hence, with equation (3.3), we obtain

$$\begin{aligned} dh(X_t) = {} & \sum_{i,j=1}^{n} (\partial_{i,j} h)(X_t)\big(dX_t^{ij} - (F_1 X_t)^{ij}\,dt\big) + \sum_{i,j=1}^{n} (\partial_{i,j} h)(X_t)(F_1 X_t)^{ij}\,dt \\ & + \frac{1}{2} \sum_{i,j,k,l=1}^{n} (\partial_{k,l}\partial_{i,j} h)(X_t) \cdot H_{ijkl}(X_t)\,dt. \end{aligned} \tag{3.4}$$

Moreover, h is a matrix coefficient of some finite tensor power $\varrho^{\otimes,a} \otimes \bar{\varrho}^{\otimes,b}$ of ϱ and $\bar{\varrho}$. Hence, by Corollary 2.9, h is in the domain of the generator L associated with $(\mu_t)_{t\geqslant 0}$. As

$$\sum_{i,j=1}^{n} (\partial_{i,j} h)(X_t)\big(dX_t^{ij} - (F_1 X_t)^{ij}\,dt\big)$$

is the differential of a local martingale by condition (2) and Proposition 2.8, equation (3.4) implies that

$$dh(X_t) - Lh(X_t)\,dt \tag{3.5}$$

is the differential of a local martingale if and only if

$$\Big\{ \sum_{i,j=1}^{n} (\partial_{i,j} h)(X_t)(F_1 X_t)^{ij} - Lh(X_t) + \frac{1}{2} \sum_{i,j,k,l=1}^{n} (\partial_{k,l}\partial_{i,j} h)(X_t) \cdot H_{ijkl}(X_t) \Big\}\,dt \tag{3.6}$$

is the differential of a local martingale. The latter is possible if and only if (3.6) is equal to zero a.s. (see 5.3.2 and 5.3.5 of [20]), which means that the integrand disappears in this equation.

To prove this, replace the process $(X_t)_{t\geqslant 0}$ above by some Gaussian process $(Y_t)_{t\geqslant 0}$ related with $(\mu_t)_{t\geqslant 0}$ such that the support of the initial distribution of Y_0 is equal to G. By Proposition 2.10, the associated differential (3.5) belongs to a local martingale. Hence, by the conclusions above,

$$\sum_{i,j=1}^{n} (\partial_{i,j} h)(Y_t)(F_1 Y_t)^{ij} - Lh(Y_t) + \frac{1}{2} \sum_{i,j,k,l=1}^{n} (\partial_{k,l}\partial_{i,j} h)(Y_t) \cdot H_{ijkl}(Y_t) = 0 \tag{3.7}$$

for all $t \geqslant 0$ a.s. It follows that for all $x \in G$

$$(3.8)\qquad \sum_{i,j=1}^{n} (\partial_{i,j} h)(x)(F_1 x)^{ij} - Lh(x) + \frac{1}{2} \sum_{i,j,k,l=1}^{n} (\partial_{k,l} \partial_{i,j} h)(x) \cdot H_{ijkl}(x) = 0.$$

Therefore, the differential (3.6) is equal to zero for $(X_t)_{t \geqslant 0}$, and the differential (3.5) belongs to a local martingale for each h. As the functions h form all matrix coefficients of all finite tensor powers $\varrho^{\otimes,a} \otimes \bar{\varrho}^{\otimes,b}$ of ϱ, $\bar{\varrho}$, and as each irreducible unitary representation of G appears as a subrepresentation of some of these tensor powers (Theorem III.4.4 of [3]), it follows from Proposition 2.8 that condition (2) of Theorem 2.6 holds. This completes the proof. ∎

3.4. Example. For the torus $T = \{z \in \boldsymbol{C}\colon |z| = 1\}$, the set of irreducible representations may be identified with the character group $\hat{G} \equiv \boldsymbol{Z}$. Taking the identity as faithful representation ϱ, we obtain from equation (3.8) for $h(x) = x^n$ that Gaussian semigroups $(\mu_t)_{t \geqslant 0}$ on T satisfy

$$\hat{\mu}_t(n) = \int_T z^n \, d\mu_t(z) = \exp\{-tF_n\} \qquad \text{for } n \in \boldsymbol{Z},$$

where the coefficients F_n satisfy

$$(3.9)\qquad F_n = nF_1 + \tfrac{1}{2} n(n-1)(F_2 - 2F_1) \qquad (n \in \boldsymbol{Z}).$$

Hence the solutions of (3.9) are given by

$$(3.10)\qquad F_n = c_1 n^2 + i \cdot c_2 n \qquad (n \in \boldsymbol{Z},\ c_1,\ c_2 \text{ constants}).$$

Hence equation (3.8) is equivalent to the well-known description of the Fourier transforms of Gaussian semigroups; cf. Section 5.2 of [8]. For $c_1 \geqslant 0$ and $c_2 \in \boldsymbol{R}$ in equation (3.16), there exists an associated Gaussian convolution semigroup $(\mu_t)_{t \geqslant 0}$, and Theorem 3.3 leads to a martingale characterization of the associated Gaussian processes. It is even possible here to drop the continuity assumption:

3.5. Corollary. *Let $c_1 \geqslant 0$ and $c_2 \in \boldsymbol{R}$. Let $(X_t)_{t \geqslant 0}$ be a stochastic process on T such that $(\exp\{(c_1 n^2 + ic_2 n)t\} \cdot X_t^n)_{t \geqslant 0}$ is a martingale for $n \in \{1, 2\}$. Then $(X_t)_{t \geqslant 0}$ is a Gaussian process on T (associated with the constants c_1, c_2).*

Proof. Let $0 \leqslant s \leqslant t$. As $|X_s| = |X_t| = 1$, we have

$$E(|X_s - X_t|^4) = 6 - 4E(X_t \overline{X}_s + X_s \overline{X}_t) + E(X_t^2 \overline{X}_s^2 + X_s^2 \overline{X}_t^2).$$

Moreover, if $(\mathscr{F}_t)_{t \geqslant 0}$ denotes the canonical filtration of $(X_t)_{t \geqslant 0}$, we obtain

$$E(X_t \overline{X}_s) = E(E(X_t \overline{X}_s \mid \mathscr{F}_s)) = E(E(X_t \mid \mathscr{F}_s) \overline{X}_s) = \exp\{(c_1 + ic_2)(t-s)\}$$

and, by similar arguments, $E(X_t^2 \overline{X}_s^2) = \exp\{(4c_1 + 2ic_2)(t-s)\}$. Hence

$$\begin{aligned} E(|X_s - X_t|^4) &= 6 - 8\exp\{c_1(t-s)\}\cos(c_2(t-s)) + 2\exp\{4c_1(t-s)\}\cos(2c_2(t-s)) \\ &= O((s-t)^2) \qquad \text{for } s - t \to 0. \end{aligned}$$

Hence, by Kolmogorov's criterion (see [18]), $(X_t)_{t\geq 0}$ is continuous after restriction to a subset of the underlying probability space with outer measure 1. Theorem 3.3 now completes the proof. ∎

Corollary 3.5 can be easily extended to the d-dimensional torus T^d. For this, fix a drift vector $c := (c_1, \ldots, c_d) \in \mathbf{R}^d$ and a positive semidefinite diffusion matrix $D := (d_{i,j})_{i,j=1,\ldots,d}$. Then, according to usual conventions, the associated convolution semigroup $(\mu_t)_{t\geq 0} \subset M^1(T^d)$ is characterized via its Fourier transform with

$$\hat{\mu}_t(n) := \int_{T^d} z_1^{-n_1} \ldots z_d^{-n_d}\, d\mu_t(z_1, \ldots, z_d) = \exp\{t(icn^t - nDn^t/2)\}$$

for $n \in \mathbf{Z}^d \simeq (T^d)^\wedge$ (where ^{t} means the transpose). With this notation, the following martingale characterization holds:

3.6. Corollary. *A stochastic process $(X_t := (X_{t,1}, \ldots, X_{t,d}))_{t\geq 0}$ on T^d is a Gaussian process associated with the drift vector c and the diffusion matrix D if and only if for all $n \in \mathbf{Z}_+^d$ with $n_1 + \ldots + n_d \leq 2$ the processes $(\exp\{t(icn^t + nDn^t/2)\} X_{t,1}^{n_1} \ldots X_{t,d}^{n_d})_{t\geq 0}$ are martingales.*

Proof. It suffices to check the "if" part. Here Corollary 3.5 yields that all components of X may be assumed to be continuous. Theorem 3.3 now completes the proof. ∎

To drop the continuity assumption in Theorem 3.3 for arbitrary compact Lie groups, we need some preparations. Denote the trace and the transpose of a matrix A by $\operatorname{tr} A$ and A^T, respectively.

3.7. Lemma. *Let ϱ be a finite-dimensional unitary representation of a locally compact group G on $\mathbf{C}^n$ with the usual scalar product. Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on G, and $(X_t)_{t\geq 0}$ a stochastic process on G with filtration $(\mathcal{F}_t)_{t\geq 0}$. Then:*

(1) If $(\tilde{\varrho}(\mu_t)^{-1} \varrho(X_t))_{t\geq 0}$ is a matrix-valued martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$, then for all $0 \leq s \leq t$

$$E\big(\operatorname{tr}(\varrho(X_t X_s^{-1}))\big) = \operatorname{tr}\big(\tilde{\varrho}(\mu_{t-s})\big).$$

(2) If $(\tilde{\pi}(\mu_t)^{-1} \pi(X_t))_{t\geq 0}$ is an operator-valued martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$ for $\pi \in \{\varrho, \varrho \otimes \varrho, \varrho \otimes \bar{\varrho}\}$, then

$$H(s, t) := E\Big(\big\{\operatorname{tr}\big((\varrho(X_s) - \varrho(X_t))\overline{(\varrho(X_s) - \varrho(X_t))^T}\big)\big\}^2\Big)$$

can be written as

$$H(s, t) = H(|s-t|, 0) = A_\varrho \cdot |s-t| + O(|s-t|^2) \quad \text{for } |s-t| \to 0,$$

where the constant A_ϱ satisfies $A_\varrho = 0$ if and only if

$$\big|\det(\varrho \otimes \varrho(\mu_t))^\sim\big| \cdot \big|\det(\varrho \otimes \bar{\varrho}(\mu_t))^\sim\big| = |\det \tilde{\varrho}(\mu_t)|^{4\cdot n} \quad \text{for } t \geq 0.$$

Proof. (1) Using Lemma 2.5 (2) and $\varrho(X_s)\cdot\overline{\varrho(X_s)}^T = \mathrm{Id}$, we obtain

$$E\big(\mathrm{tr}\,(\varrho(X_t X_s^{-1}))\big) = E\big(E\big(\mathrm{tr}\,(\varrho(X_t)\overline{\varrho(X_s)}^T)\mid \mathscr{F}_s\big)\big)$$

$$= \sum_{i,j=1}^{n} E\big(E\big(\varrho(X_t)^{ij}\overline{\varrho(X_s)^{ij}}\mid \mathscr{F}_s\big)\big) = \sum_{i,j=1}^{n} E\big(E\big(\varrho(X_t)^{ij}\mid \mathscr{F}_s\big)\cdot\overline{\varrho(X_s)^{ij}}\big)$$

$$= \sum_{i,j=1}^{n} E\big((\tilde{\varrho}(\mu_{t-s})\varrho(X_s))^{ij}\overline{\varrho(X_s)^{ij}}\big) = E\big(\mathrm{tr}\,(\tilde{\varrho}(\mu_{t-s})\varrho(X_s)\overline{\varrho(X_s)}^T)\big) = \mathrm{tr}\,(\tilde{\varrho}(\mu_{t-s})).$$

(2) Assume $0 \leqslant s \leqslant t$ without loss of generality. Using $\varrho(X_s)\cdot\overline{\varrho(X_s)}^T = \mathrm{Id}$, we obtain

$$\begin{aligned} H(s, t) &= E\big(\{2n - \mathrm{tr}\,(\varrho(X_t X_s^{-1}) + \varrho(X_s X_t^{-1}))\}^2\big) \\ &= 4n^2 - 4n\,E\big(\mathrm{tr}\,(\varrho(X_t X_s^{-1}) + \varrho(X_s X_t^{-1}))\big) \\ &\quad + E\big(\{\mathrm{tr}\,(\varrho(X_t X_s^{-1}) + \varrho(X_s X_t^{-1}))\}^2\big) \\ &= 4n^2 - 8n\,\mathrm{Re}\,E\big(\mathrm{tr}\,(\varrho(X_t X_s^{-1}))\big) + 2\,\mathrm{Re}\,E\big(\mathrm{tr}\,(\varrho\otimes\varrho(X_t X_s^{-1}))\big) \\ &\quad + 2E\big(\mathrm{tr}\,(\varrho\otimes\bar{\varrho}(X_t X_s^{-1}))\big). \end{aligned}$$

If we apply part (1) to ϱ, $\varrho\otimes\varrho$, and $\varrho\otimes\bar{\varrho}$, we obtain

$$H(s, t) = 4n^2 - 8n\,\mathrm{Re}\,\mathrm{tr}\,\tilde{\varrho}(\mu_{t-s}) + 2\,\mathrm{Re}\,\mathrm{tr}\,((\varrho\otimes\varrho)^{\sim}(\mu_{t-s})) + 2\,\mathrm{Re}\,\mathrm{tr}\,((\varrho\otimes\bar{\varrho})^{\sim}(\mu_{t-s})). \tag{3.11}$$

For $\pi \in \{\varrho, \varrho\otimes\varrho, \varrho\otimes\bar{\varrho}\}$, denote the generator of the semigroup $(\tilde{\pi}(\mu_t))_{t\geqslant 0}$ by F_π. A power series expansion of $\exp\{tF_\pi\} = \tilde{\pi}(\mu_t)$ in equation (3.11) leads to $H(s, t) = A_\varrho(t-s) + O((t-s)^2)$ for $t-s \to 0$ with

$$A_\varrho := -8n\,\mathrm{Re}\,\mathrm{tr}\,F_\varrho + 2\,\mathrm{Re}\,\mathrm{tr}\,F_{\varrho\otimes\varrho} + 2\,\mathrm{Re}\,\mathrm{tr}\,F_{\varrho\otimes\bar{\varrho}}.$$

As $A_\varrho = 0$ is equivalent to $|\det\exp\{F_\varrho\}|^{4\cdot n} = |\det\exp\{F_{\varrho\otimes\varrho}\}|\cdot|\det\exp\{F_{\varrho\otimes\bar{\varrho}}\}|$, the lemma follows. ∎

3.8. Remarks. Before we proceed with the main application of Lemma 3.7, we mention that Lemmas 3.7 and 2.5 lead to an alternative proof of parts of Section 3.2:

(1) Let $(\mu_t)_{t\geqslant 0}$ be a Gaussian convolution semigroup on some locally compact group G. Then, for any finite-dimensional unitary representation ϱ of G,

$$|\det(\varrho\otimes\varrho(\mu_t))^{\sim}|\cdot|\det(\varrho\otimes\bar{\varrho}(\mu_t))^{\sim}| = |\det\tilde{\varrho}(\mu_t)|^{4\cdot\dim\varrho} \quad (t \geqslant 0). \tag{3.12}$$

In fact, it is clear from the definition of a Gaussian semigroup that the image $(\varrho(\mu_t))_{t\geqslant 0}$ of a Gaussian semigroup $(\mu_t)_{t\geqslant 0}$ is a Gaussian semigroup on some unitary group $U(n)$. As

$$\{\mathrm{tr}\,((A-B)\overline{(A-B)}^T)\}^{1/2}$$

is the Euclidean distance on $U(n)$, it follows from the definition of a Gaussian semigroup that, for any Gaussian process $(X_t)_{t\geq 0}$ associated with $(\mu_t)_{t\geq 0}$, the function H of Lemma 3.7 satisfies $H(s,t)=o(|s-t|)$ for $|s-t|\to 0$. Equality (3.12) is now a consequence of 2.5 and 3.7.

(2) Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on a compact group G. If (3.12) holds for each irreducible unitary representation ϱ of G or some faithful one, then each associated stochastic process with independent stationary increments is continuous on a set with outer probability one.

In fact, let $(X_t)_{t\geq 0}$ be such a process with independent stationary increments. Then Lemmas 2.5 and 3.7 imply that, for each irreducible representation ϱ of G on some Hilbert space H_ϱ with faithful trace tr_ϱ,

$$E\left(\left\{\mathrm{tr}_\varrho\left((\varrho(X_s)-\varrho(X_t))\overline{(\varrho(X_s)-\varrho(X_t))^T}\right)\right\}^2\right)=O(|s-t|^2) \quad \text{for } |s-t|\to 0.$$

Hence, by Kolmogorov's criterion, $(\varrho(X_t))_{t\geq 0}$ is continuous with outer probability one. The faithful case is now clear. Otherwise, remember that G admits at most countably many (equivalence classes of) irreducible representations. Hence G may be regarded as a compact subgroup of $\prod_{\varrho\ \text{irreducible}} B(H_\varrho)$, and the statement follows.

Theorem 3.3 now leads to the following extension of Corollary 3.5:

3.9. THEOREM. *Let $(\mu_t)_{t\geq 0}$ be a Gaussian convolution semigroup on a compact Lie group G. Let ϱ be a faithful, finite-dimensional unitary representation of G. Then the following statements are equivalent for a stochastic process $(X_t)_{t\geq 0}$ on G with filtration $(\mathscr{F}_t)_{t\geq 0}$:*

(1) *$(X_t)_{t\geq 0}$ is a Gaussian process on G related to $(\mu_t)_{t\geq 0}$.*

(2) *If $\pi\in\{\varrho,\ \varrho\otimes\varrho,\ \varrho\otimes\bar{\varrho}\}$, then $(\tilde{\pi}(\mu_t)^{-1}\pi(X_t))_{t\geq 0}$ is an operator-valued $(\mathscr{F}_t)_{t\geq 0}$-martingale.*

(3) *If π is an irreducible representation of G contained in ϱ, $\varrho\otimes\varrho$, or $\varrho\otimes\bar{\varrho}$, then $(\tilde{\pi}(\mu_t)^{-1}\pi(X_t))_{t\geq 0}$ is an operator-valued martingale with respect to $(\mathscr{F}_t)_{t\geq 0}$.*

Proof. In view of Lemma 2.5 and Theorem 3.3, it suffices to check that a process $(X_t)_{t\geq 0}$ on G with property (2) or (3) is continuous with outer probability one. But this follows from Lemma 3.7 and Kolmogorov's criterion in the same way as above. ∎

3.10. EXAMPLE. The unitary group $SU(2)$ admits exactly one (equivalence class of an) irreducible unitary representation ϱ_n of dimension n for each $n\in N$; see Section II.5 of [3]. In particular, ϱ_2 is the identity with $\varrho_2\otimes\varrho_2\equiv\varrho_1\oplus\varrho_3$ and $\bar{\varrho}_2=\varrho_2$. Furthermore, ϱ_3 is the usual covering map from $SU(2)$ onto $SO(3)$. As ϱ_1 is the trivial representation (which carries no information), Theorem 3.9 leads to:

3.11. COROLLARY. *Let $(\mu_t)_{t\geq 0}$ be a Gaussian convolution semigroup on $SU(2)$. Then a stochastic process $(X_t)_{t\geq 0}$ on $SU(2)$ is a Gaussian process related to $(\mu_t)_{t\geq 0}$ if and only if the processes $(\tilde{\varrho}_n(\mu_t)^{-1}\cdot\varrho_n(X_t))_{t\geq 0}$ are operator-valued martingales for $n=2,3$.*

3.12. EXAMPLE. The orthogonal group $SO(3)$ has exactly one (equivalence class of an) irreducible unitary representation ϱ_n of dimension $2n-1$ for $n \in N$; see Section II.5 of [3]. In particular, ϱ_2 is the identity with $\varrho_2 \otimes \varrho_2 \equiv \varrho_1 \oplus \varrho_2 \oplus \varrho_3$ and $\bar{\varrho}_2 = \varrho_2$. Hence Theorem 3.9 leads to:

3.13. COROLLARY. *Let $(\mu_t)_{t \geqslant 0}$ be a Gaussian convolution semigroup on $SO(3)$. Then a stochastic process $(X_t)_{t \geqslant 0}$ on $SO(3)$ is a Gaussian process related to $(\mu_t)_{t \geqslant 0}$ if and only if the processes $(\tilde{\varrho}_n(\mu_t)^{-1} \cdot \varrho_n(X_t))_{t \geqslant 0}$ are operator-valued martingales for $n = 2, 3$.*

3.14. Remarks. (1) If ϱ is a real, faithful, finite-dimensional unitary representation on a compact Lie group G, then $\bar{\varrho} = \varrho$ always holds, i.e., in this case the martingale conditions (2) and (3) in Theorems 3.3 and 3.9 are equivalent, respectively. Concrete examples are given by the identity representations of the rotation groups $SO(n)$.

(2) Similarly to Corollary 3.6 for the multidimensional torus, Corollaries 3.11 and 3.13 can be extended to finite mixed direct products of the groups T, $SO(3)$, and $SU(2)$ by taking ϱ as the associated direct product of the corresponding identity representations. It turns out again that in this case the continuity assumption in Theorem 3.3 can be completely omitted.

In the end of this paper we extend Theorem 3.3 to arbitrary compact groups. Recall that each compact group G is (isomorphic to) the projective limit of compact Lie groups; see Section 28.61 of [7]. This means that there is a directed system $(H_\alpha)_{\alpha \in A}$ of compact normal subgroups of G with $H_\alpha \subset H_\beta$ for $\alpha \geqslant \beta$ such that G/H_α is a Lie group for each α, and such that $\bigcap_{\alpha \in A} H_\alpha = \{e\}$ holds. For $\alpha \in A$ we denote the canonical homomorphism from G onto G/H_α by p_α.

3.15. THEOREM. *Let $(\mu_t)_{t \geqslant 0}$ be a Gaussian convolution semigroup on a compact group G. Assume that the systems $(H_\alpha)_{\alpha \in A}$ and $(p_\alpha)_{\alpha \in A}$ are given as above. Then the following statements are equivalent for a continuous stochastic process $(X_t)_{t \geqslant 0}$ on G:*

(1) *$(X_t)_{t \geqslant 0}$ is a Gaussian process related to $(\mu_t)_{t \geqslant 0}$.*

(2) *For each $\alpha \in A$ there is a faithful finite-dimensional unitary representation ϱ_α of G/H_α such that for $\pi_\alpha \in \{\varrho_\alpha, \varrho_\alpha \otimes \varrho_\alpha\}$ the process*

$$((\pi_\alpha \circ p_\alpha)^\sim (\mu_t)^{-1} \cdot (\pi_\alpha \circ p_\alpha)(X_t))_{t \geqslant 0}$$

is an operator-valued martingale with respect to the canonical filtration of $(X_t)_{t \geqslant 0}$.

Proof. Due to Lemma 2.5, it suffices to prove (2) $\Rightarrow$ (1). Assume that (2) holds. Then, for each $\alpha \in A$, $(p_\alpha(\mu_t))_{t \geqslant 0} \subset M^1(G/H_\alpha)$ is a Gaussian semigroup on G/H_α. Hence, by Theorems 3.3 and 2.6, $(p_\alpha(X_t))_{t \geqslant 0}$ is a Gaussian process on G/H_α, and

$$((\varrho \circ p_\alpha)^\sim (\mu_t)^{-1} \cdot (\varrho \circ p_\alpha)(X_t))_{t \geqslant 0}$$

is an operator-valued martingale for each finite-dimensional unitary representation ϱ of G/H_α. As this holds for all α, and as each finite-dimensional unitary representation π of G takes the form $\pi = \varrho \circ p_\alpha$ for suitable α and ϱ, it follows from Theorem 2.6 that $(X_t)_{t\geqslant 0}$ is a Gaussian process related to $(\mu_t)_{t\geqslant 0}$. ■

Theorem 3.9 can be extended in a similar way; we leave this to the reader. We only mention that Corollary 3.6 can be extended to the infinite-dimensional torus T^N in the obvious way.

3.16. Remark. Theorem 3.3 can be extended from the setting of faithful finite-dimensional unitary representations of compact Lie groups to arbitrary faithful finite-dimensional representations of locally compact groups. In fact, for the proof it suffices to study the group $GL(n, \boldsymbol{R})$ with the identity representation. In the proof one now proceeds as in the proof of Theorem 3.3 except that one has to take test functions $h \in C_0^2(\boldsymbol{R}^{n^2})$ (which are contained in the domain of the generator associated with $(\mu_t)_{t\geqslant 0}$). The result then follows with Proposition 2.10. We remark that one has to be careful in this proof with respect to the following two points: One needs that all second moments $\int_{GL(n,\boldsymbol{R})} x_{i,j}^2\, d\mu_t(x)$ exist as continuous functions in $t \geqslant 0$ for the Gaussian convolution semigroup $(\mu_t)_{t\geqslant 0}$ under consideration. Moreover, one has to take real tensor products in condition (2) of Theorem 3.3 (as the test functions h above are $\boldsymbol{R}$-differentiable only).

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