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THE LADDER VARIABLES OF A MARKOV RANDOM WALK

BY

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Abstract. Given a Harris chain $(M_n)_{n \geqslant 0}$ on any state space $(\mathscr{S}, \mathfrak{S})$ with essentially unique stationary measure ξ, let $(X_n)_{n \geqslant 0}$ be a sequence of real-valued random variables which are conditionally independent, given $(M_n)_{n \geqslant 0}$, and satisfy

$$P(X_k \in \cdot \mid (M_n)_{n \geqslant 0}) = Q(M_{k-1}, M_k, \cdot)$$

for some stochastic kernel $Q: \mathscr{S}^2 \times \mathfrak{B} \to [0, 1]$ and all $k \geqslant 1$. Denote by S_n the n-th partial sum of this sequence. Then $(M_n, S_n)_{n \geqslant 0}$ forms a so-called Markov random walk with driving chain $(M_n)_{n \geqslant 0}$. Its stationary mean drift is given by $\mu = E_\xi X_1$ and assumed to be positive in which case the associated (strictly ascending) ladder epochs

$$\sigma_0 = \inf\{k \geqslant 0: S_k \geqslant 0\},$$

$$\sigma_n = \inf\{k > \sigma_{n-1}: S_k > S_{\sigma_{n-1}}\} \quad \text{for } n \geqslant 1,$$

and the ladder heights $S_n^* = S_{\sigma_n}$ for $n \geqslant 0$ are a.s. positive and finite random variables. Put $M_n^* = M_{\sigma_n}$. The main result of this paper is that $(M_n^*, S_n^*)_{n \geqslant 0}$ and $(M_n^*, \sigma_n)_{n \geqslant 0}$ are again Markov random walks (with positive increments, thus so-called *Markov renewal processes*) with Harris recurrent driving chain $(M_n^*)_{n \geqslant 0}$. The difficult part is to verify the Harris recurrence of $(M_n^*)_{n \geqslant 0}$. Denoting by ξ^* its stationary measure, we also give necessary and sufficient conditions for the finiteness of $E_{\xi^*} S_1^*$, $E_\xi S_1^*$ and $E_{\xi^*} \sigma_1$ in terms of μ or the recurrence-type of $(M_n)_{n \geqslant 0}$ or $(M_n^*)_{n \geqslant 0}$.

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1. Introduction and main result. The main purpose of this paper is the derivation of a fundamental result on the probabilistic structure of the sequence of ladder variables associated with a Markov random walk (MRW) with positive drift. The result is used in [3] to provide a coupling proof of the Markov renewal theorem that in some respects improves on the one given in [1], and further in [4] for the derivation of renewal theorems for a certain class of random walks with m-dependent increments.

For an ordinary zero-delayed random walk $(S_n)_{n \geq 0}$ with i.i.d. increments $X_1, X_2, \ldots$ with positive mean, we know that the sequence

$$\sigma_0 = 0 \quad \text{and} \quad \sigma_n = \inf\{k > \sigma_{n-1}: S_k > S_{\sigma_{n-1}}\} \text{ for } n \geq 1$$

of strictly ascending ladder epochs and the associated sequence $(S_{\sigma_n})_{n \geq 0}$ of ladder heights both have again i.i.d. increments. In Markov renewal theory, where $(S_n)_{n \geq 0}$ is governed by a temporally homogeneous Markov chain $(M_n)_{n \geq 0}$, one can also easily conclude that $(M_{\sigma_n}, \sigma_n)_{n \geq 0}$ and $(M_{\sigma_n}, S_{\sigma_n})_{n \geq 0}$ both constitute MRW's with positive increments, called *Markov renewal processes* (MRP's), providing all σ_n are a.s. finite. However, its central result, the Markov renewal theorem, additionally assumes the Harris recurrence of $(M_n)_{n \geq 0}$, and the use of ladder variables for its proof in order to reduce to the case of positive increments further requires the Harris recurrence of $(M_{\sigma_n})_{n \geq 0}$, which, surprisingly, does not seem to follow by straightforward arguments and which to show is the major purpose of the present paper.

A precise statement of the main result, Theorem 1 below, requires describing the basic setup in more detail: Given a measurable space $(\mathscr{S}, \mathfrak{S})$ with countably generated σ-field $\mathfrak{S}$ and a transition kernel $\boldsymbol{P}: \mathscr{S} \times (\mathfrak{S} \otimes \mathfrak{B}) \to [0, 1]$, $\mathfrak{B}$ the Borel σ-field on $\boldsymbol{R}$, let $(M_n, X_n)_{n \geq 0}$ be an associated Markov chain, defined on a probability space $(\Omega, \mathscr{A}, P)$, with state space $\mathscr{S} \times \boldsymbol{R}$, i.e.

$$(1.1) \qquad P\big(M_{n+1} \in A, X_{n+1} \in B \mid (M_j, X_j)_{0 \leq j \leq n}\big) = \boldsymbol{P}(M_n, A \times B) \text{ a.s.}$$

for all $n \geq 0$ and $A \in \mathfrak{S}$, $B \in \mathfrak{B}$. Thus (M_{n+1}, X_{n+1}) depends on the past only through M_n, and $(M_n)_{n \geq 0}$ forms a Markov chain with state space $\mathscr{S}$ and a transition kernel $\boldsymbol{P}(x, A) \stackrel{\text{def}}{=} \boldsymbol{P}(x, A \times \boldsymbol{R})$. It can be shown that, given $(M_j)_{j \geq 0}$, the X_n, $n \geq 0$, are conditionally independent with

$$(1.2) \qquad P\big(X_n \in B \mid (M_j)_{j \geq 0}\big) = Q(M_{n-1}, M_n, B) \text{ a.s.}$$

for all $n \geq 1$, $B \in \mathfrak{B}$ and an appropriate kernel $Q: \mathscr{S}^2 \times \mathfrak{B} \to [0, 1]$. Let throughout a canonical model be given with probability measures $P_{x,y}$, $x \in \mathscr{S}$, $y \in \boldsymbol{R}$, on $(\Omega, \mathscr{A})$ such that $P_{x,y}(M_0 = x, X_0 = y) = 1$. For any distribution (or σ-finite measure) λ on $\mathscr{S} \times \boldsymbol{R}$ put

$$P_\lambda(\cdot) = \int_{\mathscr{S} \times \boldsymbol{R}} P_{x,y}(\cdot)\, \lambda(dx, dy)$$

in which case (M_0, X_0) has initial distribution λ under P_λ. For $x \in \mathscr{S}$ and σ-finite measures ν on $\mathscr{S}$, we write for short E_x, E_ν instead of $E_{x,0}$, $E_{\nu \otimes \delta_0}$, respectively, where δ_0 is Dirac measure at 0. Finally, P and E are used for probabilities and expectations, respectively, that do not depend at all on initial conditions.

The MRW associated with $(M_n, X_n)_{n\geqslant 0}$ is defined by $(M_n, S_n)_{n\geqslant 0}$, where $S_n = X_0+\ldots+X_n$ for each $n \geqslant 0$. We always assume that $(M_n)_{n\geqslant 0}$ is Harris recurrent (see Section 2) with (essentially unique) stationary measure ξ and $\mu = E_\xi X_1 > 0$, in which case $S_n \to \infty$ $P_{x,y}$-a.s. for every $(x, y)\in \mathscr{S}\times \boldsymbol{R}$. Consequently, the strictly ascending ladder epochs

$$\begin{aligned} &\sigma_0 = \inf\{n \geqslant 0\colon\ S_n \geqslant 0\} \text{ and} \\ &\sigma_n = \inf\{k > \sigma_{n-1}\colon\ S_k > S_{\sigma_{n-1}}\} \text{ for } n \geqslant 1 \end{aligned} \tag{1.3}$$

are a.s. finite under each $P_{x,y}$ and the associated ladder heights $S_n^* = S_{\sigma_n}$ well-defined positive random variables. The slightly different definition for σ_0 appears in order to have $\sigma_0 = 0$ in the zero-delayed case $S_0 = 0$. We put $M_n^* = M_{\sigma_n}$ for each $n \geqslant 0$ and denote by ν_n and X_n^* the increments of $(\sigma_n)_{n\geqslant 0}$ and $(S_n^*)_{n\geqslant 0}$, respectively. The essentially unique stationary measure of $(M_n^*)_{n\geqslant 0}$, the existence of which follows from Theorem 1 (i) below, is denoted by ξ^*.

Finally, we have to define the lattice-type of $(M_n, S_n)_{n\geqslant 0}$. Following Shurenkov [10], the latter as well as $\boldsymbol{P}$ are called *d-arithmetic* if $d > 0$ is the maximal number for which there exists a function $\gamma\colon \mathscr{S} \to [0, d)$, called a *shift function*, such that

$$P\big(X_1 \in \gamma(x)-\gamma(y)+d\boldsymbol{Z} \mid M_0 = x,\ M_1 = y\big) = 1 \quad \xi\otimes \boldsymbol{P}\text{-a.s.,} \tag{1.4}$$

where $\xi\otimes\boldsymbol{P}$ is given through $\xi\otimes \boldsymbol{P}(A\times B) = \int_A \boldsymbol{P}(x, B)\,\xi(dx)$ for $A, B\in \mathscr{S}$. If no such d exists, $(M_n, S_n)_{n\geqslant 0}$ and $\boldsymbol{P}$ are called *nonarithmetic.* Our main result now reads as follows:

THEOREM 1. *Given an MRW $(M_n, S_n)_{n\geqslant 0}$ with Harris recurrent driving chain $(M_n)_{n\geqslant 0}$ and positive drift $\mu = E_\xi X_1$, the following assertions hold:*

(i) *$(M_n^*)_{n\geqslant 0}$ forms a Harris chain which is further positive recurrent if the same holds true for $(M_n)_{n\geqslant 0}$.*

(ii) *$(M_n^*, \sigma_n)_{n\geqslant 0}$ and $(M_n^*, S_n^*)_{n\geqslant 0}$ are MRP's, their lattice-type being that of $(M_n, n)_{n\geqslant 0}$ and $(M_n, S_n)_{n\geqslant 0}$, respectively, with the same shift function if arithmetic.*

(iii) *$E_\xi S_1^* < \infty$ iff $E_{\xi^*} S_1^* < \infty$ iff $\mu < \infty$.*

(iv) *$E_{\xi^*}\sigma_1 < \infty$ iff $(M_n)_{n\geqslant 0}$ is positive recurrent.*

In view of (ii) it is natural to ask for the lattice-span of $(M_n\ n)_{n\geqslant 0}$, which turns out to be the period of the driving chain $(M_n)_{n\geqslant 0}$ as will be shown in the final section including the definition of the shift function. In case where $(M_n)_{n\geqslant 0}$ is aperiodic the latter is 0, so that $(M_n, n)_{n\geqslant 0}$ and $(M_n^*, \sigma_n)_{n\geqslant 0}$ then constitute 1-arithmetic MRP's with shift function 0.

The subsequent corollary collects a number of convergence results that follow directly from Theorem 1 in combination with well-known ergodic theorems for Harris chains or stationary sequences (see also Remark (b) below). It is thus stated without proof.

COROLLARY 1. *Given the situation of Theorem* 1, *if* $(M_n^*)_{n \geqslant 0}$ (*or even* $(M_n)_{n \geqslant 0}$) *is positive recurrent with stationary distribution* ξ^*, *then, as* $n \to \infty$,

(i) $n^{-1} S_n^* \to E_{\xi^*} S_1^*$ P_x-*a.s. for all* $x \in \mathscr{S}$;

(ii) $E_{\xi^*} |n^{-1} S_n^* - E_{\xi^*} S_1^*| \to 0$ *if* $\mu < \infty$;

(iii) $n^{-1} \sigma_n \to E_{\xi^*} \sigma_1$ P_x-*a.s. for all* $x \in \mathscr{S}$;

(iv) $E_{\xi^*} |n^{-1} \sigma_n - E_{\xi^*} \sigma_1| \to 0$.

If $(M_n^*)_{n \geqslant 0}$ *is further aperiodic* (*and thus ergodic*), *then additionally*

(v) $\|P_x((\tau_n, X_n^*) \in \cdot) - P_{\xi^*}((\tau_1, X_1^*) \in \cdot)\| \to 0$ *for all* $x \in \mathscr{S}$ –

holds true as $n \to \infty$, *where* $\tau_n = \sigma_n - \sigma_{n-1}$ *and* $\|\cdot\|$ *denotes total variation norm.*

Remarks. (a) All previous results remain true if the ladder epochs σ_n are replaced by $\sigma_0(a) = \sigma_0$ and $\sigma_n(a) = \inf\{k > \sigma_{n-1}(a): S_k - S_{\sigma_{n-1}(a)} > a\}$ for $n \geqslant 1$ and arbitrary $a \in (0, \infty)$.

(b) It is a trivial consequence of the transition structure of $(M_n, X_n)_{n \geqslant 0}$ that, given the Harris recurrence of $(M_n)_{n \geqslant 0}$, the same holds true for $(M_n, X_{n+1})_{n \geqslant 0}$. Moreover, if ξ is the essentially unique stationary measure of $(M_n)_{n \geqslant 0}$, then $(M_n, X_{n+1})_{n \geqslant 0}$ is stationary under P_ξ (in the measure-theoretic sense if P_ξ is not a probability distribution), and thus $P_\xi((M_0, X_1) \in \cdot)$ the essentially unique stationary measure of $(M_n, X_{n+1})_{n \geqslant 0}$. Consequently, the validity of Theorem 1 (i) further gives the Harris recurrence of $(M_n^*, \nu_{n+1})_{n \geqslant 0}$, $(M_n^*, X_{n+1}^*)_{n \geqslant 0}$ and $(M_n^*, X_{n+1}^*, \nu_{n+1})_{n \geqslant 0}$ as well as their stationarity under P_{ξ^*}. This can be used in combination with certain ergodic theorems for such sequences to prove the convergence results of Corollary 1.

(c) According to part (i) of Theorem 1, the positive recurrence of $(M_n)_{n \geqslant 0}$ entails the same for $(M_n^*)_{n \geqslant 0}$. The following simple example shows that the converse may fail: Let $(M_n)_{n \geqslant 0}$ be any irreducible, null-recurrent Markov chain on N_0 with stationary measure $\xi = (\xi_j)_{j \geqslant 0}$ and $X_n = 1_{\{0\}}(M_n)$. Then $(M_n, S_n)_{n \geqslant 0}$ constitutes an MRW with drift $\mu = E_\xi X_1 = \xi_0 > 0$. Its ladder epochs occur each time the driving chain hits state 0, i.e., $M_n^* = 0$ for every $n \geqslant 1$.

(d) Let us further point out that $(M_n^*)_{n \geqslant 0}$ does not need to be aperiodic. A counterexample is given in [7] where strictly ascending ladder epochs for sums of 1-dependent indicator functions are investigated. More precisely, given i.i.d. Bernoulli variables $Y_0, Y_1, \ldots$ with $P(Y_0 = 1) = 1 - P(Y_0 = 0) = p$, where $p \in (0, 1)$, let $X_n = 1_{\{Y_n \neq Y_{n+1}\}}$ and $M_n = (Y_n, Y_{n+1})$ for $n \geqslant 0$. Then $(X_n)_{n \geqslant 0}$ forms a stationary 1-dependent sequence, $(M_n)_{n \geqslant 0}$ an ergodic (positive recurrent and aperiodic) Harris chain (see Section 2 for details), and $(M_n, S_n)_{n \geqslant 0}$ an MRW. Now observe that a ladder epoch occurs at n iff $M_n = (Y_n, Y_{n+1}) = (0, 1)$ or $(1, 0)$, and that these two states are attained in alternating order at consecutive ladder epochs. Hence $(M_n^*)_{n \geqslant 0}$ is a 2-periodic discrete Markov chain with state space $\{(0, 1), (1, 0)\}$ and transition matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. As a consequence, unless $p = \frac{1}{2}$, the distribution of τ_n under $P_{(0,1)}$ as well as $P_{(1,0)}$ does not converge to a stationary limit, but is either geometric with parameter p or with parameter $1-p$, in alternating order for each $n \geqslant 1$. The ladder height increments X_n^* are, of course, always equal to 1 here.

(e) The hypotheses that $(M_n)_{n\geqslant 0}$ is positive recurrent and $\mu = E_\xi X_1 < \infty$ are not enough to imply $E_\xi \sigma_1 < \infty$ as demonstrated by these two examples: Let $(W_n)_{n\geqslant 0}$ be a 1-arithmetic discrete renewal process with positive increments Y_n, which are thus independent for $n \geqslant 0$ and further identically distributed for $n \geqslant 1$. Suppose $\nu \overset{\text{def}}{=} EY_1 \in (0, \infty)$ and $EY_1^2 = \infty$. Let $(M_n)_{n\geqslant 0}$ be the associated sequence of forward recurrence times, i.e. $M_n = W_{\tau(n)} - n$, where $\tau(n) = \inf\{k \geqslant 0\colon W_k > n\}$. It is well known, see e.g. [5], that $(M_n)_{n\geqslant 0}$ forms an ergodic discrete Markov chain with state space N and stationary distribution $\hat{\xi} = (\hat{\xi}_k)_{k\geqslant 1}$ defined by $\hat{\xi}_k = \nu^{-1} P(Y_1 > k-1)$. Note that $M_0 = W_0$ and that $\hat{\xi}$ has infinite mean because

$$\sum_{k\geqslant 1} k\hat{\xi}_k = (2\nu)^{-1} EY_1(Y_1+1) = \infty.$$

We will now consider two different MRW's with the same driving chain $(M_n)_{n\geqslant 0}$ as just defined. In the first example, $E_{\hat{\xi}}\sigma_1$ is finite while being infinite in the second one.

(1) Put $S_0 = X_0 = 0$ and $X_n = \mathbf{1}_{\{k,k+1,\ldots\}}(M_n)$ for $n \geqslant 1$ and some fixed $k \geqslant 1$. Then $\mu = E_{\hat{\xi}} X_1 > 0$ follows from $p_k \overset{\text{def}}{=} P(Y_1 \geqslant k) > 0$ for each $k \geqslant 1$ (recall that $EY_1^2 = \infty$). Plainly, the X_n are all equal to 1 if $k = 1$, in which case $\sigma_1 = 1$ and $E_{\hat{\xi}}\sigma_1 < \infty$ trivially follow. But the latter holds true also for every other $k \geqslant 2$. In fact, the ladder epochs are all those renewal times W_n that come with $Y_{n+1} \geqslant k$ (notice $M_{W_n} = Y_{n+1}$) so that

$$\sigma_1 = \mathbf{1}_{\{Y_0\geqslant k\}} + \sum_{n\geqslant 1} W_{n-1}\mathbf{1}_{\{Y_0<k,\ldots,Y_{n-1}<k,Y_n\geqslant k\}} \leqslant \mathbf{1}_{\{Y_0\geqslant k\}} + k\sum_{n\geqslant 1} n\mathbf{1}_{\{Y_0<k,\ldots,Y_{n-1}<k,Y_n\geqslant k\}}$$

with stationary mean

$$E_{\hat{\xi}}\sigma_1 \leqslant 1 + k\sum_{n\geqslant 1} n(1-p_k)^{n-1} p_k < \infty.$$

(2) If $X_n = \mathbf{1}_{\{1\}\times N}(M_{n-1}, M_n)$ for $n \geqslant 1$, then S_n has zero jumps except for those n where a renewal takes place, i.e. when $W_k = n$ for some $k \geqslant 0$. But this means nothing but $\sigma_n = W_{n-1}$ for $n \geqslant 1$ providing $S_0 = X_0 = 0$. Hence

$$E_{\hat{\xi}}\sigma_1 = E_{\hat{\xi}} W_0 = E_{\hat{\xi}} M_0 = \sum_{k\geqslant 1} k\hat{\xi}_k = \infty.$$

The further presentation is organized as follows: The proof of Theorem 1 will be given in Section 3 after the derivation of an important auxiliary result (Theorem 2) on the Harris recurrence of a related excursion chain. Some basic facts on Harris recurrence and regeneration are collected in Section 2 including the definition of a particular regeneration scheme for $(M_n)_{n\geqslant 0}$ which will be needed thereafter.

2. Harris recurrence and regeneration. For a moment we only consider the Markov chain $(M_n)_{n\geqslant 0}$ with state space $(\mathscr{S}, \mathfrak{S})$ and r-step transition kernel $\boldsymbol{P}_r$ $(\boldsymbol{P} = \boldsymbol{P}_1)$. $(M_n)_{n\geqslant 0}$ is called *Harris recurrent* or just a *Harris chain* if it has a *recurrence set* $\mathfrak{R}$, i.e. $P_x(M_n \in \mathfrak{R} \text{ i.o.}) = 1$ for all $x \in \mathscr{S}$, such that for some $\alpha \in (0, 1]$, $r \geqslant 1$, and a distribution φ on $\mathscr{S}$ the minorization condition

$$\boldsymbol{P}_r(x, \cdot) \geqslant \alpha\varphi \quad \text{for all } x \in \mathfrak{R} \tag{2.1}$$

holds true. Given the latter condition, $\mathfrak{R}$ is called a *regeneration set* because it induces a regenerative structure for $(M_n)_{n\geqslant 0}$ that divides the chain into stationary (possibly except for the first one) 1-dependent cycles. This has been shown in the fundamental paper by Athreya and Ney [6] for $r = 1$ in which case the cycles are even independent; see also [9] for a similar technique. Indeed, (2.1) is equivalent to the existence of a *sequence* $(\tau_n)_{n\geqslant 0}$ *of regeneration epochs*, characterized through the following four conditions:

(R.1) $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \infty$ a.s. under each P_λ.

(R.2) There is a filtration $(\mathscr{F}_n)_{n\geqslant 0}$ such that $(M_n)_{n\geqslant 0}$ is Markov-adapted and each τ_k a stopping time with respect to $(\mathscr{F}_n)_{n\geqslant 0}$.

(R.3) Under each P_x, $x \in \mathscr{S}$, the M_{τ_n} are independent for $n \geqslant 0$ and further identically distributed with common distribution ζ for $n \geqslant 1$.

(R.4) $P\big((\tau_{n+j}-\tau_n, M_{\tau_n+j})_{j\geqslant 0} \in \cdot \mid \mathscr{F}_{\tau_n}\big) = P_{M_{\tau_n}}\big((\tau_j, M_j)_{j\geqslant 0} \in \cdot\big)$ P_x-a.s. for each $n \geqslant 0$ and $x \in \mathscr{S}$.

We note that the construction of $(\tau_n)_{n\geqslant 0}$ generally goes along with a re-definition of $(M_n)_{n\geqslant 0}$ on a possibly enlarged probability space. We note further that the previous conditions for regeneration epochs are weaker than those stated in [1] in that we do not require here $(\tau_{n+j}-\tau_n, M_{\tau_n+j})_{j\geqslant 0}$ to be independent of $\tau_0, \ldots, \tau_n$ for each $n \geqslant 0$. A more detailed discussion of this can be found in [2].

By defining (with ζ as in (R.3))

$$\xi(A) \overset{\text{def}}{=} E_\zeta\Big(\sum_{j=0}^{\tau_1 - 1} 1_{\{M_j \in A\}}\Big), \quad A \in \mathfrak{S}, \tag{2.2}$$

we obtain the unique (up to multiplicative constants) σ-finite stationary measure of $(M_n)_{n\geqslant 0}$ if $0 < \xi(A) < \infty$ for at least one $A \in \mathfrak{S}$, in which case we call $(\tau_n)_{n\geqslant 0}$ *regular*. Such a regular sequence $(\tau_n)_{n\geqslant 0}$ always exists and can in fact be obtained by using Athreya and Ney's "coin-tossing" procedure. Next, $\hat{\xi} = \xi/E_\zeta\tau_1$ forms the unique stationary distribution provided $E_\zeta\tau_1 < \infty$. In that case the chain is called *positive recurrent*. It is further called *aperiodic* if the distribution of τ_1 under P_ζ is aperiodic (1-arithmetic), and *ergodic* if it is aperiodic as well as positive recurrent. The latter implies that $P_\lambda(M_n\in\cdot)$ converges to ξ in total variation for each λ.

Returning to Markov renewal theory, suppose we are further given a sequence $(S_n)_{n\geq 0}$ of real-valued random variables such that $(M_n, S_n)_{n\geq 0}$ forms an MRW. Note first that then

$$E_\xi f(X_1) = E_\zeta\Big(\sum_{j=0}^{\tau_1-1} f(X_j)\Big) \tag{2.3}$$

for every function $f: \boldsymbol{R}\to\boldsymbol{R}$ such that $E_\xi f(X_1)$ exists. In particular, we have

$$\mu = E_\xi X_1 = E_\zeta\Big(\sum_{j=0}^{\tau_1-1} X_j\Big) \tag{2.4}$$

for the drift μ of $(M_n, S_n)_{n\geq 0}$ provided $E_\xi|X_1|<\infty$.

In order to prove Theorem 1 and the auxiliary result – Theorem 2 – stated in the next section, it is not enough to pick any sequence of regeneration epochs for $(M_n)_{n\geq 0}$ but rather to choose one that comes with a number of further properties concerning the bivariate chain $(M_n, X_n)_{n\geq 0}$. The existence of such a sequence $(\tau_n)_{n\geq 0}$ has been shown in [1]; see Lemmata 3.1–3.3 therein.

Given $\mathfrak{R}$, r and φ as in (2.1), there are φ-positive subsets C, D of $\mathfrak{R}$, some $c\in(0,\infty)$, a filtration $(\mathscr{F}_n)_{n\geq 0}$ and a sequence $(\tau_n)_{n\geq 0}$ of regeneration epochs such that in addition to (R.1)–(R.4):

(R.5) $(M_n, X_n)_{n\geq 0}$ is Markov adapted and each τ_k-r a stopping time with respect to $(\mathscr{F}_n)_{n\geq 0}$.

(R.6) $(M_{\tau_n+j}, X_{\tau_n+j+1})_{j\geq 0}$ and $\mathscr{F}_{\tau_n-r}$ are independent for each $n\geq 1$.

(R.7) $P(M_{\tau_n-r}\in C)=1$ for all $n\geq 1$ and $\zeta = P(M_{\tau_1}\in\cdot) = \varphi(\cdot\cap D)/\varphi(D)$.

(R.8) $|X_{\tau_n-r+j}|\leq c$ for each $1\leq j\leq r$ and $n\geq 1$.

(R.9) If P has lattice-span $d\geq 0$ with shift function γ, then $P_\zeta(S_{\tau_1}-\gamma(M_0)+\gamma(M_{\tau_1})\in\cdot)$ is of the same lattice-type and

$$E_x\big|E\big(\exp(2\pi it(S_{\tau_1}-S_0))\mid M_0, M_{\tau_1}\big)\big|<1 \quad \text{for each } 0<|t|\in 1/d \text{ and } x\in\mathscr{S}.$$

Indeed, C, $D\subset\mathfrak{R}$ and $c\in(0,\infty)$ are chosen in such a way that

$$P_x(|X_1|\leq c,\ldots,|X_r|\leq c\mid M_1=y)\geq\beta \text{ a.s.}$$

for all $(x,y)\in C\times D$ and some $\beta>0$. Condition (R.8) will be crucial for our considerations in the following section.

Let us finally note that, given any (regular) sequence $(\tilde\tau_n)_{n\geq 0}$ of regeneration epochs for $(M_n)_{n\geq 0}$ of the form

$$\tilde\tau_{n+1}-\tilde\tau_n = f\big((M_{\tilde\tau_n+k})_{k\geq 0}, L_n\big), \quad n\geq 0,$$

f a suitable function, and $(L_n)_{n\geq 1}$ a sequence of i.i.d. random variables (governing randomization) independent of the "rest of the world", one can always switch to a new one satisfying (R.9) by defining $\tau_0 = 0$ and

$$\tau_{n+1} - \tau_n = \chi_n + f\big((M_{\tau_n+\chi_n+k})_{k\geqslant 0}, L_n\big), \qquad n \geqslant 0,$$

where the χ_n, $n \geqslant 0$, are i.i.d. geometric (1/2) variables independent of the "rest of the world" as well. This is an immediate consequence of Lemma 3 in the final section and already stated here because we will use it in Section 3 for the proof of Theorem 1 (ii).

3. An excursion chain and the proof of Theorem 1. The proof of Theorem 1 is essentially furnished by Theorem 2 below, which states the Harris recurrence of a further Markov chain $(M_n, Z_n)_{n\geqslant 0}$, called the *chain of (negative) excursions* associated with $(M_n, S_n)_{n\geqslant 0}$.

Let $Z_0 = \mathbf{1}_{\{S_0>0\}} + S_0 \mathbf{1}_{\{S_0\leqslant 0\}}$ and

$$(3.1) \qquad Z_n = \begin{cases} 1 & \text{if } S_n > \max\limits_{0\leqslant j<n} S_j, \\ S_n - \max\limits_{0\leqslant j<n} S_j & \text{otherwise} \end{cases}$$

for $n \geqslant 1$. $Z_n = 1$ means that a strictly ascending ladder epoch occurs at n, while $Z_n < 0$ denotes a negative *excursion* of S_n from the current record value of the random walk at n. Notice the recursive structure

$$(3.2) \quad Z_n = \begin{cases} 1 & \text{if } Z_{n-1} = 1,\ X_n > 0 \text{ or } Z_{n-1} \leqslant 0 < Z_{n-1} + X_n, \\ X_n & \text{if } Z_{n-1} = 1,\ X_n \leqslant 0, \\ Z_{n-1} + X_n & \text{if } Z_{n-1} < 0,\ Z_{n-1} + X_n \leqslant 0 \end{cases}$$

for $n \geqslant 1$, which immediately implies that $(M_n, Z_n)_{n\geqslant 0}$ forms a Markov chain with state space $\mathscr{E} = \mathscr{S} \times \big((-\infty, 0] + \{1\}\big)$. Let $\mathfrak{E}$ be the associated σ-field over $\mathscr{E}$ induced by $\mathfrak{S}\otimes\mathfrak{B}$.

THEOREM 2. *$(M_n, Z_n)_{n\geqslant 0}$ forms a Harris chain which is positive recurrent iff the same holds true for $(M_n)_{n\geqslant 0}$. Furthermore, there exists a regular sequence of regeneration epochs $(T_n)_{n\geqslant 0}$ for both chains such that $Z_{T_n} = 1$ for all $n \geqslant 1$.*

Given the Harris recurrence of $(M_n, Z_n)_{n\geqslant 0}$, it is easy to show the existence of a regular sequence of regeneration epochs $(T_n)_{n\geqslant 0}$ such that $Z_{T_n} = 1$ for each $n \geqslant 1$. It is also easily seen that $(T_n)_{n\geqslant 0}$ then forms a sequence of regeneration epochs for $(M_n)_{n\geqslant 0}$. Hence the crucial point of the second assertion of Theorem 2 is that $(T_n)_{n\geqslant 0}$ is again regular for $(M_n)_{n\geqslant 0}$.

The proof of Theorem 2 will be based on the following two lemmas. Put $\hat{P}_{x,z} = P(\cdot \mid M_0 = y, Z_0 = z)$ for $(x, z) \in \mathscr{E}$ and $\hat{P}_x = \hat{P}_{x,1}$. Let further $(\tau_n)_{n\geqslant 0}$ be a sequence of regeneration epochs for $(M_n)_{n\geqslant 0}$ such that (R.1)–(R.9) hold true. It then follows from (R.4) that $(M_{\tau_n}, Z_{\tau_n})_{n\geqslant 0}$ forms a temporally homogeneous Markov chain.

LEMMA 1. *For all* $B \in \mathfrak{E}$, $Q((x, z), B) \overset{\text{def}}{=} \hat{P}_{x,y}((M_{\tau_n}, Z_{\tau_n}) \in B \text{ i.o.})$ *does not depend on* $(x, z) \in \mathscr{E}$ *and is either* 0 *or* 1.

Proof. Fixing an arbitrary $B \in \mathfrak{E}$, it suffices to verify that $Q((x, z), B) = Q(B)$ does not depend on (x, z) because then, with $\nu(k)$ denoting the k-th hitting time of B, the strong Markov property implies

$$Q(B) = \int_{\{\nu(k) < \infty\}} \hat{P}_{M_{\tau_{\nu(k)}}, Z_{\tau_{\nu(k)}}}((M_{\tau_n}, Z_{\tau_n}) \in B \text{ i.o.}) \, d\hat{P}_{x,z} = Q(B) \hat{P}_{x,z}(\nu(k) < \infty).$$

Thus $Q(B) = 0$ or $\hat{P}_{x,z}(\nu(k) < \infty) = 1$ for all $(x, z) \in \mathscr{E}$ and $k \geqslant 1$, which is the same as $Q(B) = 1$.

To prove $Q((x, z), B) = Q(B)$, we use a simple coupling argument: Given $(M_n, X_n, Z_n)_{n \geqslant 0}$ with initial conditions (x, y, z) and regeneration sequence $(\tau_n)_{n \geqslant 0}$ as previously stated, let (x', y', z') be any other initial state vector. On a possibly enlarged probability space, we can then construct a chain $(M'_n, X'_n)_{n \geqslant 0}$ with regeneration sequence $(\tau'_n)_{n \geqslant 0}$, the same transition law as $(M_n, X_n)_{n \geqslant 0}$ and initial state (x', y') such that

(3.3) $\quad (M_n, X_{n+1})_{n \geqslant \tau_1} = (M'_n, X'_{n+1})_{n \geqslant \tau'_1}$ and $(\tau_n - \tau_1)_{n \geqslant 1} = (\tau'_n - \tau'_1)_{n \geqslant 1}$.

Next, given $Z'_0 = z'$, the sequence $(Z'_n)_{n \geqslant 0}$ is completely determined by $(M'_n, X'_n)_{n \geqslant 0}$, which in combination with (3.3) easily shows

$$(M_n, X_n, Z_n)_{n \geqslant T} = (M'_n, X'_n, Z'_n)_{n \geqslant T'},$$
$$\text{where } T = \inf\{n \geqslant \tau_1 \colon S_n - S_{\tau_1} > Z_{\tau_1}^- \vee (Z'_{\tau'_1})^-\}$$

and T' is similarly defined with $(S_n - S_{\tau_1})_{n > \tau_1}$ replaced by $(S'_n - S'_{\tau'_1})_{n > \tau'_1}$. Namely, at T and T' ladder epochs occur for $(M_n, S_n)_{n \geqslant 0}$ and $(M'_n, S'_n)_{n \geqslant 0}$, respectively, which gives $Z_T = Z'_{T'} = 1$. But the "Z"-sequences start from scratch each time they hit state 1, whence, by (3.3), indeed, $Z_{T+n} = Z'_{T'+n}$ for all $n \geqslant 0$ holds true. Finally, the assertion $Q((x, z), B) = Q(B)$ is now an immediate consequence of the fact that in particular

$$(M_{\tau_n}, Z_{\tau_n})_{n \geqslant \nu} = (M'_{\tau'_n}, Z'_{\tau'_n})_{n \geqslant \nu'},$$

where $\nu = \inf\{n \geqslant 1 \colon \tau_n \geqslant T\}$ and $\nu' = \inf\{n \geqslant 1 \colon \tau'_n \geqslant T'\}$.

LEMMA 2. *There is a constant* $b \in (0, \infty)$ *such that*

$$\hat{P}_{x,z}(Z_{\tau_n} \geqslant -b \text{ i.o.}) = \hat{P}_{x,z}(\liminf_{n \to \infty} Z_{\tau_n} > -\infty) = 1$$

for all $(x, z) \in \mathscr{E}$.

Proof. We will show the existence of $b \in (0, \infty)$ and $m \geqslant 1$ such that

(3.4) $$E(Z_{\tau_{n+m}-r} - Z_{\tau_n - r} \mid \mathscr{F}_{\tau_n - r}) > 0 \text{ a.s.}$$

on $A_{n,b} \overset{\text{def}}{=} \{Z_{\tau_n - r} < -b\}$. For then combining $\sup_{n \geq 1} |Z_{\tau_n} - Z_{\tau_n - r}| \leq 1 + rc$, by (R.8), with a straightforward generalization of Theorem 9.4.1 in Meyn and Tweedie [8], we conclude that

$$\hat{P}_{x,z}(\liminf_{n\to\infty} Z_{\tau_n} > -\infty) = \hat{P}_{x,z}(\liminf_{n\to\infty} Z_{\tau_n - r} > -\infty) = 1$$

for all $(x, z) \in \mathscr{E}$, which in turn together with Lemma 1 further gives

$$\hat{P}_{x,z}(Z_{\tau_n} \geq -b \text{ i.o.}) = 1 \quad \text{for all } (x, z) \in \mathscr{E} \text{ and some } b \in (0, \infty).$$

For $k \geq 0$ and $l \geq 1$, put

$$Y_{k,l} = \sum_{j=\tau_k+1}^{\tau_{k+l}-r} X_j^- \quad \text{and} \quad W_{k,l} = \sup_{\tau_k + 1 \leq j \leq \tau_{k+l} - r} S_j - S_{\tau_k}.$$

By (R.6), the $(Y_{k,l}, W_{k,l})$, $k \geq 1$, are identically distributed under each $\hat{P}_{x,z}$ with distribution $P_\zeta((Y_{0,l}, W_{0,l}) \in \cdot)$. Moreover, $\mu > 0$ and (2.3) give $E_\zeta Y_{0,l} \leq l E_\xi X_1^- < \infty$. Next, by another appeal to (R.8),

$$E(Z_{\tau_{n+m}-r} - Z_{\tau_n - r} \mid \mathscr{F}_{\tau_n - r}) \geq E(Z_{\tau_{n+m}-r} - Z_{\tau_n} \mid \mathscr{F}_{\tau_n - r}) - rc \text{ a.s.} \tag{3.5}$$

on $A_{n,rc}$, because no ladder epoch occurs at $\tau_n - r + j$, $1 \leq j \leq r$. A ladder epoch does not either occur at $\tau_n + 1, \ldots, \tau_{n+m} - r$ on

$$B_n \overset{\text{def}}{=} \{W_{n,n+m} \leq -Z_{\tau_n}, Z_{\tau_n} \leq 0\} \cup \{W_{n,n+m} \leq 0, Z_{\tau_n} = 1\},$$

implying

$$Z_{\tau_{n+m}-r} - Z_{\tau_n} = S_{\tau_{n+m}-r} - S_{\tau_n} \tag{3.6}$$

on that event. On B_n^c we will make use of

$$Z_{\tau_{n+m}-r} - Z_{\tau_n} \geq -Y_{n,n+m}, \tag{3.7}$$

which holds because $-Y_{n,n+m}$ bounds the maximal possible negative excursion between $\tau_n + 1$ and $\tau_{n+m} - r$. We further have

$$P(W_{n,n+m} \leq 0, Z_{\tau_n} = 1 \mid \mathscr{F}_{\tau_n - r}) = 0 \text{ a.s.} \quad \text{on } A_{n,rc},$$

which together with $|Z_{\tau_n} - Z_{\tau_n - r}| \leq rc$ on that event gives

$$\{W_{n,n+m} \leq -Z_{\tau_n - r} - rc\} \cap A_{n,rc} \subset B_n \cap A_{n,rc} \subset \{W_{n,n+m} \leq -Z_{\tau_n - r} + rc\} \cap A_{n,rc}.$$

By using this we obtain upon setting $Z_{\tau_n - r} = z$

$$\begin{aligned} &E\big((S_{\tau_{n+m}-r} - S_{\tau_n})^+ \mathbb{1}_{B_n} \mid \mathscr{F}_{\tau_n - r}\big) \\ &\quad \geq E\big((S_{\tau_{n+m}-r} - S_{\tau_n})^+ \mathbb{1}_{\{W_{n,n+m} \leq -z - rc\}} \mid \mathscr{F}_{\tau_n - r}\big) \\ &\quad = E_\zeta S_{\tau_m - r}^+ \mathbb{1}_{\{W_{0,m} \leq -z - rc\}} \text{ a.s.} \end{aligned} \tag{3.8}$$

Similarly,

(3.9) $$E\big((S_{\tau_{n+m}-r}-S_{\tau_n})^- \mathbf{1}_{B_n} \mid \mathscr{F}_{\tau_n-r}\big) \leqslant E\big((S_{\tau_{n+m}-r}-S_{\tau_n})^- \mid \mathscr{F}_{\tau_n-r}\big) = E_\zeta S^-_{\tau_m-r} \text{ a.s.},$$

(3.10) $$E\big((Z_{\tau_{n+m}-r}-Z_{\tau_n}) \mathbf{1}_{B_n^c} \mid \mathscr{F}_{\tau_n-r}\big) \geqslant -E\big(Y_{n,n+m} \mathbf{1}_{\{W_{n,n+m}>-z-rc\}} \mid \mathscr{F}_{\tau_n-r}\big) = -E_\zeta Y_{0,m} \mathbf{1}_{\{W_{0,m}>-z-rc\}} \text{ a.s.}$$

Next $E_\zeta S_{\tau_n-r} \geqslant E_\zeta S_{\tau_n} - rc = n\mu - rc \to \infty$ as $n \to \infty$ allows us to pick a sufficiently large m so that

(3.11) $$E_\zeta S_{\tau_m-r} > 3rc.$$

Given such m, we can further fix $b \geqslant rc$ large enough such that, by monotone convergence,

(3.12) $$E_\zeta S^+_{\tau_m-r} \mathbf{1}_{\{W_{0,m} \leqslant -z-rc\}} \geqslant E_\zeta S^+_{\tau_m-r} - rc$$

and, upon using $E_\zeta Y_{0,m} < \infty$,

(3.13) $$E_\zeta Y_{0,m} \mathbf{1}_{\{W_{0,m}>-z-rc\}} < rc \quad \text{for all } z \leqslant b.$$

By combining (3.5)–(3.13), we finally obtain on $A_{n,b} \subset A_{n,rc}$ with $z = Z_{\tau_n-r}$

$$\begin{aligned} E(Z_{\tau_{n+m}-r}-Z_{\tau_n-r} \mid \mathscr{F}_{\tau_n-r}) &\geqslant E(Z_{\tau_{n+m}-r}-Z_{\tau_n} \mid \mathscr{F}_{\tau_n-r}) - rc \\ &\geqslant E_\zeta S^+_{\tau_m-r} \mathbf{1}_{\{W_{0,m} \leqslant -z-rc\}} - E_\zeta S^-_{\tau_m-r} - E_\zeta Y_{0,m} \mathbf{1}_{\{W_{0,m}>-z-rc\}} - rc \\ &\geqslant E_\zeta S_{\tau_m-r} - 3rc > 0 \text{ a.s.}, \end{aligned}$$

which is the desired conclusion (3.4).

Proof of Theorem 2. Given $(\tau_n)_{n \geqslant 0}$ with associated filtration $(\mathscr{F}_n)_{n \geqslant 0}$ such that (R.1)–(R.9) hold true, Lemma 2 and $\mu > 0$ yield the existence of some $b > 0$ such that $\eta_0 = T_0 = 0$ and

$$\eta_n = \inf\{\tau_k > T_{n-1} + r \colon Z_{\tau_k-r} \geqslant -b\},$$
$$T_n = \inf\{k > \eta_n \colon S_k - S_{\eta_n} > b + rc\}$$

for $n \geqslant 1$ are a.s. finite $(\mathscr{F}_n)_{n \geqslant 0}$-times under each $\hat{P}_{x,z}$. By (R.8), we then infer that $Z_{\eta_n} \geqslant -b-rc$, which in turn implies $Z_{T_n} = 1$ for all $n \geqslant 1$. We further see with the help of (R.6) that, again under each $\hat{P}_{x,z}$, the M_{η_n} are independent for $n \geqslant 0$ and identically distributed for $n \geqslant 1$ with common distribution $\zeta = P(M_{\tau_1} \in \cdot)$. The same follows for the bivariate sequence $(M_{T_n}, Z_{T_n})_{n \geqslant 0}$, the common distribution for $n \geqslant 1$ being $\lambda \otimes \delta_1$,

$$\lambda \stackrel{\text{def}}{=} P_\zeta(M_{\tau(b+rc)} \in \cdot), \qquad \tau(b+rc) \stackrel{\text{def}}{=} \inf\{k \geqslant 1 \colon S_k - S_0 > b + rc\}.$$

From these facts we can easily conclude that $(T_n)_{n \geqslant 0}$ forms indeed a sequence of regeneration epochs for $(M_n, Z_n)_{n \geqslant 0}$, and thus Harris recurrence of the latter chain.

To prove the asserted equivalence choose now any *regular* sequence $(T_n)_{n\geqslant 0}$ of regeneration epochs for $(M_n, Z_n)_{n\geqslant 0}$ so that

$$\phi \overset{\text{def}}{=} \hat{E}_\zeta\Big(\sum_{j=0}^{T_1-1} \mathbb{1}_{\{(M_j,Z_j)\in\cdot\}}\Big), \qquad \zeta \overset{\text{def}}{=} P\big((M_{T_1}, Z_{T_1})\in\cdot\big),$$

forms the pertinent essentially unique σ-finite stationary measure. We note that the previously defined T_n need not be regular. Now $(T_n)_{n\geqslant 0}$ also forms a sequence of regeneration epochs for the marginal sequence $(M_n)_{n\geqslant 0}$ for which we only note concerning (R.2) that M_{n+1} depends on (M_n, Z_n) only through M_n. As a consequence, $\hat{E}_\zeta T_1$ must be infinite if $(M_n)_{n\geqslant 0}$ is null recurrent. To complete the proof of the equivalence it is therefore enough to show $\hat{E}_\zeta T_1 < \infty$ if $(M_n)_{n\geqslant 0}$ is positive recurrent, which we assume hereafter. We denote by ξ its stationary distribution.

Define

$$N_n(A) = \sum_{j=1}^{n} \mathbb{1}_{\{(M_j,Z_j)\in A\}}, \qquad A\in\mathfrak{E},$$

and $N_n = N_n(\mathcal{S}\times\{1\})$, the number of strictly ascending ladder epochs up to time n. As the first step, we show

$$\liminf_{n\to\infty} n^{-1} N_n \geqslant c \text{ a.s.} \qquad \text{for some } c>0;$$

here and in the following "a.s." means "$\hat{P}_{x,z}$-a.s. for all $(x, z)\in\mathcal{E}$". To that end, put $X'_n = X_n \wedge a$, where a is so large that $\mu' \overset{\text{def}}{=} E_\xi X'_1 > 0$. Assume that S'_n, Z'_n, N'_n, σ'_n have the obvious meanings and observe that each ladder epoch σ'_n for $(S'_n)_{n\geqslant 0}$ is also one for $(S_n)_{n\geqslant 0}$. Hence $N_n \geqslant N'_n$. Now consider the event

$$A = \{\liminf_{n\to\infty} n^{-1} N'_n \leqslant \mu'/3a\}.$$

Put $\tau'(b) = \inf\{n\geqslant\sigma_0\colon S'_n - S'_{\sigma'_0} > b\}$ for $b>0$. Since $(S'_n)_{n\geqslant 0}$ has a positive recurrent driving chain $(M_n)_{n\geqslant 0}$, straightforward arguments yield $n^{-1} S_n \to \mu'$ a.s., and then further

$$\frac{\tau'(b)}{b} \to \frac{1}{\mu'} \in (0, \infty) \text{ a.s.} \qquad (b\to\infty).$$

On the other hand, we have

$$\{N'_n \leqslant \mu' n/2a\} \subset \{\max_{1\leqslant j\leqslant n} (S'_j - S'_{\sigma'_0}) \leqslant \mu' n/2\} \subset \{\tau'(\mu' n/2) > n\}$$

by the boundedness of the X'_n, and therefore

$$\hat{P}_{x,z}(A) \leqslant \hat{P}_{x,z}\big(\limsup_{b\to\infty} b^{-1}\tau'(b) > 2/\mu'\big) = 0 \text{ a.s.}$$

Consequently,

(3.14) $$\liminf_{n\to\infty} n^{-1} N_n \geqslant \liminf_{n\to\infty} n^{-1} N_n' \geqslant \mu'/3a > 0 \text{ a.s.}$$

As the second step, we next prove the existence of a set $A_0 \in \mathfrak{S}$ such that

(3.15) $$\phi(A_0 \times \{1\}) \in (0, \infty) \quad \text{and} \quad \liminf_{n\to\infty} n^{-1} N_n(A_0 \times \{1\}) > 0 \text{ a.s.}$$

Since ϕ is σ-finite, we can find a partition $(A_k)_{k\geqslant 1}$ of $\mathscr{S}$ such that $\phi(A_k \times \{1\}) \in (0, \infty)$ for all $k \geqslant 1$. By the ergodic theorem,

$$\lim_{n\to\infty} n^{-1} N_n(B \times \mathbb{R}) = \lim_{n\to\infty} n^{-1} \sum_{j=1}^{n} \mathbb{1}_{\{M_j \in B\}} = \xi(B) \text{ a.s.}$$

for all $B \in \mathfrak{S}$. Hence we can choose k_1 sufficiently large such that

$$\limsup_{n\to\infty} n^{-1} N_n\Big(\big(\bigcup_{k>k_1} A_k\big) \times \{1\}\Big) \leqslant \hat{\xi}\big(\bigcup_{k>k_1} A_k\big) < \mu'/6a \text{ a.s.}$$

By combining this with (3.14) and upon setting $A_0 \stackrel{\text{def}}{=} \bigcup_{k=1}^{k_1} A_k$, we obtain $\phi(A_0 \times \{1\}) < \infty$ as well as

$$0 < \mu'/6a \leqslant \liminf_{n\to\infty} n^{-1} N_n - \limsup_{n\to\infty} n^{-1} N_n\Big(\big(\bigcup_{k>k_1} A_k\big) \times \{1\}\Big)$$

$$\leqslant \liminf_{n\to\infty} n^{-1} N_n(A_0 \times \{1\}),$$

which together give (3.15).

Finally, since $n^{-1} T_n \to \hat{E}_\xi T_1$ and $n^{-1} N_{T_n}(A_0 \times \{1\}) \to \phi(A_0 \times \{1\}) < \infty$ a.s. by the strong law of large numbers for random walks with stationary 1-dependent increments (use the 1-dependence of the cycles for the latter assertion), we conclude that

$$\infty > \lim_{n\to\infty} n^{-1} N_{T_n}(A_0 \times \{1\}) \geqslant (\lim_{n\to\infty} n^{-1} T_n) \cdot (\liminf_{n\to\infty} T_n^{-1} N_{T_n}(A_0 \times \{1\}))$$

$$\geqslant (\mu'/6a) \hat{E}_\xi T_1,$$

and hence $\hat{E}_\xi T_1 < \infty$.

It remains to prove the existence of a regular sequence of regeneration epochs $(T_n)_{n\geqslant 0}$ for both $(M_n, Z_n)_{n\geqslant 0}$ and $(M_n)_{n\geqslant 0}$ such that $Z_{T_n} = 1$ for all $n \geqslant 1$. To that end notice first that $\mathscr{S} \times \{1\}$ forms a recurrence set for $(M_n, Z_n)_{n\geqslant 0}$, and thus contains a regeneration set $\mathfrak{R} \times \{1\}$; see Theorem 5.2.2 in [8]. Clearly, $\mathfrak{R}$ can be chosen such that $\xi(\mathfrak{R}) \in (0, \infty)$, where ξ denotes the essentially unique stationary measure of $(M_n)_{n\geqslant 0}$. Let $(T_n)_{n\geqslant 0}$ be the resulting regular sequence of regeneration epochs when using Athreya and Ney's "coin-

-tossing" procedure. It is easily seen that the T_n are also regeneration epochs for $(M_n)_{n\geqslant 0}$, but we must still verify their regularity for the latter chain, i.e. the existence of some $A \in \mathfrak{S}$ such that

$$\hat{E}_{\zeta\otimes\delta_1}\Big(\sum_{j=0}^{T_1-1} \mathbb{1}_{\{M_j\in A\}}\Big) \in (0, \infty), \qquad \zeta = P(M_{T_1} \in \cdot).$$

We will do so for $A = \mathfrak{R}$. Consider the MRW $(M_{\tau_n}, S_{\tau_n})_{n\geqslant 0}$ with positive Harris recurrent driving chain $(M_{\tau_n})_{n\geqslant 0}$, where $\tau_0 = 0$ and τ_k denotes the k-th visit of $(M_n)_{n\geqslant 0}$ to $\mathfrak{R}$ for $k \geqslant 1$. The stationary distribution of $(M_{\tau_n})_{n\geqslant 0}$ is $\xi_{\mathfrak{R}} = \xi(\cdot\cap\mathfrak{R})/\xi(\mathfrak{R})$. Let $(M_{\tau_n}, Z_n^{\mathfrak{R}})_{n\geqslant 0}$ be the associated excursion chain. Although $(Z_n^{\mathfrak{R}})_{n\geqslant 0}$ and $(Z_{\tau_n})_{n\geqslant 0}$ are generally different sequences, we have $Z_n^{\mathfrak{R}} = Z_{\tau_n}$ if $Z_{\tau_n} = 1$, and thus $T_n = \tau_{\varrho_n}(A)$ for suitable ϱ_n. It is easily checked that $(\varrho_n)_{n\geqslant 0}$ forms a regular sequence of regeneration epochs for $(M_{\tau_n}, Z_n^{\mathfrak{R}})_{n\geqslant 0}$, the regularity following from

$$\hat{E}_{\zeta\otimes\delta_1}\Big(\sum_{k=0}^{\varrho_1-1} \mathbb{1}_{\{M_{\tau_k}\in\mathfrak{R}, Z_k^{\mathfrak{R}}=1\}}\Big) = \hat{E}_{\zeta\otimes\delta_1}\Big(\sum_{k=0}^{T_1-1} \mathbb{1}_{\{M_k\in\mathfrak{R}, Z_k=1\}}\Big) = \phi(\mathfrak{R}\times\{1\}) < \infty.$$

We know from the previous part of the proof that $(M_{\tau_n}, Z_n^{\mathfrak{R}})_{n\geqslant 0}$ is positive recurrent because this is true for $(M_{\tau_n})_{n\geqslant 0}$. Consequently, $\hat{E}_{\zeta\otimes\delta_1}\varrho_1 < \infty$, which together with $\hat{P}_{\zeta\otimes\delta_1}(M_{T_1}\in\mathfrak{R}) = 1$ further yields

$$\infty > \hat{E}_{\zeta\otimes\delta_1}\varrho_1 = \hat{E}_{\zeta\otimes\delta_1}\Big(\sum_{k=0}^{\varrho_1-1} \mathbb{1}_{\{M_{\tau_k}\in\mathfrak{R}\}}\Big) = \hat{E}_{\zeta\otimes\delta_1}\Big(\sum_{k=0}^{T_1-1} \mathbb{1}_{\{M_k\in\mathfrak{R}\}}\Big) > 0,$$

that is the desired result.

Proof of Theorem 1. (i) Notice first that it suffices to prove the assertions for $(M_n^*)_{n\geqslant 0}$ because then the same follows immediately for the other three chains when observing that, for each $n \geqslant 1$, (X_{n+1}^*, ν_{n+1}) depends on M_n^* and the history $(M_j^*, X_j^*, \nu_j)_{0\leqslant j<n}$ only through M_n^*.

We know now by Theorem 2 that $(M_n, Z_n)_{n\geqslant 0}$ has a sequence $(T_n)_{n\geqslant 0}$ of regeneration times, with associated filtration $(\mathscr{F}_n)_{n\geqslant 0}$ according to (R.2), such that each T_n is also a ladder epoch $\sigma_{T_n^*}$, say, for $(M_n, S_n)_{n\geqslant 0}$. It follows that $(T_n^*)_{n\geqslant 0}$ forms a sequence of regeneration times for $(M_n^*)_{n\geqslant 0}$ with associated filtration $(\mathscr{F}_{\sigma_n})_{n\geqslant 0}$ because T_n^* is clearly a stopping time with respect to that filtration and $Z_{T_n} = 1$ for every $n \geqslant 1$. This proves the Harris recurrence of $(M_n^*)_{n\geqslant 0}$.

If $(M_n)_{n\geqslant 0}$ is positive recurrent, i.e. $E_\zeta T_1 < \infty$, $\zeta = P(M_{T_1}\in\cdot)$, then $T_1^* \leqslant T_1$ implies $E_\zeta T_1^* < \infty$. Combining this with $M_{T_1^*}^* = M_{T_1}$, i.e. $\zeta = P(M_{T_1^*}^* \in \cdot)$, we conclude the positive recurrence of $(M_n^*)_{n\geqslant 0}$.

(ii) Using the strong Markov property, we infer that $(M_n^*, S_n^*)_{n\geqslant 0}$ and $(M_n^*, \sigma_n)_{n\geqslant 0}$ both constitute MRP's. Hence it remains to verify the lattice-type assertions. Let d denote the lattice-span of $(M_n, S_n)_{n\geq 0}$ and d^* that of $(M_n^*, S_n^*)_{n\geq 0}$ with associated shift functions γ and γ^*, respectively. Since the latter process forms a subsequence of the former one, we have $d^* \geq d$.

For the reverse inequality suppose $d^* > 0$ and note first that

$$(3.16)\quad \left|E\left(\exp(2\pi i(S_{T_n^*}^* - S_0^*)/d^*) \mid M_0^*, M_{T_n^*}^*\right)\right| = \left|E\left(\exp(2\pi i(S_{T_n} - S_0)/d^*) \mid M_0, M_{T_n}\right)\right| = 1 \quad P_{\xi^*}\text{-a.s.}$$

for all $n \geq 1$. On the other hand, recalling the final paragraph of Section 2, we may assume without loss of generality that $(T_n)_{n\geq 0}$ satisfies (R.9) in addition to (R.1)–(R.4). Consequently,

$$E_x\left|E\left(\exp(2\pi it(S_{T_1} - S_0)) \mid M_0, M_{T_1}\right)\right| < 1$$

for all $0 < |t| < 1/d$ and $x \in \mathcal{S}$, and thus

$$(3.17)\quad P_{\xi^*}\left(\left|E\left(\exp(2\pi it(S_{T_1} - S_0)) \mid M_0, M_{T_1}\right)\right| < 1\right) > 0$$

for all $0 < |t| < 1/d$. Now (3.16) and (3.17) together show that $d^* \leq d$, i.e. $(M_n, S_n)_{n\geq 0}$ and its subsequence $(M_n^*, S_n^*)_{n\geq 0}$ do indeed have the same lattice-span d. The simple argument (use the telescoping structure of condition (1.4)) that they must then also have the same shift function γ can be omitted.

In order to see that $(M_n^*, \sigma_n)_{n\geq 0}$ and $(M_n, n)_{n\geq 0}$ are of the same lattice-type, we note first that the latter process contains the former one as a subsequence. Moreover, $(T_n)_{n\geq 0}$ and $(T_n^*)_{n\geq 0}$ are again sequences of regeneration epochs for $(M_n, n)_{n\geq 0}$ and $(M_n^*, \sigma_n)_{n\geq 0}$, respectively. Hence the necessary arguments are the same as before involving the application of (R.9) for $(M_n, n)_{n\geq 0}$ instead of $(M_n, S_n)_{n\geq 0}$. We do not supply the details again.

(iii) If $\mu = E_\xi X_1 = E_\zeta S_{T_1} < \infty$, then $S_1^* \leq S_{T_1^*}^* = S_{T_1}$ clearly implies $E_\xi S_1^* < \infty$. For the converse it suffices to note that $X_1^+ \leq S_{\sigma_1} = S_1^*$ P_ξ-a.s.

The second equivalence follows directly from $E_{\xi^*} S_1^* = E_\zeta S_{T_1^*}^* = E_\zeta S_{T_1} = E_\xi X_1 = \mu$, where (2.4) and $S_{T_1^*}^* = S_{\sigma_{T_1^*}} = S_{T_1}$ have been utilized.

(iv) Here the assertion follows from $E_{\xi^*} \sigma_1 = E_\zeta \sigma_{T_1^*} = E_\zeta T_1$, where again (2.4) has been used.

4. The lattice-type of $(M_n, n)_{n\geq 0}$. Our purpose of this final section is to determine the lattice-type of the MRP $(M_n, n)_{n\geq 0}$, and thus of $(M_n, \sigma_n)_{n\geq 0}$ according to Theorem 1 (ii). The key is provided by the following lemma which characterizes the lattice-span of an arbitrary MRW $(M_n, S_n)_{n\geq 0}$ with Harris recurrent driving chain via *geometric sampling*. It is essentially a sharpened reformulation of Lemma A.6 in [3].

LEMMA 3. *Let $(M_n, S_n)_{n\geqslant 0}$ be an MRW with Harris recurrent driving chain $(M_n)_{n\geqslant 0}$ and lattice-span $d\in[0, \infty]$. Let ξ denote the stationary measure of $(M_n)_{n\geqslant 0}$ and η be an independent (under each P_x) geometric (1/2) variable. Then*

$$\begin{aligned}(4.1)\quad d^{-1} &= \inf\{t>0\colon E_x\left|E\left(\exp(2\pi i t S_\eta) \mid M_0, M_\eta\right)\right| = 1 \text{ for some } x\in\mathcal{S}\}\\ &= \inf\{t>0\colon E_x\left|E\left(\exp(2\pi i t S_\eta) \mid M_0, M_\eta\right)\right| = 1 \text{ for } \xi\text{-almost all } x\in\mathcal{S}\},\end{aligned}$$

where as usual $\inf\emptyset \overset{\text{def}}{=} \infty$, $\infty^{-1} \overset{\text{def}}{=} 0$ *and* $0^{-1} \overset{\text{def}}{=} \infty$.

Proof. It is evident that the first infimum is not bigger than the second one, which in turn is not bigger than d^{-1}. On the other hand, it is shown in Lemma A.6 in [3] that

$$E_x\left|E\left(\exp(2\pi i t S_\eta) \mid M_0, M_\eta\right)\right| = 1 \qquad \text{for some } x\in\mathcal{S} \text{ and } t>0$$

implies $d^{-1}\leqslant t$, whence (4.1) follows.

Now consider an arbitrary Harris chain $(M_n)_{n\geqslant 0}$, let $d\in N$ be its period, and denote by $\mathscr{C}_0, \ldots, \mathscr{C}_{d-1}$ its cyclic classes indexed in the correct transitional order. Put $\mathscr{C}_{kd+r} = \mathscr{C}_r$ for all $k\in N$ and define

$$\gamma_M\colon \mathcal{S}\to\{0, \ldots, d-1\}, \qquad x\mapsto \sum_{r=1}^{d-1}(d-r)\mathbb{1}_{\mathscr{C}_r}(x).$$

Notice that it is enough for the following result that $\mathscr{C}_0, \ldots, \mathscr{C}_{d-1}$, and thus also γ_M are only determined up to ξ-null sets. Notice further that $\gamma_M\equiv 0$ in the aperiodic case ($d=1$).

LEMMA 4. *Given a Harris chain $(M_n)_{n\geqslant 0}$ with period $d\in N$, the associated MRP $(M_n, n)_{n\geqslant 0}$ is d-arithmetic with shift function γ_M.*

Proof. Let d^* denote the lattice-span of $(M_n, n)_{n\geqslant 0}$. From the equality

$$E(e^{2\pi i t\eta} \mid M_0, M_\eta) = \sum_{n\geqslant 1} e^{2\pi i t n} P(\eta = n \mid M_0, M_\eta)$$

and the obvious fact that $P(\eta = kd+r \mid M_0, M_\eta) = 0$ P_ξ-a.s. for all $k\in N_0$ and $r=0, \ldots, d-1$ on the complement of $\{(M_0, M_\eta)\in A_r\}$, $A_r \overset{\text{def}}{=} \sum_{j=0}^{d-1}\mathscr{C}_j\times\mathscr{C}_{j+r}$, we infer that

$$\begin{aligned}(4.2)\quad & E(e^{2\pi i t\eta} \mid M_0, M_\eta)\\ &= \sum_{r=0}^{d-1}\Big(\mathbb{1}_{A_r}(M_0, M_\eta)\, e^{2\pi i t r}\sum_{k\geqslant 0} e^{2\pi i t k d} P(\eta = kd+r \mid M_0, M_\eta)\Big),\end{aligned}$$

and then $|E(e^{2\pi i\eta/d} \mid M_0, M_\eta)| = 1$ P_ξ-a.s. showing that $d^*\geqslant d$ with Lemma 4.

To get the reverse inequality, let us denote by γ the shift function of $(M_n, n)_{n\geqslant 0}$ and by ζ any probability measure equivalent to ξ. Then $P_\zeta(\eta-\gamma(M_0)+\gamma(M_\eta)\in d^* Z) = 1$ in combination with (4.2) and the independence of η and $(M_n)_{n\geqslant 0}$ gives

$$\begin{aligned} 1 &= E_\zeta \exp(2\pi i(\eta-\gamma(M_0)+\gamma(M_\eta))/d^*) \\ &= E_\zeta E\big(\exp(2\pi i(\eta-\gamma(M_0)+\gamma(M_\eta))/d^*) \mid M_0, M_\eta\big) \\ &= \sum_{r=0}^{d-1} \sum_{k\geqslant 0} \int_{\substack{\{(M_0,M_\eta)\in A_r,\\ \eta=kd+r\}}} \exp(2\pi i(r-\gamma(M_0)+\gamma(M_\eta))/d^*) \exp(2\pi ikd/d^*)\, dP_\zeta \\ &= \sum_{(r,k)\neq(0,0)} 2^{-kd-r} \int_{\{(M_0,M_{kd+r})\in A_r\}} \exp(2\pi i(r-\gamma(M_0)+\gamma(M_{kd+r}))/d^*) \\ &\qquad\qquad \times \exp(2\pi ikd/d^*)\, dP_\zeta, \end{aligned}$$

which can only hold true if $\exp(2\pi ikd/d^*) = 1$ *for all* $k\in N$, which in turn yields that d^* is a factor of d, in particular $d^* \leqslant d$. Together with $d^* \geqslant d$ from the previous part we thus obtain the asserted $d^* = d$. The reader can easily check that γ_M is a pertinent shift function.

Remark. Let us finally take another brief look at Janson's [7] example mentioned earlier in Remark (d) of Section 1. There $(M_n^*)_{n\geqslant 0}$ is 2-periodic on the state space $\{(0, 1), (1, 0)\}$ and $(M_n^*, S_n^*)_{n\geqslant 0} = (M_n^*, n)_{n\geqslant 0}$. Hence the latter process has lattice-span 2 and shift function $\gamma(0, 1) = 0 = 1-\gamma(1, 0)$. Theorem 1 (ii) in combination with Lemma 4 shows now that $(M_n, S_n)_{n\geqslant 0}$ is also 2-arithmetic, the shift function being an extension of the afore-mentioned one by setting $\gamma(0, 0) = 1-\gamma(1, 1) = 1$. Finally, the obvious aperiodicity of $(M_n)_{n\geqslant 0}$ implies that $(M_n, n)_{n\geqslant 0}$, and therefore $(M_n^*, \sigma_n)_{n\geqslant 0}$ is 1-arithmetic with shift function 0.

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