ON BOUNDEDNESS AND CONVERGENCE OF SOME BANACH SPACE VALUED RANDOM SERIES

BY

RAFAł SZTENCEL (WARSZAWA)

Abstract. Let \( (f_i) \) and \( (g_i) \) be sequences of independent symmetric random variables and \( (x_i) \) a sequence of elements from a Banach space. We prove that under certain assumptions the a.s. boundedness of the series \( \sum x_i f_i \) implies the a.s. convergence of \( \sum x_i g_i \) in every Banach space.

If \( f_i \) are identically distributed, \( E|f_i| \) is finite, \( g_i \) are identically distributed and non-degenerate, then the above implication fails in \( c_0 \).

If \( f_i \) are equidistributed and there is a sequence \( (a_n) \) such that

\[
a_n^{-1} \sum_{i=1}^n |f_i| \to 1 \text{ in probability},
\]

then there is a sequence \( (x_i) \) in \( c_0 \) such that \( \sum x_i f_i \) is a.s. bounded, but does not converge a.s. In particular, if \( f_i \) are \( p \)-stable with \( E e^{i\eta x} = e^{-|x|^p} \), then for \( p < 1 \) the a.s. boundedness of the series implies its a.s. convergence, but for \( p \geq 1 \) it fails.

The origin of this paper is the following Garling's question:

Let \( (\eta_i)_{i \in \mathbb{N}} \) be a sequence of \( p \)-stable random variables (r.v.) with characteristic function \( e^{-|x|^p} \), \( p \in (0, 2) \), and \( (x_i) \) a sequence in a Banach space \( E \). If the series \( \sum_{i \in \mathbb{N}} \eta_i x_i \) is a.s. bounded, then is it a.s. convergent?

Some general results are obtained; it turns out that the answer is positive for \( p \in (0, 1) \) and negative for \( p \in [1, 2) \).

1. Preliminaries. We begin with some known facts.

1.1. Definition. Let \( (q_i) \) and \( (\xi_i) \) be two sequences of independent symmetric real-valued r.v. The sequence \( (q_i) \) is dominated by \( (\xi_i) \) if there exist constants \( K \) and \( L \) such that for every \( t \) and \( i \)

\[
P(|q_i| > t) \leq K P(L|\xi_i| > t).
\]
The forthcoming theorem is an easy corollary to a result stated in [3]. The proof in the sequel with a better constant than in [3] is due to S. Kwapien and seems to be new.

1.2. THEOREM. Let $X_1, X_2, \ldots, X_n$ be independent symmetric $E$-valued r.v. Then for every $t \in \mathbb{R}$

$$P \left( \left\| \sum_{i=1}^{n} a_i X_i \right\| > t \right) \leq 2P(\max_{i} |a_i| \left\| \sum_{i=1}^{n} X_i \right\| > t).$$

Proof. We can assume that $0 \leq a_1 \leq \ldots \leq a_n = 1$. Put $a_0 = 0$, $b_k = a_k - a_{k-1}$ for $k = 1, 2, \ldots, n$, $S_k = \sum_{i=k}^{n} X_i$. Then

$$\sum_{i=1}^{n} a_i X_i = \sum_{k=1}^{n} b_k S_k, \quad \sum_{k=1}^{n} b_k = 1.$$

Consequently, if $\left\| \sum_{i=1}^{n} a_i X_i \right\| > t$, then $\max_k \left\| S_k \right\| > t$. Therefore we have

$$P \left( \left\| \sum_{i=1}^{n} a_i X_i \right\| > t \right) \leq P(\max_{k} \left\| S_k \right\| > t) \leq 2P(\left\| S_1 \right\| > t),$$

which completes the proof.

1.3. THEOREM (E. Rychlik, oral communication). If $(\xi_i)$ is dominated by $(\zeta_i)$ with constants $K$ and $L$, where $K \in \mathbb{N}$, then for every $x_1, x_2, \ldots, x_n \in E$ and $t \in \mathbb{R}$

$$P \left( \left\| \sum_{i \in I} q_i x_i \right\| > t \right) \leq 2K P(K \left\| \sum_{i \in I} \xi_i x_i \right\| > t).$$

Proof. We may assume without loss of generality that $L = 1$. Let $\psi_i^k (i = 1, 2, \ldots, n; k = 1, 2, \ldots, K)$ be r.v. such that

(i) $P(\psi_i^k = 1) = 1 - P(\psi_i^k = 0) = 1/K$,

(ii) $\psi_1^1 + \ldots + \psi_i^k = 1$ for $i = 1, 2, \ldots, n$,

(iii) $\psi_i^k, \ldots, \psi_n^k$ are independent for fixed $k$.

We prove that

$$P \left( \left\| \sum_{i \in I} q_i x_i \right\| > t \right) \leq KP(K \left\| \sum_{i \in I} q_i \psi_i^1 x_i \right\| > t) \leq 2KP(\left\| \sum_{i \in I} \xi_i x_i \right\| > t).$$

The first inequality can be rewritten in the form

$$P \left( \left\| \sum_{i \in I} q_i \psi_i^1 x_i + \ldots + \sum_{i \in I} q_i \psi_i^K x_i \right\| > t \right) \leq \sum_{j=1}^{K} P \left( \left\| \sum_{i \in I} q_i \psi_i^j x_i \right\| > \frac{t}{K} \right).$$

Now it is obvious that if the event on the left-hand side takes place, then some of $K$ events on the right-hand side must take place. Therefore (*) holds.
The second inequality is a consequence of 1.1. We prove that
\[ P\left(\left\| \sum \varphi_i \psi_i^j x_i \right\| > t\right) \leq 2P\left(\left\| \sum \xi_i x_i \right\| > t\right). \]
We have
\[ P\left(\left\| \varphi_i \psi_i^j \right\| > t\right) = \frac{1}{K} \left(\left\| \varphi_i \right\| > t\right) \leq P\left(\left\| \xi_i \right\| > t\right). \]

Then it is not hard to see that there are r.v. \( \varphi_i \) and \( \xi_i \) on a probability space \((\Omega', \mathcal{F}', P')\) such that
(i) \( |\varphi_i| \leq 1 \),
(ii) the sequences \((\xi_i)_{i \leq n}\) and \((\xi_i)_{i \leq n}\) are identically distributed,
(iii) the sequences \((\varphi_i \xi_i)_{i \leq n}\) and \((\varphi_i \xi_i)_{i \leq n}\) are identically distributed.

Let \((\xi_i)_{i \leq n}\) be a Bernoulli sequence on a probability space \((\Omega'', \mathcal{F}'', P'')\). Then
\[ P\left(\left\| \sum \varphi_i \psi_i^j x_i \right\| > t\right) = P\left(\left\| \sum \varphi_i \xi_i x_i \right\| > t\right) = P' \times P''\left(\left\| \sum \varphi_i \xi_i x_i \right\| > t\right) \]
\[ \leq 2P' \times P''\left(\max \left\| \varphi_i \right\| \sum \xi_i x_i > t\right) \leq 2P\left(\left\| \sum \xi_i x_i \right\| > t\right). \]

The proof is completed.

As a simple consequence we obtain

1.4. Theorem (Jain and Marcus [2]). If \((\varphi_i)\) is dominated by \((\xi_i), (x_i) \subseteq E\), then the convergence of \(\sum \xi_i x_i\) in \(L^p\) for some \(p \in [0, \infty)\) implies the convergence of \(\sum \varphi_i x_i\) in \(L^p\).

1.5. Remark. If \((\varphi_i)\) and \((\xi_i)\) are sequences of i.i.d. r.v. and the assertion of Theorem 1.4 holds for \(p = 0\) and every Banach space \(E\), then \((\varphi_i)\) is dominated by \((\xi_i)\).

2. The main result.

2.1. Theorem. Assume that \((\varphi_i)\) and \((\xi_i)\) satisfy the following assumptions:
(i) \((\varphi_i)\) is dominated by \((\xi_i)\),
(ii) for every \(\alpha > 0\) there exist constants, \(K\) and \(L\) such that (i) holds and \(KL < \alpha\).

Then for every Banach space \(E\) and \((x_i) \subseteq E\) the a.s. boundedness of \(\sum \xi_i x_i\) implies the a.s. convergence of \(\sum \varphi_i x_i\).

Proof. Suppose that \(\sum \varphi_i x_i\) does not converge a.s.; then it does not converge in probability. So we can find \(\alpha > 0\) and \(n_1 < m_1 < n_2 < m_2 < \cdots\) such that \(P\left(\left\| \sum_{n_k < i < m_k} \varphi_i x_i \right\| > \alpha\right) > \alpha\). Put
\[ U_k^x = \left\| \sum_{n_k < i < m_k} \varphi_i x_i \right\|, \quad U_k^\xi = \left\| \sum_{n_k < i < m_k} \xi_i x_i \right\|, \]
\[ S_n = \sum_{i \leq n} \xi_i x_i, \quad M = \sup_n \|S_n\|. \]
Note that $\sup_k U_k^\varepsilon \leq 2M$. Since $M < \infty$ a.s., there is $\lambda$ such that 

$$P(2M < \lambda) > 0.$$ 

Hence 

$$0 < P(2M < \lambda) \leq P(\sup_k U_k^\varepsilon \leq \lambda) = \prod_{k=1}^{\infty} (1 - P(U_k^\varepsilon > \lambda)).$$ 

Therefore $\sum_k P(U_k^\varepsilon > \lambda) < \infty$. By assumptions, (i) holds with $K$ and $L$ such that $\alpha/KL > \lambda$. It is easy to see that $K$ can be chosen to be natural. Then 1.3 yields 

$$\alpha < P(U_k^\varepsilon > \alpha) < 2KP(KL U_k^\varepsilon > \alpha) < 2KP(U_k^\varepsilon > \lambda).$$ 

But $P(U_k^\varepsilon > \lambda) \to 0$ as $k \to \infty$, a contradiction. This completes the proof.

**2.2. Remark.** One can prove the following converse: 

If $\{q_i\}$ and $\{\xi_i\}$ are sequences of i.i.d. r.v. and the assertion of Theorem 2.1 holds, then for every $L > 0$ there exists a constant $K$ such that for every $t$ and $i$

$$P(|q_i| > t) \leq KP(L|\xi_i| > t).$$

**2.3. Corollary.** Let $\xi, \eta_1, \eta_2, \ldots$ be i.i.d. symmetric r.v. such that $P(|\eta| > t) \sim t^{-p}$ for $t \to \infty$, $p \in (0, 1)$, e.g. $p$-stable r.v. Let $(x_i) \in E$. Then the a.s. boundedness of the series $\sum \eta_i x_i$ implies its a.s. convergence.

**Proof.** Fix $t_0$ such that for $t > t_0$ and for some $C$

$$C^{-1} t^{-p} \leq P(|\eta| > t) \leq Ct^{-p}.$$ 

If $0 < L \leq 1$, then for $t > t_0$ we have 

$$C^2 L^{-p} P(L|\eta| > t) \geq C t^{-p} \geq P(|\eta| > t).$$ 

So it suffices to take $K$ such that $K \geq C^2 L^{-p}$ and $KP(L|\eta| > t_0) \geq 1$, e.g.

$$K = \lfloor \max(C^2, C^{-1} i_0) L^{-p} \rfloor + 1.$$ 

Then $KL \sim L^{1-p}$, whence $KL$ can be made arbitrarily small, which completes the proof.

The following theorem answers Garling's problem in the negative for $p \in (1, 2)$.

**2.4. Theorem.** Let $\xi, \xi_1, \xi_2, \ldots$ be i.i.d. symmetric r.v. and let $q, q_1, q_2, \ldots$ be i.i.d. symmetric with $P(q = 0) < 1$. If $E|\xi| < \infty$, then there are a Banach space $E$ and a sequence $(x_i) \in E$ such that $\sum \xi_i x_i$ is a.s. bounded but $\sum q_i x_i$ is not a.s. convergent.
Proof. Assume $E|\xi| = 1$ and put

$$q_n = P\left(\frac{1}{n} \sum_{i=1}^{n} |\xi_i| > 2\right).$$

By the weak law of large numbers we have $q_n \to 0$, so we can choose $n_1 < n_2 < \ldots$ such that

$$\sum_i q_n_i \leq \frac{1}{4}.$$

Put $m_i = n_1 + \ldots + n_i$ and let $E = \left(\ell_{n_1}^1 \times \ell_{n_2}^1 \times \ldots\right)_{c_0}$ be the set of all sequences $(a_i)$ such that

$$\sum_{m_{k-1} < i \leq m_k} |a_i| \to 0 \quad \text{and} \quad \|\langle a_i \rangle\| = \sup_k \sum_{m_{k-1} < i \leq m_k} |a_i|.$$

Note that $E$ is isometric to a subspace of $c_0$. Put $x_k = (1/n_i)e_k$ for $m_{k-1} < k \leq m_k$, where $e_k$ is the $k$-th unit vector. If $(\varepsilon_i)$ is a Bernoulli sequence, then $\sum \varepsilon_i x_i$ does not converge a.s. because

$$\left\| \sum_{m_{k-1} < k \leq m_k} e_k x_k \right\| = 1.$$

Hence, by Theorem 1.4, $\sum q_i x_i$ does not converge a.s. It remains to show that $\sum \xi_i x_i$ is a.s. bounded. Let $S_n$ be the $n$-th partial sum, $M = \sup_n \|S_n\|$. Then we have

$$P(\sup \|S_i\| > 2) \leq P(\sup \|S_i\| > 2) \leq 2P(\|S_{m_k}\| > 2) = 2P\left(\left(\frac{1}{n_1} \sum_{i \leq n_1} |\xi_i| > 2\right) \cup \ldots \cup \left(\frac{1}{n_k} \sum_{m_{k-1} < i \leq m_k} |\xi_i| > 2\right)\right) \leq 2 \sum_i q_n_i \leq \frac{1}{2}.$$

Hence $P(M > 2) \leq \frac{1}{2}$, and then $P(M < \infty) = 1$. This completes the proof.

The following theorem gives a negative answer to Garling's question for $p = 1$.

2.5. Theorem. Let $\xi, \xi_1, \xi_2, \ldots$ be i.i.d. symmetric r.v. such that

$$\frac{E|\xi| \cdot I_{|\xi| \leq t}}{t P(|\xi| > t)} \to \infty \quad \text{as} \quad t \to \infty. \quad \text{(**)}$$

Then there are a Banach space $E$ and a sequence $(x_i) \subset E$ such that $\sum \xi_i x_i$ is a.s. bounded but does not converge a.s.
Proof. If (**) holds, then there is \((a_n)_{n \in \mathbb{N}}\) such that

\[
\frac{1}{a_n} \sum_{i \leq n} |\xi_i| \to 1 \text{ in probability}
\]

(cf. [1]). Let \(E\) be as in the proof of Theorem 2.4. Further reasoning is quite similar: put

\[
a_n = P\left(\frac{1}{a_n} \sum_{i \leq n} |\xi_i| > 2\right),
\]

choose \(n_1 < n_2 < \ldots\) such that \(\sum_{i} a_{n_i} \leq \frac{1}{4}\), and put \(x_k = (l/a_{n_k})e_k\) for \(m_{i-1} < k \leq m_i\). It is clear that \(\sum \xi_i x_i\) is a.s. bounded, but does not converge a.s. since

\[
P\left(\left\| \sum_{m_{i-1} < k \leq m_i} \xi_k x_k \right\| > \frac{1}{2}\right) \to 1 \text{ as } i \to \infty.
\]

This completes the proof.

2.6. Remark. The a.s. boundedness of \(\sum \xi_i x_i\), where \(\xi_i\) are 1-stable r.v., implies the convergence of \(\sum e_i x_i\), which is in contrast with the case of \(p > 1\).

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Added in proof. Let \((X_i)\) be a sequence of independent \(E\)-valued r.v. and \((\theta_i)\) i.i.d. real r.v. Assume that for every \(i\) and \(\varepsilon > 0\) there are \(y_1, \ldots, y_k \in E\) such that

\[
d\left(\mathcal{L}(X_i), \mathcal{L}\left(\sum_{j \leq k} \theta_j y_j\right)\right) < \varepsilon,
\]

where \(d\) is the Prokhorov distance. If the a.s. boundedness of \(\sum x_i \theta_i\) implies its a.s. convergence, the same holds for \(\sum X_i\). Typical examples are \(p\)-stable or semistable symmetric r.v. if \(p < 1\).

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Department of Mathematics
University of Warsaw
00-901 Warszawa, PKiN, Poland

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