THE SECOND ORDER OPTIMALITY OF TESTS AND ESTIMATORS FOR MINIMUM CONTRAST FUNCTIONALS. II

BY

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This paper is a continuation of part I (see [7]). It presumes that the reader is familiar with the concepts and notation introduced there. Part II contains lemmas and proofs of the results given in part I.

9. Some auxiliary results. First we derive some asymptotic expansions which are needed in the proofs.

Let \( P_\ast \in \mathfrak{P} \) and \( \Delta > 0 \). Let \( P_n \in \mathfrak{P}, n \in N \), be a sequence fulfilling

\[
\begin{align*}
\kappa_0 (P_n) &= \kappa_0 (P_\ast) - n^{-1/2} \Delta \\
p_n &:= 1 - n^{-1/2} \Delta \sigma_{00}^{-1} (P_\ast) f_0 (\cdot, P_\ast) + n^{-1} \bar{r}_n
\end{align*}
\]

and admitting a \( P_\ast \)-density

\[
(9.3) \quad P_\ast (\bar{r}_n^2) = o (n).
\]

Assume that the following regularity conditions are fulfilled:

\[
\begin{align*}
(9.4) \quad M_4 (P_\ast f^x (\cdot, \kappa (P_\ast))) \quad &\text{for } |x| = 1, 2, \\
M_2 (P_\ast f^x (\cdot, \kappa (P_\ast))) \quad &\text{for } |x| = 3;
\end{align*}
\]

\[
\begin{align*}
(9.5) \quad L_4 (\kappa (P_\ast), P_\ast) \quad &\text{for } f^x: X \times T \to R \text{ if } |x| = 2, \\
L_2 (\kappa (P_\ast), P_\ast) \quad &\text{for } f^x: X \times T \to R \text{ if } |x| = 3.
\end{align*}
\]

If a fixed \( p \)-measure \( P_\ast \) is given, we omit the argument \( P_\ast \) in expressions depending on \( P_\ast \), if this is convenient.

We first derive an asymptotic expansion for \( \kappa (P_n), n \in N \). By a Taylor expansion of \( t \to f^{(0)} (x, t) \) about \( t = \kappa (P_\ast) \), we infer from (9.2)-(9.5) that

\[
(9.6) \quad P_n \left( f^{(0)} (\cdot, \kappa (P_\ast) + n^{-1/2} \Delta a) \right) = o (n^{-1/2})
\]
for
\begin{equation}
(9.7)
  a_l := -\sigma_{00}^{-1} A_{ii} A_{0j} F_{i,j}, \quad l = 0, \ldots, p.
\end{equation}

Let \( g_n, n \in \mathbb{N} \), be defined by
\[ g_n(t) := P_n(f^*(\cdot, t)). \]

By condition (9.5), \( g_n \) is differentiable in some neighborhood \( V(x(P_*)) \) of \( x(P_*) \), and the order of differentiation and integration may be interchanged. As \( P_n \to P_* \), \( n \in \mathbb{N} \), in the strong topology, (8.4) implies the existence of a constant \( \lambda_0 > 0 \) and of an \( \varepsilon \)-neighborhood \( V_\varepsilon(x(P_*)) \) of \( x(P_*) \) such that for all sufficiently large \( n \in \mathbb{N} \)
\begin{equation}
(9.8)
  \|g_n(t) - g_n(t')\| \geq \lambda_0 \|t - t'\| \quad \text{for } t, t' \in V_\varepsilon(x(P_*)).
\end{equation}

By (8.5), \( x(P_n) \in V_{\varepsilon/2}(x(P_*)) \) for all sufficiently large \( n \in \mathbb{N} \). Since \( V_{\varepsilon/2}(x(P_n)) \subset V_\varepsilon(x(P_*)) \) and \( g_n(x(P_n)) = 0 \), (9.8) implies the existence of a \( \delta \)-neighborhood \( V_\delta(0) \) such that \( g_n^{-1} \) exists on \( V_\delta(0) \) for all sufficiently large \( n \in \mathbb{N} \), and
\begin{equation}
(9.9)
  \|g_n^{-1}(v) - g_n^{-1}(v')\| \leq \frac{1}{\lambda_0} \|v - v'\| \quad \text{for } v, v' \in V_\delta(0).
\end{equation}

As \( g_n(x(P_*)) + n^{-1/2} \Delta a \) is in \( V_\delta(0) \) for sufficiently large \( n \in \mathbb{N} \) by (9.6), it follows from (9.6) and (9.9) that
\begin{equation}
(9.10)
  x(P_n) = x(P_*) + n^{-1/2} \Delta a + n^{-1/2} R_n,
\end{equation}
where, by (9.1),
\begin{equation}
(9.11)
  R_{n,i} = O(n^0) \quad \text{for } l = 1, \ldots, p, \quad R_{n,0} = 0.
\end{equation}
(Notice that \( a_0 = -1 \).)

By a Taylor expansion and (9.10),
\[ F_{ij}(P_n) = F_{ij} + n^{-1/2} \sigma_{00}^{-1} (A_{0k} F_{ijk} - A_{kl} A_{0p} F_{i,p} F_{j,k}) + o(n^{-1/2}), \]
and therefore
\begin{equation}
(9.12)
  A_{0i}(P_n) = A_{0i} + n^{-1/2} \Delta e_i + o(n^{-1/2}),
\end{equation}
where
\begin{equation}
(9.13)
  e_i := \sigma_{00}^{-1} A_{ii} A_{0r} A_{0s} (A_{pq} F_{q,r} F_{s,p} - F_{s,r}), \quad i = 0, \ldots, p.
\end{equation}

Furthermore,
\begin{equation}
(9.14)
  F_{ij}(P_n) = F_{ij} + n^{-1/2} \Delta \sigma_{00}^{-1} A_{0k} (A_{ip} F_{k,p} (F_{i,j} + F_{j,i}) + F_{i,j,k}) + o(n^{-1/2}).
\end{equation}

By (9.12) and (9.14),
\begin{equation}
(9.15)
  \sigma_{00}(P_n) = \sigma_{00} + n^{-1/2} \Delta c + o(n^{-1/2}),
\end{equation}

where, by (9.1),
\begin{equation}
(9.16)
  c = O(n^0) \quad \text{for } l = 1, \ldots, p, \quad c_0 = 0.
\end{equation}
(Notice that \( a_0 = -1 \).)
where
\begin{equation}
(9.16) \quad c := A_{0l} A_{0j} A_{0v} \left( A_{kq} F_{v,q}(4F_{ik,j}-F_{i,k} A_{ls} F_{skj})-2F_{i,j,0} \right).
\end{equation}

From (9.15), by a Taylor expansion of \( x \to x^{1/2} \) about \( x = \sigma_{00} \), we get
\begin{equation}
(9.17) \quad \sigma_{0}(P_{n}) = \sigma_{0} + \frac{1}{2} n^{-1/2} \Delta \sigma_{0}^{-1} c + o(n^{-1/2}).
\end{equation}

If in (9.2) we take \( \tilde{r}_{n} = \Delta^{2} h + n^{-1/2} r_{n} \) with
\begin{equation}
(9.18) \quad M_{2} (P_{*} \ast h)
\end{equation}
and
\begin{equation}
(9.19) \quad P_{*}(r_{n}^{2}) = o(n),
\end{equation}
similarly as in (9.6)-(9.11) we obtain
\begin{equation}
(9.20) \quad \kappa(P_{n}) = \kappa(P_{*}) + n^{-1/2} \Delta a + n^{-1} \Delta^{2} b + o(n^{-1}),
\end{equation}
where \( a_{l} \) \((l = 0, \ldots, p)\) are given by (9.7), and
\begin{equation}
(9.21) \quad b_{l} = -A_{il} \left( (\sigma_{00} A_{0k} F_{k,ij} a_{j} + \frac{1}{2} a_{j} a_{k} F_{ijk}) + P_{*}(hf^{(0)}) \right), \quad l = 0, \ldots, p.
\end{equation}

Moreover, \( b_{0} = 0 \) by (9.1).

The essential point of the following lemma is that the power function of the sequence of c.r. \( \{ F_{n}(, x, \kappa_{0}(P_{*})-n^{-1/2} \Delta) > 0 \} \) does not depend on the polynomial \( M \) occurring in the stochastic expansion of \( F_{n}(, t_{0}). \)

\begin{equation}
(9.22) \quad \text{LEMMA. Let } P_{n} \epsilon \Psi, n \epsilon \mathbb{N}, \text{ be a sequence fulfilling (9.1)-(9.3). Let } F_{n},
\end{equation}

\begin{equation}
\text{and } M_{3}(P_{n}, \kappa_{0}(P_{*})) \text{ be a sequence of test functions for } \kappa_{0} \text{ of type } S \text{ which is asymptotically similar of level } \alpha + o(n^{-1/2}) \text{ for } U_{n,\delta}(P_{*}) \text{ for every } \delta \epsilon (0, 1).
\end{equation}

Then
\begin{equation}
P_{*}\{ F_{n}(, \kappa_{0}(P_{*})-n^{-1/2} \Delta) > 0 \} = \pi_{n}(\Delta, \alpha) + o(n^{-1/2}),
\end{equation}
where \( \pi_{n}(\Delta, \alpha) \) is given by (5.7).

This holds true under conditions (9.4) and (9.5).

\begin{proof}
We first note that, by Lemma (9.35), \( P_{n} \epsilon U_{n,\delta}(P_{*}) \) for all sufficiently large \( n \epsilon \mathbb{N} \) if
\begin{equation}
\delta > 2 \left( 1 - \Phi \left( \frac{1}{2} \sigma_{0}^{-1} \Delta \right) \right).
\end{equation}

Furthermore, we may assume without loss of generality that \( U_{n,\delta}(P_{*}) \subset U_{*} \) for all \( n \epsilon \mathbb{N} \).

By a Taylor expansion of \( t \to f^{*}(, t) \) about \( t = \kappa(P_{*}) \) for \( |\alpha| = 1 \), we infer from (9.3) and (9.4) that \( M_{3}(\{ P_{*} \ast f_{0}(, P_{n}): n \epsilon \mathbb{N} \}) \) is fulfilled.

Let
\begin{equation}
f_{0,n} := f_{0}(, P_{n}) - P_{*}(f_{0}(, P_{n})), \quad g_{n} := g(, P_{n}) - P_{*}(g(, P_{n})).
\end{equation}

Since \( f_{0}(, P_{*}) \) and \( g_{1}(, P_{*}) \) are \( P_{*} \)-uncorrelated, by (4.11), (9.3) and (4.14) we have
\begin{align}
\tag{9.23} P_\ast(g_i(\cdot, P_n)) &= P_\ast(g_i(\cdot, P_n)) - n^{-1/2} \Delta \sigma_{00}^{-1} P_\ast(f_0(\cdot, P_n) g_i(\cdot, P_n)) + o(n^{-1/2}) \\
&= P_\ast(g_i(\cdot, P_n)) + o(n^{-1/2}).
\end{align}

Hence \( P_\ast(g_i(\cdot, P_n)) = 0 \) implies
\begin{align}
\tag{9.24} P_\ast(g_i(\cdot, P_n)) = o(n^{-1/2}).
\end{align}

Therefore, for \( \tilde{g}_{i,n}(x) := n^{-1/2} \sum_{i=1}^{n} g_{i,n}(x_i) \) we have
\begin{align}
\tag{9.25} \tilde{g}_{i,n} = \tilde{g}_i(\cdot, P_n) + o(n^0).
\end{align}

Moreover, by a Taylor expansion of \( t \to f^{(i)}(\cdot, t) \) about \( x_0(P_n) \), (9.10) and (9.12),
\begin{align}
\tag{9.26} P_\ast(f_0(\cdot, P_n)) = n^{-1/2} \Delta - n^{-1} \Delta^2 (\frac{1}{2} a_j a_k A_{0i} F_{ikj} + e_i a_j F_{ij}) + o(n^{-1}).
\end{align}

Thus, for \( \tilde{f}_{0,n}(x) := n^{-1/2} \sum_{i=1}^{n} f_{0,n}(x_i) \) we get
\begin{align}
\tag{9.27} \tilde{f}_{0,n} = f_0(\cdot, P_n) - \Delta + n^{-1/2} \Delta^2 (\frac{1}{2} a_j a_k A_{0i} F_{ikj} + e_i a_j F_{ij}) + o(n^{-1/2}).
\end{align}

Using (9.17), (9.25) and (9.27) and the fact that \( F_n(\cdot, x_0(P_n) - n^{-1/2} \Delta) \) is asymptotically similar of level \( \alpha + o(n^{-1/2}) \) for \( P_n \), from (4.8) we obtain
\begin{align}
\tag{9.28} F_n(\cdot, x_0(P_n) - n^{-1/2} \Delta)

= \tilde{f}_{0,n} + N_a \sigma_0 + \Delta - n^{-1/2} (\Delta^2 (\frac{1}{2} a_j a_k A_{0i} F_{ikj} + e_i a_j F_{ij}) + \\
+ \frac{1}{2} \Delta \sigma_0^{-1} A + M (\tilde{f}_{0,n} + \Delta, \tilde{g}_n, P_n) + \\
+ n^{-1/2} o_n(\frac{1}{2}) \text{ with respect to } P_\ast.
\end{align}

Let \( \sigma_n := P_\ast(f_{0,n}^2)^{1/2} \). By a Taylor expansion, from (9.10) and (9.12) we obtain
\begin{align}
\tag{9.29} \sigma_n^2 &= \sigma_{00} + n^{-1/2} \Delta (A_{0i} a_j a_k F_{ikj} + A_{0j} e_i F_{ij}) + o(n^{-1/2}).
\end{align}

Thus, by a Taylor expansion of \( x \to x^{1/2} \) about \( x = \sigma_{00} \),
\begin{align}
\tag{9.30} \sigma_n = \sigma_0 + \frac{1}{2} n^{-1/2} \sigma_0^{-1} \Delta (A_{0i} a_j a_k F_{ikj} + A_{0j} e_i F_{ij}) + o(n^{-1/2}).
\end{align}

In virtue of conditions (4.10)-(4.15), Lemma (9.63), Lemma 5.25 in [8], p. 20, and (9.28) we get
\begin{align}
\tag{9.31} P_\ast \{ F_n(\cdot, x_0(P_n) - n^{-1/2} \Delta) > 0 \}

= \Phi((N_a \sigma_0 + \Delta) \sigma_n^{-1}) - n^{-1/2} \sigma_n^{-1} \varphi(N_a + \Delta \sigma_n^{-1}) (k(-N_a \sigma_0 - \Delta) - \\
- \int dv \varphi_2(v) M(-N_a \sigma_0, v, P_\ast) + \\
+ \Delta (\frac{1}{2} a_j a_k A_{0i} F_{ikj} + e_i a_k F_{ik}) - \frac{1}{2} \sigma_n^{-1} N_a)) + o(n^{-1/2}),
\end{align}
where \( k(t) := \frac{1}{2} \sigma_{00}^{-1} P_* (f_0^2)(1 - \sigma_{00}^{-1} t^2) \) and \( \Sigma_0 \) is the covariance matrix of \( P_* \).

Using a Taylor expansion, from (9.30) we obtain

\[
\Phi((N_z \sigma_0 + \Delta) \sigma_0^{-1}) = \Phi(N_z + \Delta \sigma_0^{-1}) - 
- n^{-1/2} \sigma_0^{-3} \varphi(N_z + \Delta \sigma_0^{-1}) (N_z + \Delta \sigma_0^{-1}) \Delta (A_{0i} A_{0j} a_k F_{ik,j} + F_{i,j} A_{0j} e_i) + o(n^{-1/2}).
\]

For \( \Delta = 0 \) and \( P_n = P_* \), making use of (9.31) and the fact that \( P_* \{ F_n(\cdot, x(P_*)) > 0 \} = x + o(n^{-1/2}) \) we get

\[
\int dv \varphi_{z_0}(v) M(-N_z \sigma_0, v, P_*) = k(-N_z \sigma_0).
\]

The assertion of the lemma now follows easily from (9.31)-(9.33).

Remark. A result corresponding to (9.22) can be obtained for \( \Delta < 0 \).

Lemma. Let \( P_n \in \Psi, \ n \in N, \) be a sequence admitting a \( P_* \)-density

\[
p_n = 1 + n^{-1/2} \Delta g + n^{-1} \tilde{r}_n.
\]

Assume that \( \tilde{r}_n = \Delta^2 h + n^{-1/2} r_n \) with

\[
M_{5/2}^g(P_* \ast h),
\]

\[
M_{5/2}^g (\{P_* \ast r_n : n \in N\}).
\]

If \( \varphi_n, n \in N, \) is asymptotically of level \( x + o(n^{-1/2}) \) for \( P_n \), then

\[
P_n^\alpha (\varphi_n) \leq \Phi(N_z + \Delta \sigma) + 
+ n^{-1/2} \varphi(N_z + \Delta \sigma) \sigma^{-1} \Delta \left( \Delta (P_* (gh) - \frac{1}{\delta} P_* (g^3)) + \frac{1}{\delta} P_* (g^3) N_z \sigma^{-1} \right) + 
+ o(n^{-1/2}),
\]

where \( \sigma := P_* (g^2)^{1/2} \).

This holds true under the following regularity conditions:

\[
M_{5/2}^g (P_* \ast g),
\]

\[
C(P_* \ast g).
\]

If (9.38), \( P_*(r_n^2) = o(n) \), \( g = - \sigma_{00}^{-1} f_0 \), and (9.1) are fulfilled, we obtain

\[
P_n^\alpha(\varphi_n) \leq \pi_n(\Delta, x) + o(n^{-1/2}),
\]

where \( \pi_n(\Delta, x) \) is given by (5.7).

Proof. For \( r \in R \) let

\[
D_n(r) := \{ x \in X^n : \prod_{i=1}^n P_n(x_i) \leq r \}, \quad r_{n,x} := \inf \{ r \in R : P_n(D_n(r)) \geq x \}.
\]

We have \( P_n^\alpha(D_n(r_{n,x})) \geq x \).
Let now \( \varphi_n, n \in N \), be asymptotically of level \( \alpha + o(n^{-1/2}) \) for \( P_n \). Let \( \alpha_n := \max \{ \alpha, P_n(\varphi_n) \} \). We have

\[
\alpha \leq \alpha_n \leq \alpha + o(n^{-1/2}).
\]

Since \( \alpha \alpha_n^{-1} \varphi_n \) is of level \( \alpha \), by the Neyman-Pearson lemma we obtain

\[
P_n^*(\alpha \alpha_n^{-1} \varphi_n) \leq P_n^*(D_n(r_{n,z})).
\]

Therefore

\[
P_n^*(\varphi_n) \leq P_n^*(D_n(r_{n,z}))+o(n^{-1/2}).
\]

Let

\[
A_n := \{ \varrho \mid |g| \leq \frac{1}{2} n^{1/2} \text{ and } |r_n| \leq \frac{1}{2} n \},
\]

\[
B_n := A_n^n.
\]

By the definition of \( P_n \), Markov's inequality and Hölder's inequality, we obtain for \( Q_n = P_n \) and \( Q_n = P_n^* \)

\[
Q_n(B_n^c) \leq n(Q_n(\varrho |g| > \frac{1}{2} n^{1/2}) + Q_n(|r_n| > \frac{1}{2} n)) = o(n^{-1/2}).
\]

Hence for \( Q_n = P_n \) and \( Q_n = P_n^* \) we have

\[
Q_n^*(D(r_{n,z})) = Q_n^* \{ x \in B_n : \sum_{i=1}^n \log p_n(x_i) \leq r'_{n,z} \} + o(n^{-1/2})
\]

for some suitably chosen \( r'_{n,z} \in R \).

For notational convenience let

\[
k_n := \Delta g + n^{-1/2} r_n.
\]

From a Taylor expansion of \( \log \) we obtain

\[
\log p_n = n^{-1/2} k_n - \frac{1}{2} n^{-1} k_n^2 + \frac{1}{8} n^{-3/2} k_n^3 + n^{-3/2} k_n^3 v(n^{-1/2} k_n),
\]

where

\[
v(y) := \int_0^1 (1-u)^2 ((1-uy)^{-2}-1) du.
\]

For \( |y| \leq \frac{1}{2} \) we have

\[
|v(y)| \leq 2 |y|.
\]

From (9.46) and (9.47) for \( x \in B_n \) we obtain

\[
\sum_{i=1}^n \log p_n(x_i) = n^{-1/2} \Delta \sum_{i=1}^n g(x_i) + n^{-1} \Delta^2 \sum_{i=1}^n (h(x_i) - \frac{1}{2} g^2(x_i)) + n^{-1/2} \Delta^3 \left( \frac{1}{3} P_n(g^3) - P_n(gh) \right) + n^{-3/2} \sum_{i=1}^n R_n(x_i),
\]
where
\[ R_n = r_n + \frac{1}{3} k_n^2 - \frac{1}{2} n^{-3/2} r_n^2 + k_n^2 v(n^{-1/2} k_n) - n^{-1/2} \Delta g r_n - n^{-1} \Delta^2 h r_n - \Delta^3 (g h - P_\star (g h) + P_\star (g^3)) - \frac{1}{2} \Delta^4 n^{-1/2} h^2. \]

From Lemmas (9.57) and (9.58) we obtain
\[ n^{-3/2} \sum_{v=1}^{n} R_n(x_v) 1_{B_n}(x) = n^{-1/2} o_n(\frac{1}{2}) \]
with respect to \( P_\star \) and with respect to \( P_n \), since by (9.48) we have
\[ n^{-3/2} \left| \sum_{v=1}^{n} k_n^3(x_v) v(n^{-1/2} k_n(x_v)) 1_{B_n}(x) \right| \leq 2n^{-2} \sum_{v=1}^{n} k_n^4(x_v). \]

As \( n^{-3/2} P_\star (g r_n) = O(n^{-3/2}) \), we infer from (9.37)-(9.40) and Lemma (9.65) that
\[ n^{-3/2} P_n(\tilde{g}) - \Delta P_\star (g^2) - n^{-1/2} \Delta^2 P_\star (g h) + \frac{1}{2} (g^2)^{-1} + \frac{1}{2} \Delta P_\star (g^3)) < s \]
\[ = \Phi(s \sigma_n^{-1}) + n^{-1/2} \varphi(s \sigma_n^{-1}) H(s) + o(n^{-1/2}) \]
uniformly for \( s \in \mathbb{R} \), and
\[ n^{-3/2} P_\star (g + n^{-1/2} \Delta (\tilde{h} - \frac{1}{2} (g^2)^{-1}) < s \]
\[ = \Phi(s \sigma_n^{-1}) + n^{-1/2} \varphi(s \sigma_n^{-1}) H(s) + o(n^{-1/2}) \]
uniformly for \( s \in \mathbb{R} \), where \( \sigma_n^2 := P_\star (g^2) + n^{-1/2} \Delta P_\star (g^3) \), and
\[ H(s) := \sigma_n^{-3} \left( \frac{1}{6} P_\star (g^3)(1 - s^2 \sigma_n^{-2}) + \frac{1}{2} P_\star (g^3) - P_\star (g h) s \sigma_n^{-1} \right). \]

Therefore, from (9.45) and (9.50) by Lemma (9.63) it follows that for \( Q_n = P_\star \) and \( Q_n = P_n \)
\[ Q_n^n(D_n(r_n,x)) = Q_n^n(C_n,x) + o(n^{-1/2}), \]
where
\[ C_n,x := \{ \tilde{g} + n^{-1/2} \Delta (\tilde{h} - \frac{1}{2} (g^2)^{-1}) < c_{n,x} \}\]
with
\[ c_{n,x} := r_n,x \Delta^{-1} + \frac{1}{3} \Delta P_\star (g^2) + n^{-1/2} \Delta^2 (P_\star (g h) - \frac{1}{2} P_\star (g^3)). \]

As \( P_\star^n(C_n,x) = \alpha + o(n^{-1/2}) \), from a uniform version of Lemma 7 in [5], p. 1016, we obtain
\[ c_{n,x} = \Delta (N_x \sigma^{-1} - \frac{1}{6} n^{-1/2} \sigma^{-2} P_\star (g^3)(1 - N_x^2) + \Delta^2 (P_\star (g^2) + n^{-1/2} \sigma^{-1} N_x P_\star (g h) + n^{-1/2} \Delta^3 (2 P_\star (g h) + \frac{1}{2} P_\star (g^3)) \].
The assertion of the lemma now follows from (9.42), (9.52), (9.54) and (9.55). Relation (9.41) follows immediately from (9.21).

(9.56) Remark. In the case $\lambda < 0$ and $g = -\sigma_{oo}^{-1} f_0$, in the same way as in Lemma (9.35) one can derive

$$P_n^\varphi \geq \pi_n(\lambda, \alpha) + o(n^{-1/2}).$$

The following lemma is an immediate consequence of Lemma 6.3 in [4], p. 152.

(9.57) Lemma. Let $\mathcal{Q}_n$, $n \in \mathbb{N}$, be families of $p$-measures. Let $s \in [0, \infty)$ and $a > \frac{1}{2}$. Let $h_n(\cdot, Q): X \to \mathbb{R}, Q \in \mathcal{Q}_n, n \in \mathbb{N}$, be measurable functions fulfilling

$$M_{(a+1)/2}^s \left( \{ P_n h_n(\cdot, Q): n \in \mathbb{N}, P, Q \in \mathcal{Q}_n \} \right).$$

Assume that one of the following conditions is satisfied:

$$a > 1$$

or

$$a \leq 1 \quad \text{and} \quad \sup_{P \in \mathcal{Q}_n} |P(h_n(\cdot, Q))| = o(n^{a-1}).$$

Then there exist $\delta > 0$ and, for every $c > 0$, a constant $B$ depending on

$$\sup_{m \in \mathbb{N}} \sup_{P \in \mathcal{Q}_n} \sup_{Q \in \mathcal{Q}_n} P(h_n(\cdot, Q)^{(a+1)/2}) \quad \text{and} \quad \sup_{P \in \mathcal{Q}_n} |P(h_n(\cdot, Q))|$$

such that

$$\sup_{P \in \mathcal{Q}_n} P^n \left\{ x \in X^n: n^{-a} \left| \sum_{v=1}^{n} h_n(\cdot, Q) \right| > c \right\} \leq Bn^{-(a+\delta)}.$$

(9.58) Lemma. Let the assumptions of Lemma (9.57) be satisfied for $s = \frac{1}{2}$, $\mathcal{Q}_n = \{ P_n \}^\varphi$ and $h_n(\cdot, Q) = h_n$. Let $P_n, n \in \mathbb{N}$, be a sequence of $p$-measures admitting a $P_n$-density (9.36) such that

$$M_3(P_n * g),$$

$$M_{3/2}^s \left( \{ P_n * f_n: n \in \mathbb{N} \} \right).$$

Then

$$n^{-a} \sum_{v=1}^{n} h_n(x_v) = o_n(\frac{1}{n})$$

with respect to $P_n$.

Proof. Let $A_n$ be determined by (9.43). Let a $p$-measure on $\mathcal{A}$ be defined by

$$Q_n(A) := P_n(A \cap A_n)/P_n(A_n), \quad A \in \mathcal{A}.$$
Since $Q_n(h_n) = P_*(h_n) + O(n^{-1/2})$ if $a \leq 1$, and the $P_*$-density of $Q_n$ is bounded by $3/2$, from Lemma (9.57) we obtain

\[(9.62) \quad Q_n \{ n^{-a} \sum_{v=1}^{n} h_v(x_v) > c \} = o(n^{-1/2}). \]

The assertion now follows from (9.44) and (9.62).

(9.63) Lemma. Let $Q_n, n \in N$, be families of $p$-measures over $\mathcal{A}$. Let $h_n(\cdot, Q): X^n \to R$ and $g_n(\cdot, Q): X^n \to R$, $n \in N$, $Q \in Q_n$, be measurable functions fulfilling

\[ h_n(\cdot, Q) = g_n(\cdot, Q) + n^{-1/2} o_n(\frac{1}{n}) \]

with respect to $Q_n$.

Let $H_n(\cdot, Q)$ and $G_n(\cdot, Q)$ be the distribution functions of $Q^n * h_n(\cdot, Q)$ and $Q^n * g_n(\cdot, Q)$, respectively.

If

\[(9.64) \quad |H_n(s, Q) - H_n(s', Q)| \leq c |s - s'| + o(n^{-1/2}) \]

uniformly for $s, s' \in R$ and $Q \in Q_n$, then

\[ G_n(s, Q) = H_n(s, Q) + o(n^{-1/2}) \]

uniformly for $s \in R$ and $Q \in Q_n$.

(9.64) is in particular fulfilled if $H_n(\cdot, Q)$ admits an Edgeworth expansion of order $n^{-1/2}$, uniformly for $Q \in Q_n$.

Proof. Choose $c_n, n \in N$, such that $c_n \downarrow 0$ and

\[ Q^n \{ n^{1/2} |h_n(\cdot, Q) - g_n(\cdot, Q)| > c_n \} = o(n^{-1/2}) \]

uniformly for $Q \in Q_n$.

Then from (9.64) we obtain

\[ G_n(s, Q) \leq Q^n \{ h_n(\cdot, Q) < s + n^{-1/2} c_n \} + Q^n \{ n^{1/2} |h_n(\cdot, Q) - g_n(\cdot, Q)| \geq c_n \} = H_n(s, Q) + o(n^{-1/2}) \]

uniformly for $s \in R$ and $Q \in Q_n$.

In the same way one can show that $G_n(s, Q) \geq H_n(s, Q) + o(n^{-1/2})$.

(9.65) Lemma. Let $P_n, n \in N$, be a sequence of $p$-measures fulfilling (9.36), (9.59), and (9.60). Let $h_1: X \to R$ and $h_2: X \to R$ be measurable functions for which the following regularity conditions are fulfilled:

\[(9.66) \quad P_*(h_1) = P_*(h_2) = 0, \]

\[(9.67) \quad M_3(P_*(h_1)), \quad M_{3/2}(P_*(h_2)), \]

\[(9.68) \quad C(P_*(h_1)). \]
Then
\[ P_n \{ \tilde{h}_1 - n^{1/2} P_n(h_1) + n^{-1/2} (\tilde{h}_2 - \Delta P_*(gh_2)) < s \} \]
\[ = \Phi(s\sigma_n^{-1}) + n^{-1/2} \varphi(s\sigma_n^{-1}) H(s) + o(n^{-1/2}) \]
uniformly for \( s \in \mathbb{R} \), where
\[ \sigma_n^2 := P_*(h_1^2) + n^{-1/2} \Delta P_*(h_1^2 g), \]
\[ H(s) := \sigma^{-3} \left( \frac{1}{6} P_*(h_1^3) (1 - s^2 \sigma^{-2}) - P_* (h_1 h_2) s \right) \]
with \( \sigma^2 := P_*(h_1^2) \).

Proof. Let \( A_n \) and \( Q_n \) be defined by (9.43) and (9.61), respectively. By (9.36), (9.59), (9.60), (9.66) and (9.67) we have
\[ Q_n(h_1) - P_n(h_1) = o(n^{-1}), \quad Q_n(h_2) - n^{-1/2} \Delta P_*(gh_2) = o(n^{-1/2}). \]
Thus, from (9.67), (9.68) and from Theorem 1 in [2], p. 650, applied for \( \tilde{h}_1 - n^{-1/2} Q_n(h_1) + n^{-1/2} (\tilde{h}_2 - n^{-1/2} Q_n(h_2)) \), we obtain
\[ Q_n^* \{ \tilde{h}_1 - n^{1/2} P_n(h_1) + n^{-1/2} (\tilde{h}_2 - \Delta P_*(gh_2)) < s \} \]
\[ = \Phi(s\sigma_n'^{1/2}) + n^{-1/2} \varphi(s\sigma_n'^{1/2}) H_n(s) + o(n^{-1/2}) \]
uniformly for \( s \in \mathbb{R} \), where \( \sigma_n'^2 \) is the variance of \( Q_n \# h_1 \), and
\[ H_n(s) := \sigma_n'^{-3} \left( \frac{1}{6} Q_n(h_1^3) (1 - s^2 \sigma_n'^{-2}) - Q_n(h_1 h_2) s \right). \]
Since \( \sigma_n' - \sigma_n = o(n^{-1/2}) \) and \( Q_n(h_1^2 h_2^2) \rightarrow P_*(h_1^2 h_2^2) \), \( n \in \mathbb{N} \), for all \((\alpha_1, \alpha_2)\) such that \( \alpha_1 + 2\alpha_2 \leq 3 \), the assertion of the lemma follows from (9.69) and (9.44).

(9.70) Lemma. Assume that for some strong neighborhood \( U_0 \) of \( P_0 \) in \( \mathfrak{P} \) the following regularity conditions are fulfilled:
(9.71) \( K_{3/2} (\chi(P_0), U_0) \) for \( f : X \times T \rightarrow \mathbb{R} \),
(9.72) \( M_3^* \left( \{ P \# f^\alpha ( , \chi(Q)) : P, Q \in U_0 \} \right) \) for \( |\alpha| = 1, 2, 3 \),
(9.73) \( L_{3/2}^* (\chi(P_0), U_0) \) for \( f^\alpha : X \times T \rightarrow \mathbb{R} \) if \( |\alpha| = 3 \).

Then, for \( i = 0, \ldots, p \),
\[ n^{1/2} (x_i^{(n)} - \chi_i(P)) = f_0 ( , P) + n^{-1/2} M_i ( \tilde{f}, \tilde{f}^*, P) + n^{-1/2} o_n (\tilde{f}) \]
with respect to \( U_{n,\delta}(P_0) \) for every \( \delta \in (0, 1) \), where
\[ M_i ( \tilde{f}, \tilde{f}^*, P) = - \frac{1}{2} A_{ij} F_{jkl} \tilde{f}_i ( , P) \tilde{f}_j ( , P) + \tilde{f}_i ( , P) \tilde{f}^{(1)} ( , P). \]

Proof. The proof follows the pattern of the proof of Theorem 5 in [1], p. 298ff. The crucial point is to show that
\[ \| x^{(n)} - \chi(P) \| = o_n (\tilde{f}) \]
with respect to \( U_{n,\delta}(P_0) \) for every \( \delta \in (0, 1) \).
If we copy the proof in [3], p. 79, for the case \( K = \{ \chi(P_\star) \} \), we obtain immediately
\[
(9.76) \quad \| \chi^{(n)} - \chi(P_\star) \| = o_n(\frac{1}{n})
\]
with respect to \( U_{n, \delta}(P_\star) \) for every \( \delta \in (0, 1) \).

Since \( P \to \chi(P) \) is continuous by General Assumption (8.5), relation (9.76) implies (9.75).

10. Proofs. In order not to overload the paper with technicalities, the proofs are given for fixed \( \Delta \). Uniformity in \( \Delta \) can be obtained by exactly the same reasoning if uniform versions of the lemmas are used.

Proof of Theorem (4.16). (i) By General Assumption (8.5), \( P \to \chi(P) \) is continuous. Hence condition (4.21) implies the existence of \( g \) with \( M_{3/2} \left( \{ P \star g : P \in U_\star \} \right) \) such that, for some strong neighborhood \( U'_\star \subset U_\star \) of \( P_\star \),
\[
(10.1) \quad |f^{(ij)}(\cdot, \chi(P)) - f^{(ij)}(\cdot, \chi(P_\star))| \leq \| \chi_k(P) - \chi_k(P_\star) \| \| f^{(ik)}(\cdot, \chi(P)) + \| \chi(P) - \chi(P_\star) \|^2 g, \]
\[
(10.2) \quad |f^{(ik)}(\cdot, \chi(P)) - f^{(ik)}(\cdot, \chi(P_\star))| \leq \| \chi(P) - \chi(P_\star) \| g.
\]

Hence it follows easily that \( P \to F_{ijk}(P), P \to F_{ij}(P) \) and \( P \to A_{ij}(P) \) are continuous at \( P_\star \) in the strong topology.

Thus the coefficients of the polynomials \( M_i(\cdot, P) \) defined in (9.74) are continuous at \( P_\star \).

(ii) By condition (4.23), for every \( P \in U'_\star \) there exists a \( P \)-linearly independent subsystem \( \{ f_0(\cdot, P), g_1(\cdot, P), \ldots, g_m(\cdot, P) \} \) of \( \{ f_i(\cdot, P), i = 0, \ldots, p \} \) generating the same space and fulfilling
\[
(10.3) \quad C \left( \{ P \star f_0(\cdot, P), g(\cdot, P) : P \in U'_\star \} \right).
\]

Without loss of generality we may assume that \( f_0(\cdot, P_\star) \) and \( g_i(\cdot, P_\star) \) are \( P_\star \)-uncorrelated. Otherwise, we replace \( g_i(\cdot, P) \) by
\[
g_i(\cdot, P) := g_i(\cdot, P) - P_\star(\cdot) f_0(\cdot, P_\star) g_i(\cdot, P_\star) s_0^{-1} f_0(\cdot, P).
\]

Notice that (10.3) and the following statements remain valid for \( g_i(\cdot, P) \). Moreover, there exists a polynomial \( M(\cdot, P) \) the coefficients of which are continuous at \( P_\star \) such that
\[
(10.4) \quad M(\tilde{f}_0(\cdot, P), g(\cdot, P), P) = M_0(\tilde{f}_0^*, \tilde{f}_0^*, P) + \tilde{k}(\cdot, P) \quad P^a\text{-a.e.}
\]

From Lemma (9.70), (4.3), (4.6), and (10.4) we get
\[
F_n(\cdot, \chi_0(P)) = \tilde{f}_0(\cdot, P) + N_\sigma \sigma_0(P) + 
\]
\[
+ n^{-1/2} M(\tilde{f}_0(\cdot, P), \tilde{g}(\cdot, P), P) - n^{-1/2} c_0(P) + n^{-1/2} o_n(\frac{1}{n})
\]
with respect to \( U_{n, \delta}(P_\star) \).
By Lemma 3 in [6], p. 245, we see from the choice of $c_\varepsilon$ (cf. (4.5)) that $F_n$, $n \in \mathbb{N}$, is asymptotically similar of level $\alpha + \alpha(n^{-1/2})$. 

Proof of Proposition (4.25) (i).

(a) Let $V(\kappa(P_\#))$ be given by condition (4.30). Then we infer from (9.76) that for every $\delta \in (0, 1)$

$$P^n\{\varepsilon^{(n)}(x) \notin V(\kappa(P_\#))\} = o(n^{-1/2})$$

uniformly for $P \in U_{n, \delta}(P_\#)$.

Furthermore, it follows from General Assumption (8.5) that there exists a strong neighborhood $U'_* \subset U_*$ such that $\kappa(P) \in V(\kappa(P_\#))$ for $P \in U'_*$. Thus for $\varepsilon^{(n)}(x) \in V(\kappa(P_\#))$ and $P \in U'_*$, by a Taylor expansion of $t \to n^{-1} \sum_{v=1}^n f^{(ij)}(x, t)$ about $\kappa(P)$, we obtain

$$F_{ij}^{(n)}(P) - F_{ij}(P) = n^{-1/2} f^{(ij)}(\cdot, \kappa(P)) + F_{ijk}(P) \varepsilon^{(n)}(x) - \kappa(P) + R_n(\cdot, P),$$

where

$$|R_n(x, P)| \leq \|\varepsilon^{(n)}(x) - \kappa(P)\| \left\|\left(\sum_{v=1}^n f^{(ijk)}(x, \kappa(P)) - F_{ijk}(P)\right)_{k=0, \ldots, p}\right\| + \|\varepsilon^{(n)}(x) - \kappa(P)\|^2 \frac{1}{2} n^{-1} \sum_{v=1}^n g(\varepsilon_v),$$

$g$ being the function which occurs in $L_2(\kappa(P_\#), U_\#)$ for $f^\alpha$ if $|\alpha| = 3$.

By Lemma (9.57), (9.75) and General Assumption (8.5) we have

$$\|\varepsilon^{(n)} - \kappa(P)\| = n^{-1/4 - \varepsilon} o_n(\frac{1}{n})$$

with respect to $U_{n, \delta}(P_\#)$ for every $\delta \in (0, 1)$ and some sufficiently small $\varepsilon > 0$, and

$$\left\|\left(\sum_{v=1}^n f^{(ijk)}(\cdot, \kappa(P))\right)_{k=0, \ldots, p}\right\| = n^{-1/4 - \varepsilon} o_n(\frac{1}{n})$$

with respect to $U_{n, \delta}(P_\#)$ for every $\delta \in (0, 1)$.

Thus, by Lemma (9.57), (10.5), (10.6), (10.8), and (10.9), we get

$$R_n(\cdot, P) = n^{-1/2} o_n(\frac{1}{n})$$

with respect to $U_{n, \delta}(P_\#)$ for every $\delta \in (0, 1)$.

Let

$$\varphi_{ij}(\cdot, P) := f^{(ij)}(\cdot, \kappa(P)) - F_{ij}(P) - F_{ijk}(P) f_k(\cdot, P).$$

Using Lemma (9.57), we obtain

$$F_{ij}^{(n)}(P) - F_{ij}(P) = n^{-1/2} \varphi_{ij}(\cdot, P) + n^{-1/2} o_n(\frac{1}{n})$$

with respect to $U_{n, \delta}(P_\#)$ for every $\delta \in (0, 1)$. 

The second order optimality. II

In a similar way as above one can show that

\[(10.12) \quad F^{(n)}_{ij} - F_{ij}(P) = n^{-1/2} \left( (f^{(0)} f^{(0)})^{-1}(\cdot, x(P)) + (F_{i,j,k}(P) + F_{j,i,k}(P)) \cdot k(\cdot, P) \right) + n^{-1/2} a_n(\frac{1}{2}) \]

with respect to \( U_{n,\delta}(P) \) for every \( \delta \in (0,1) \).

(\beta) Let

\[ C_n := \{ x \in X^n : F^{(n)}(x) \text{ is invertible} \} . \]

As \( P \to F_{ij}(P) \) is continuous because of condition \((4.29)\) and the continuity of \( P \to x(P) \), we have

\[ C_n \subset \{ \| F^{(n)}_{ij} - F_{ij}(P) \|_{i,j=0,\ldots,p} \geq d \} \]

for some \( d > 0 \) and for all \( P \) in some neighborhood \( U'' \subset U \).

Thus

\[(10.13) \quad P^n(C_n) = o(n^{-1/2}) \]

uniformly for all \( P \in U_{n,\delta}(P) \) for every \( \delta \in (0,1) \).

Putting

\[ \alpha_{ij}(\cdot, P) := -A_{ii}(P) A_{jk}(\cdot, P) \]

we obtain from \((10.11)\)

\[(10.14) \quad F^{(n)}_{ij}(A_{ii}(P) + n^{-1/2} A_{ij}(\cdot, P)) = \delta_{ii} + n^{-1/2} a_n(\frac{1}{2}) \]

with respect to \( U_{n,\delta}(P) \) for every \( \delta \in (0,1) \) and, therefore, by \((10.13)\),

\[(10.15) \quad A^{(n)}_{ij} = A_{ij}(P) + n^{-1/2} A_{ij}(\cdot, P) + n^{-1/2} a_n(\frac{1}{2}) \]

with respect to \( U_{n,\delta}(P) \) for every \( \delta \in (0,1) \).

From \((10.12), (10.15)\), and a Taylor expansion of \( x \to x^{1/2} \) about \( x = \sigma_{00} \) we obtain

\[(10.16) \quad \sigma^{(n)}_{0} = \sigma_{0}(P) + n^{-1/2} k(\cdot, P) + n^{-1/2} a_n(\frac{1}{2}) \]

with respect to \( U_{n,\delta}(P) \) for every \( \delta \in (0,1) \), where \( k(\cdot, P) \) is given by \((4.27)\).

**Proof of Proposition \((4.25)\) (ii).** The proof is a simple application of Lemma \((9.57)\) and will be omitted.

**Proof of Theorem \((5.1)\).** The theorem follows immediately from Lemmas \((9.22)\) and \((9.35)\) applied for \( P_{n,A} \), \( n \in N \), \( 0 < A \leq A_0 \).

**Proof of Corollary \((5.11)\).** The corollary follows immediately from Theorems \((4.16)\) and \((5.1)\) if we establish that for every \( \Delta (0 < \Delta \leq \Delta_0) \) there exists a sequence \( P_{n,A} \in \Phi \), \( n \in N \), fulfilling \((5.2)-(5.5)\). We restrict ourselves to prove the assertion for fixed \( \Delta > 0 \).
By (5.12), there exists \( \varepsilon \in (0, 1) \) such that \( M_{(\theta+\varepsilon)/2}(P_{\ast} \ast f^2(\cdot, \kappa(P_{\ast}))) \) is fulfilled for \(|x| = 1\). Let
\[
\beta := \frac{3 + \varepsilon/4}{6 + 3\varepsilon/4} \in (0, \frac{1}{2}) \quad \text{and} \quad k'_{n,t} := f^{(i)}1_{|f^{(i)}(0)\leq \beta|}.
\]
Since \( P_{\ast}(f^{(0)}) = 0 \), we obtain
\[
(10.17) \quad P_{\ast}(k'_{n,t}) = o(n^{-3/2}).
\]
Let, furthermore, \( k_{n,t} := k'_{n,t} - P_{\ast}(k'_{n,t}) \) and let \( a \) be defined by (9.7).

From (10.17) and a Taylor expansion of \( t \to f^{(i)}(\cdot, t) \) about \( \kappa(P_{\ast}) \) we obtain
\[
(10.18) \quad P_{\ast}(f^{(0)} - k_{n,t}) f^{(i)}(\cdot, \kappa(P_{\ast}) + n^{-1/2} a)) = O(n^{-1}).
\]
Let \( F_{n} \) be a matrix defined by
\[
F_{n,i,j} := P_{\ast}(k_{n,t} f^{(i)}(\cdot, \kappa(P_{\ast}) + n^{-1/2} a)), \quad i, j = 0, \ldots, p.
\]
By a Taylor expansion and (10.18) we have
\[
F_{n,i,j} = F_{i,j} + n^{-1/2} \Delta a_{k} F_{i,k} + O(n^{-1}).
\]
Thus, \( F_{n} \) is invertible if \( n \) is sufficiently large, and the inverse, say \( B_{n} \), admits the expansion
\[
(10.19) \quad B_{n,ij} = B_{ij} + n^{-1/2} e_{ij} + O(n^{-1}),
\]
where \((B_{ij})_{i,j=0,\ldots,p}\) is the inverse of \((F_{i,j})_{i,j=0,\ldots,p}\), and
\[
(10.20) \quad e_{ij} := -B_{jk} B_{li} F_{l,k} a_{p}.
\]
Let now \( a_{n,j}, n \in \mathbb{N}, j = 0, \ldots, p \), be defined by
\[
a_{n,j} := n^{1/2} B_{n,ij} P_{\ast}(f^{(0)}(\cdot, \kappa(P_{\ast}) + n^{-1/2} \Delta a)).
\]
From (10.19) we obtain
\[
(10.21) \quad a_{n,j} = \Delta a_{0}^{-1} A_{0} + n^{-1/2} A_{2} (e_{ji} F_{ik} a_{k} + B_{ji} F_{ikl} a_{l} a_{j}) + n^{-1} R_{n,j},
\]
where \( R_{n,j} = O(n^{0}) \).

As \( a_{n,j} \) is bounded, the signed measure \( P_{n} \), defined by the \( P_{\ast} \)-density
\[
p_{n} := 1 + n^{-1/2} a_{n,j} k_{n,j},
\]
belongs to \( \mathcal{B} \) if \( n \) is sufficiently large.

Furthermore, by a simple calculation we obtain
\[
P_{n}(f^{(0)}(\cdot, \kappa(P_{\ast}) + n^{-1/2} \Delta a)) = 0, \quad i = 0, \ldots, p,
\]
provided \( n \) is sufficiently large.

Thus
\[
\kappa(P_{n}) = \kappa(P_{\ast}) + n^{-1/2} \Delta a.
\]
It follows from (10.21) and the definition of $k_{n,j}$ that $p_n$ can be written in the form (5.3) with

$$ h := e_{ij} (F_{ik} a_k + B_{jk} F_{kl} a_k a_i) f^{(0)}, $$

$$ r_{n,a} := n^{1/2} \Delta \left( n^{1/2} \sigma_{00}^{-1} A_{0j} + \Delta e_{ij} (F_{ik} a_k + B_{jk} F_{kl} a_k a_i) \right) f^{(0)} 1_{\|f^{(0)}\| > \rho_1} + nR_{n,j} K'_{n,j} - na_{n,j} P_* (K'_{n,j}). $$

Condition (5.4) holds trivially.

By the choice of $\beta$,

$$ \int |f^{(0)}|^{(3/2 + \epsilon)/8} 1_{\|f^{(0)}\| > \rho_1} dP_* = O(n^{-(3/2 + \epsilon)/8}), $$

$$ \int f^{(0)2} 1_{\|f^{(0)}\| > \rho_1} dP_* = o(n^{-1}). $$

Hence condition (5.5) is fulfilled.

Proof of Corollary (5.15). Let $A > 0$ and $a_i := -A_{0i}^{-1} A_{0i}, i = 0, \ldots, p$. Then for sufficiently large $n \in \mathbb{N}$ we have $\theta_* + n^{-1/2} \Delta a \in \Theta$, and the sequence $P_n := P_{\theta_* + n^{-1/2} \Delta a}$ fulfills (5.2).

We have

$$ p(\cdot, \theta_* + n^{-1/2} \Delta a)/p(\cdot, \theta^*) $$

$$ = 1 + n^{-1/2} \Delta a p^{(0)}(\cdot, \theta^*)/p(\cdot, \theta^*) + \frac{1}{2} n^{-1} \Delta^2 a_i a_j p^{(0)2}(\cdot, \theta^*)/p(\cdot, \theta^*) + $$

$$ + n^{-1} \Delta^2 a_i a_j \int [(1-u) p^{(0)}(\cdot, \theta^* + un^{-1/2} \Delta a) - p^{(0)}(\cdot, \theta^*)] du. $$

Hence (5.3)-(5.5) follow easily by conditions (5.12) and (5.16).

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