ON PAUL LÉVY'S ARC SINE LAW AND SHIGA–WATANABE’S TIME INVERSION RESULT

BY

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Abstract. Let \((X_t, \mathbb{P})\) be a symmetric real-valued \(H\)-self-similar diffusion starting at 0. We characterize the distributions of \(A_t\), the time spent on \((0, a)\) before time \(t\), and \(g_t\), the time of the last visit to 0 before \(t\). This gives a simple new proof to well-known results including P. Lévy’s arc sine law for Brownian motion and Brownian bridge and similar results for symmetrized Bessel processes. Our focus is more on simplicity of proofs than on novelty of results. Section 3 contains a generalization of T. Shiga’s and S. Watanabe’s theorem on time inversion for Bessel processes. We show that their result holds also for symmetrized Bessel processes.

0. Introduction. P. Lévy showed in [8] that for a Brownian motion starting at 0 both

\[ A_t = \int_0^1 I_{B_s > 0} \, du \quad \text{and} \quad g_t = \sup \{s < 1 \mid B_s = 0\} \]

are arc sine distributed. This result has been extended mainly in two directions: on one hand for Lévy processes (see [5]), on the other hand for self-similar Markov processes (see [1], [4], [6], and [10]). \(A_t\) remains beta distributed in many cases if \((B_t)\) is replaced by a more general Lévy process (see [5]) whereas \(g_t\) is beta distributed if \((B_t)\) is replaced by any self-similar Markov process (see Dynkin [4] and Lamperti [6]). In [1] the distribution of \(A_t\) was calculated for symmetrized Bessel processes (which are \(\frac{1}{2}\)-self-similar); it turned out not to be a beta distribution.

In this note we calculate the moments of \(g_1\) (Theorem 1 in Section 2). This gives a new proof of the fact that \(g_t\) is beta distributed for any Bessel process (cf. [4] and [6]). Thereafter we use this to calculate the moments of \(A_t\). We first show, by quite elementary methods, the connection between \(A_t\) and \(g_1\) (see Lemma 4 in Section 1). Under an independence assumption (A), which we make, this gives also a connection between the moments of \(A_t\) and \(g_t\). However, to show that (A) is valid for all symmetrized Bessel processes (Theorem 2 in
Section 4) we need a generalization of T. Shiga's and S. Watanabe's time inversion result (Proposition 1 in Section 3). We believe that this result might have some interest on its own, independently of the rest of the paper. Except in the first section we are most of the time working with symmetrized Bessel processes but the results are also valid for Bessel and Brownian bridges and for symmetric $H$-self-similar diffusions on the whole real line, $H > 0$ (see Remarks 5 and 6).

1. Notation and basic ideas. Throughout this paper $((X_t, P)$ denotes a real-valued process starting at 0 with continuous paths satisfying the following properties:

1. Symmetry, i.e. $(X_t) \sim P = (-X_t) \sim P$, and that $P(X_t = 0) = 0$ for $t > 0$.
2. $((X_t, P)$ is a strong Markov process.
3. $((X_t, P)$ is $H$-self-similar under $P$, i.e.

$$a^{-H}X_{at} \overset{d}{=} (X_t) \text{ under } P \text{ for all } a > 0.$$ 

Examples: Brownian motion and more general symmetrized Bessel processes with index $\nu \in (-1, 0)$ starting at 0. Here $H = 1/2$.

By a symmetrized Bessel process with index $\nu \in (-1, 0)$ starting at 0. Here $H = 1/2$.

They form exactly the class of diffusions fulfilling the properties 1, 2 and 3 in the case $H = 1/2$. Brownian motion is a special case, corresponding to the index $\nu = -1/2$. The processes fulfilling the properties 1, 2 and 3 for $H \neq 1/2$ can be obtained similarly from $H$-self-similar diffusions on $(0, \infty)$. See also Section 3, Remark 2 in Section 2 and Remark 5 in Section 4.

The following notation will be used repeatedly. Define for all $t > 0$

$$A_t := \int_0^t 1_{(X_u > 0)} du, \quad d^t := \inf \{s > t \mid X_s = 0\}, \quad g^t := \sup \{s < t \mid X_s = 0\},$$

and

$$\tau_0 := \inf \{s > 0 \mid X_s = 0\}.$$ 

If $X_s \neq 0$ for all $s > 0$, then $d^t$ and $\tau_0 := \infty, g^t := 0$.

Remark 1. Obviously, the property 3 implies that $A_t, d^t$ and $g^t$ are equal in law to $tA_1, td_1$ and $tg_1$, respectively.

**Lemma 1.**

$$A^n_t = n! \int_0^{t_n} \ldots \int_0^{t_2} 1_{(X_{t_n} > 0, \ldots, X_{t_2} > 0)} dt_1 \ldots dt_n$$

for all $t > 0$ and $n \geq 1$.

**Proof.** We use induction based on the formula $A^n_t = n \int_0^t A^{n-1}_s dA_s$. □
Notice that neither of the properties 1, 2 or 3 were used. For each \( n \geq 1 \) we thus have by Fubini's theorem
\[
\mu_n := E[A_1^n] = n! \int_0^{t_1} \ldots \int_0^{t_n} P(X_{t_1} > 0, \ldots, X_{t_n} > 0) dt_1 \ldots dt_n.
\]
Being bounded the distribution of \( A_1 \) is determined by its moments. In order to compute these we shall use the following simple identity. Note that only the property 1 is needed.

**Lemma 2.**

\[ \mu_1 = \frac{1}{2} \text{ and } 2\mu_n = \sum_{k=1}^{n-1} \binom{n}{k} (-1)^k \mu_k + 1 \quad \text{for every odd } n \geq 3. \]

**Proof.** The property 1 shows that \( A_1 \) and \( 1-A_1 \) are identically distributed under \( P \). The binomial expansion of \((1-A_1)^n\) immediately gives \((*)\). \( \blacksquare \)

The next result, which makes use of the properties 1 and 2, gives an expression for the distribution of \( g_t \) and establishes a link between the distributions of \( g_t \) and \( A_t \).

**Lemma 3.**

\[ P(g_t \leq s) = 4P(X_s > 0, X_t > 0) - 1 \quad \text{for all } 0 < s < t. \]

**Proof.** Let \( 0 < s < t \) be given. Using the strong Markov property at time point \( d^* \) and symmetry we get
\[
4P(X_s > 0, X_t > 0) = 4P(X_t > 0) - 4P(X_s < 0, X_t > 0)
\]
\[
= 2 - 4P(X_s < 0, d^* < t, X_t > 0) = 2 - 4E[P(X_{t-u} > 0) = 0, d^* < t, X_s < 0]
\]
\[
= 2 - 2P(X_s < 0, d^* < t) = 2 - (P(X_s < 0, d^* < t) + P(X_s > 0, d^* < t))
\]
\[
= 2 - P(d^* < t) = 1 + P(d^* > t) = 1 + P(g_t \leq s). \quad \blacksquare
\]

Using this we immediately get
\[
\mu_2 = 2 \int_0^1 \int_0^1 P(X_s > 0, X_t > 0) dt \, ds = \int_0^1 \int_0^1 \left( \frac{1}{2} + \frac{1}{2} P(g_s \leq t) \right) dt \, ds = \frac{1}{2} + \frac{1}{2} E\left[ \int_0^1 (1 - g_s) dt \right] = \frac{1}{2} + \frac{1}{2} E\left[ g_s \right] ds.
\]

From now on the self-similarity (property 3) will be fundamental. As noticed in Remark 1 we have for all \( n \geq 1 \) and \( t > 0 \)
\[
E[g_t] = t \cdot E[g_1] \quad \text{and} \quad E[A^*_t] = t^n \cdot E[A^*_1].
\]

Inserting above shows that the first three moments of \( A_1 \) are determined by
\[
\mu_1 = \frac{1}{2}, \quad \mu_2 = \frac{1}{2} - \frac{1}{4} E[g_1] \quad \text{and} \quad \mu_3 = \frac{3}{2} \mu_2 - \frac{3}{2} \mu_1 + \frac{1}{2} = \frac{1}{2} - \frac{3}{8} E[g_1].
\]
In order to compute the higher moments we need the following lemma:

**Lemma 4.**

\[
E[A_t^n] = E[A_t^{n-1}] - \frac{1}{2} E[A_{t-1}^{n-1}] = E[A_t^{n-1}] - \frac{1}{2} E[g_t^{-1} \left( \int_0^1 \mathbf{1}_{X_{u,1} > 0} du \right)^{n-1}]
\]

for all \( n \geq 1 \).

**Proof.** It suffices to prove the first equality, the second one then follows by substitution. Arguing like in Lemma 3 we have for all \( n \geq 1 \) and all \( 0 < t_1 < \ldots < t_n < 1 \)

\[
2P(X_{t_1} > 0, \ldots, X_{t_{n-1}} > 0) - 2P(X_{t_1} > 0, \ldots, X_{t_n} > 0) = 2P(X_{t_1} > 0, \ldots, X_{t_{n-1}} > 0, X_{t_n} < 0) = P(X_{t_1} > 0, \ldots, X_{t_{n-1}} > 0, d_{n-1} < t_n, X_{t_n} < 0) = P(X_{t_1} > 0, \ldots, X_{t_{n-1}} > 0) - P(X_{t_1} > 0, \ldots, X_{t_{n-1}} > 0, d_{n-1} > t_n) = P(X_{t_1} > 0, \ldots, X_{t_{n-1}} > 0) - P(X_{t_1} > 0, \ldots, X_{t_{n-1}} > 0, g_1 \leq t_{n-1}) = P(X_{t_1} > 0, \ldots, X_{t_{n-1}} > 0) - P(X_{t_1}, \ldots, X_{t_{n-1}}, > 0, g_1 \leq t_{n-1}/t_n),
\]

and therefore

\[
E[A_t^n] = n! \int \ldots \int_0^{t_2} P(X_{t_1} > 0, \ldots, X_{t_n} > 0) dt_1 \ldots dt_n = \frac{1}{2} E[A_t^{n-1}] + \frac{n!}{2} \left( \int \ldots \int_0^{t_2} P(X_{u_1} > 0, \ldots, X_{u_{n-1}} > 0, g_1 \leq u_{n-1}) du_1 \ldots du_{n-1} dt_n \right) = \frac{1}{2} E[A_t^{n-1}] + \frac{n!}{2} \left( \int \ldots \int_0^{t_2} P(X_{u_1} > 0, \ldots, X_{u_{n-1}} > 0, g_1 \leq u_{n-1}) du_1 \ldots du_{n-1} \right) = \frac{1}{2} E[A_t^{n-1}] + \frac{1}{2} \left( \int \mathbf{1}_{g_1 \leq \xi_1} du \right) = \frac{1}{2} E[A_t^{n-1}] - \frac{1}{2} E[A_{t-1}^{n-1}] - E[A_t^{n-2}] + \int_0^1 \mathbf{1}_{g_1 \leq \xi_1} du = E[A_t^{n-1}] - \frac{1}{2} E[A_{t-1}^{n-1}].
\]

Under the following assumption (A), which will be characterized later in Section 4,

\[
(A) \quad g_1 \text{ and } \int_0^1 \mathbf{1}_{X_{u,1} > 0} du = \int_0^1 \mathbf{1}_{g_1 \leq \xi_1} du \text{ are independent},
\]
the identity between the first and last terms in Lemma 4 can be rewritten as

\[ \mu_n = \mu_{n-1} - \frac{1}{2} \tilde{\mu}_{n-1} E[g_{1}^{-1}] \quad \text{for all } n \geq 1, \]

where

\[ \tilde{\mu}_n = E[\tilde{A}_n^1] = E[\left( \int_{0}^{1} 1_{\{X_{nt} > 0\}} \, dt \right)^n] \quad \text{for } n \geq 0. \]

This formula contains a lot of information. To see this define \((\tilde{X}_t) := (X_{tg_1})\). We see that \((\tilde{X}_t, P)\) is a continuous process starting at 0 and satisfying the property 1. Thus \((\tilde{\mu}_n)_{n \geq 1}\) fulfills (\(*\)). Combining (\(*\)), (\(**\)) and the fact that \(\mu_1, \tilde{\mu}_0\) and \(\tilde{\mu}_1\) are (trivially) known we see that in order to recursively compute \(\mu_n\) and \(\tilde{\mu}_n\) for all \(n\) it is enough to calculate the moments of \(g_1\). Thus under the assumption (A) the distribution of \(g_1\) determines that of both \(A_1\) and \(\tilde{A}_1\).

2. The distribution of \(g_1\). As concluded above it is important to be able to compute the moments of \(g_1\) under \(P\). A first step in this direction was already taken in Lemma 3 and using this result we shall now deduce the following known result:

**Theorem 1.** Let \((X_t, P)\) be a symmetrized Bessel process of index \(\nu \in (-1, 0)\) starting at 0, that is, a symmetric diffusion on the real line which on \((0, \infty)\) behaves like an ordinary Bessel process of index \(\nu\). Then

\[ g_1 \overset{d}{=} \beta(-\nu, \nu + 1) \quad \text{under } P, \]

i.e. the distribution of \(g_1\) under \(P\) is absolutely continuous with respect to the Lebesgue measure with density

\[ t \rightarrow \frac{1}{\pi} \sin(\pi |\nu|) \cdot t^{\nu-1} \cdot (1-t)^\nu \quad \text{for } t \in (0, 1). \]

In order to prove Theorem 1 we need the following

**Lemma 5.** We have \(g_1 \overset{d}{=} 1/d^1\) under \(P\).

**Proof.** Obviously, \(P(g_1 < t) = P(d^1 > 1).\) Using the self-similarity property 3 (see Remark 1) this becomes equal to \(P(td^1 > 1) = P(1/d^1 < t).\)

Using Lemma 5 and a well-known formula related to the gamma function we obtain for the moments of \(g_1\)

\[ E[g_1^n] = \frac{1}{(n-1)!} \int_{0}^{\infty} t^{n-1} E[\exp(-td^1)] \, dt \quad \text{for } n \geq 1. \]

Write \((X_t, P_x)\) for a symmetrized Bessel process of index \(\nu\) for some \(\nu \in (-1, 0)\), starting at \(x \in \mathbb{R}\). The Markov property and self-similarity imply for all \(t > 0\)

\[ E_0[\exp(-td^1)] = e^{-t} E_0[E_{X_t}[\exp(-t\tau_0)]] = e^{-t} E_0[E_{X_t}[\exp(-\tau_0)]] = e^{-t} E_0[E_{X_t}[\exp(-\tau_0)]]. \]
Rewriting $f(x) = E_x[\exp(-\tau_0)]$ for $x \in \mathbb{R}$ we have

$$E_0[g^n_t] = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-t} E_0[f(X_t)] \, dt \quad \text{for each } n \geq 1.$$  

**Proof of Theorem 1.** Let $((X_t), (P_x)_{x \in \mathbb{R}})$ be a symmetrized Bessel process of index $v$ for some $v \in (-1, 0)$. Let further $G^1(\cdot, \cdot)$ and $(p_t(\cdot, \cdot))_{t > 0}$ denote continuous versions of the corresponding 1-Green function and the transition density with respect to the speed measure, i.e.

$$G^1(x, y) = \int_0^\infty e^{-t} p_t(x, y) \, dt \quad \text{for all } x, y \in \mathbb{R}.$$  

It is known that

$$p_t(0, 0) = \frac{|v|}{\Gamma(1+v)} \cdot 2^{-v} t^{-1-v} \quad \text{for } t > 0,$$

and therefore

$$G^1(0, 0) = \frac{\Gamma(1-v)}{\Gamma(1+v)} \cdot 2^{-v}.$$  

The well-known theory (see [2]) shows that $f = c \cdot G^1(\cdot, 0)$, where the constant $c$ is determined by the equation $f(0) = 1$, i.e.

$$c = 2^v \cdot \frac{\Gamma(n-v)}{\Gamma(1+v)}.$$  

Using the Chapman-Kolmogorov equation we get

$$E_0[f(X_t)] = c E_0\left[\int_0^\infty e^{-u} p_u(X_t, 0) \, du\right] = c \int_0^\infty e^{-u} \int p_u(z, 0) \cdot p_t(0, z) \, dz \, du$$

$$= c \int_0^\infty e^{-u} \cdot p_{t+u}(0, 0) \, du = ce^t \int e^{-u} \cdot p_u(0, 0) \, du$$

and inserting this above we obtain

$$E_0[g^n_t] = \frac{c}{(n-1)!} \int_0^\infty t^{n-1} \left\{ \int e^{-u} \cdot p_u(0, 0) \, du \right\} dt = \frac{1}{n!} \int_0^\infty t^n e^{-t} \cdot p_t(0, 0) \, dt$$

$$= \frac{|v|}{\Gamma(1+v)} \cdot 2^{-v} \int_0^\infty t^{n-v-1} e^{-t} \, dt = \frac{1}{n! |v|} \frac{\Gamma(n-v)}{\Gamma(1+v)} = \frac{1}{n!} \frac{\Gamma(n-v)}{(-v)}$$

for every $n \geq 1$. This is exactly the $n$-th moment of a beta $(-v, v+1)$ distribution. Writing now $P = P_0$ we can conclude that the result is proved. \(\blacksquare\)

**Remark 2.** Using Theorem 1 we can calculate the moments of $g_1$ corresponding to any symmetric $H$-self-similar diffusion satisfying the basic assum-
tions 1, 2 and 3. Everything depends obviously only on the radial process, which is an $H$-self-similar diffusion on $[0, \infty)$. Now let $(r_t^x, P)$ denote a Bessel process on $[0, \infty)$ of index $\nu$ starting at 0. It is known (see [6] and [7]) that any $H$-self-similar diffusion $(R_t^x, P)$ on $[0, \infty)$ starting at 0 is identical in law to a process of the form

$$R_t^x = (r_{\sigma^2 t}^x)^{2H}$$

for some (unique) $\nu$ and $\sigma > 0$. This is due to the fact that $(R_t^x, P)$ on $(0, \infty)$ is governed by the differential operator

$$A = 2\sigma^2 H^2 r^{2-(1/H)} \frac{d^2}{dr^2} + 2\sigma^2 H(H + \nu) r^{1-(1/H)} \frac{d}{dr}.$$  

Using this relation and the following identities it is clear how results for Bessel processes can be used to give general results (assume for simplicity $\sigma^2 = 1$):

$$A_1^{R^x} = \int_0^1 I_{(R^x_s > 0)} ds = \int_0^1 I_{(r_s^{2H} > 0)} ds = \int_0^1 I_{(r_s^x > 0)} ds = A_1^x.$$  

Similarly,

$$g_1^{R^x} = \sup\{s < 1 \mid R_s^x = 0\} = \sup\{s < 1 \mid (r_s^x)^{2H} = 0\}$$

$$= \sup\{s < 1 \mid r_s^x = 0\} = g_1^x.$$  

Remark 3. According to the result of Lamperti [7], 0 is a regular boundary point for $(R_t^x)$ (that is, the process can hit 0 and can be started from 0) iff

$$\frac{1}{2}(1-(1/H))(4H^2) < 2H(H + \nu) < 2H^2$$

or, equivalently,

$$-1 < \nu < 0,$$

which is exactly the case when 0 is a regular boundary point for $(r_t^x)$.

3. On a result of T. Shiga and S. Watanabe. In this section we prove a time inversion result for symmetrized Bessel processes, analogous to the result which Shiga and Watanabe [12] showed for ordinary Bessel processes on $[0, \infty)$. This is needed in Section 4 to show that (A) is valid for any symmetrized Bessel process of index $\nu \in (-1, 0)$. Their result states:

(3.1) If $(r_t, P)$ is a Bessel process on $[0, \infty)$ starting at 0, then $(r_t)$ and $(t r_1)^{1/\nu}$ are equivalent diffusions under $P$.

It is well known that a similar result is true for Brownian motion. We shall prove the following generalization:

**Proposition 1.** Let $(X_t, P)$ be a symmetrized Bessel process of index $\nu \in (-1, 0)$ starting at 0. Then $(X_t)$ and $(t X_1)^{1/\nu}$ are equivalent diffusions under $P$.

**Proof.** Let $\nu \in (-1, 0)$ be given and let $(r_t, P_0)$ be a Bessel process on $[0, \infty)$ of index $\nu$ starting at 0. Denoting by $(e_{n,h})_{n \geq 1}$ an ordering of the excur-
ions from 0 for \((r_t, \mathcal{Q}_0)\) it is clear (compare with [11], exercise (2.16), p. 449) that \((X_t)\) under \(P\) is identical in law to \((Z_t, \bar{Q})\), which is defined by

\[ Z_t = Y_n e_n (t - s_n) \quad \text{if} \quad t \in S_n, \]

\[ \bar{Q} = \mathcal{Q}_0 \times \mathcal{Q}, \quad ((Y_n, n \geq 1), \mathcal{Q}) \]

is a Rademacher sequence and \(S_n = [s_n, s_n']\) is the excursion interval corresponding to \(e_n\) for each \(n\). Notice that \(Z_t = Y_n r_t\) if \(t \in S_n\) and that \(\bigcup S_n = [0, \infty)\) since \(r_0 = 0\). Obviously,

\[ tZ_{1/t} = Y_n t e_n ((1/t) - s_n) = Y_n tX_{1/t} \quad \text{if} \quad 1/t \in S_n. \]

According to [12], \((r_t)\) and \((tr_{1/t})\) have the same distribution under \(\mathcal{Q}_0\). Now, let \(0 < t_1 < \ldots < t_n\) be given. To prove that \((X_t)\) and \((tX_{1/t})\) or, equivalently, \((Z_t)\) and \((tZ_{1/t})\) have the same finite-dimensional distributions we need to verify that for a given set of subintervals \(I_1, \ldots, I_n\) of \([0, \infty)\) and elements \(i_1, \ldots, i_n\) in \((-1, 1)\) the following two probabilities are equal:

\[ P(\{|Z_{t_k}| \in I_k, \; \text{sign} Z_{t_k} = i_k, \; k = 1, \ldots, n\}) \]

and

\[ P(\{|t_k Z_{1/t_k}| \in I_k, \; \text{sign} t_k Z_{1/t_k} = i_k, \; k = 1, \ldots, n\}). \]

We shall consider only the case \(n = 2\) because this case includes all the difficulties (except the combinatorial ones) which occur in the general case. Write

\[ P(\{|Z_{t_k}| \in I_k, \; \text{sign} Z_{t_k} = i_k, \; k = 1, 2\}) = p_1 + p_2, \]

where \(p_1\) and \(p_2\) mean

\[ p_1 = P(\{|Z_{t_k}| \in I_k, \; \text{sign} Z_{t_k} = i_k, \; k = 1, 2; \; t_1, t_2 \in \mathbb{R}\}, \]

\[ t_1, t_2 \text{ belong to different excursions of } (r_t)) \]

and

\[ p_2 = P(\{|Z_{t_k}| \in I_k, \; \text{sign} Z_{t_k} = i_k, \; k = 1, 2; \; t_1, t_2 \in \mathbb{R}\}, \]

\[ t_1, t_2 \text{ belong to the same excursion of } (r_t)) \]

Due to the independence, \(p_1\) is equal to

\[ \frac{1}{4} \mathcal{Q}_0 (r_{t_k} \in I_k, \; k = 1, 2; \; t_1, t_2 \text{ belong to different excursions of } (r_t)) \]

\[ = \frac{1}{4} \mathcal{Q}_0 (r_{t_k} \in I_k, \; k = 1, 2; \; \exists s \in (t_1, t_2) \text{ such that } r_s = 0), \]

and \(p_2\) is equal to

\[ \mathcal{Q} (i_1, i_2) \mathcal{Q}_0 (r_{t_k} \in I_k, \; k = 1, 2; \; t_1, t_2 \text{ belong to the same excursion of } (r_t)) \]

\[ = \mathcal{Q} (i_1, i_2) \mathcal{Q}_0 (r_{t_k} \in I_k, \; k = 1, 2; r_s > 0 \; \forall s \in (t_1, t_2)), \]

where

\[ \mathcal{Q} (i_1, i_2) = 1/2 \text{ if } i_1 = i_2 \quad \text{and} \quad \mathcal{Q} (i_1, i_2) = 0 \text{ if } i_1 \neq i_2. \]
Using the result of Shiga and Watanabe (3.1) we obtain
\[ Q_0(r_k \in I_k, k = 1, 2; \exists s \in (t_1, t_2) \text{ such that } r_s = 0) \]
\[ = Q_0(t_k r_1/k \in I_k, k = 1, 2; \exists s \in (t_1, t_2) \text{ such that } r_1/s = 0) \]
\[ = Q_0(t_k r_1/k \in I_k, k = 1, 2; \exists 1/s \in (t_2, 1/t_1) \text{ such that } r_1/s = 0) \]
\[ = Q_0(t_k r_1/k \in I_k, k = 1, 2; \exists s \in (t_2, 1/t_1) \text{ such that } r_s = 0) \]
\[ = Q_0(t_k r_1/k \in I_k, k = 1, 2; 1/t_1, 1/t_2 \text{ belong to different excursions of } (r_n)) \]
which, using the independence, implies that \( p_1 \) is equal to
\[ P(t_k Z_{1/n} \in I_k, \text{ sign } t_k Z_{1/n} = i_k, k = 1, 2; \]
\[ 1/t_1, 1/t_2 \text{ belong to different excursions of } (r_n). \]
Similarly, we can show that \( p_2 \) is equal to
\[ P(t_k Z_{1/n} \in I_k, \text{ sign } t_k Z_{1/n} = i_k, k = 1, 2; \]
\[ 1/t_1, 1/t_2 \text{ belong to the same excursion of } (r_n). \]
Adding up we get
\[ P(t_k Z_{1/n} \in I_k, \text{ sign } Z_{1/n} = i_k, k = 1, 2) = P(t_k Z_{1/n} \in I_k, \text{ sign } t_k Z_{1/n} = i_k, k = 1, 2), \]
which implies that the two-dimensional marginal distributions of \((Z_1)\) and \((tZ_{1/n})\) are the same. As remarked above, the same kind of argument shows that all the finite-dimensional distributions of \((Z_1)\) and \((tZ_{1/n})\) are the same. Thus \((X_1)\) and \((tX_{1/n})\) have the same distribution under \( P \). Finally, since \((X_1, P)\) is continuous and Markovian with respect to a Feller semigroup, the continuity and the strong Markov property of \((tX_{1/n})\) under \( P \) are proved by using standard arguments.

4. The independence assumption (A). In this section we shall use the result of Section 3 and show that the assumption (A) is valid:

**Theorem 2.** Let \( ((X_t), P) \) be a symmetrized Bessel process of index \( \nu \in (-1, 0) \) starting at 0. Then (A) is satisfied.

**Proof.** Define \( (Y_t) := (tX_{1/n}). \) As is well known for Brownian motion and proved in Proposition 1 (Section 3) in the general case \((X_t)\) and \((Y_t)\) are equivalent diffusions under \( P \). Define \( T := d^1(Y) \). Then \( T \) is a finite stopping time and \( P(Y_T = 0) = 1. \) The strong Markov property implies that the distribution of \((Y_{T + h})_{h > 0} \) under \( P(\cdot | T = a) \) equals the distribution of \((Y_{h > 0}) \) under \( P \) for any \( a > 1. \) Therefore, since \((Y_t) \) under \( P \), and thus also \((Y_{a + t}) \) under \( P(\cdot | T = a) \) is \( \frac{1}{2} \)-self-similar, we have for any \( a > 1 \)
\[ (Y_{a + t})_{t > 0} \overset{d}{=} (a^{-1/2} Y_{a + t})_{t > 0} \text{ under } P(\cdot | T = a), \]
which implies that

\((1-t) Y_{a+(1-t)/a} | 0 \leq t \leq 1\) and \((a^{-1/2} (1-t) Y_{a+(at)/(1-t)} | 0 \leq t \leq 1)\)

have the same distribution under \(P(\cdot | T = a)\) for all \(a > 1\). Consider the law of

\(((1-t) Y_{a+(1-t)/a} | 0 \leq t \leq 1)\) under \(P(\cdot | T = a)\).

For all \(a > 1\) this is the law of \((Y_{a+(a)(1-a)/2})_{t \geq 0}\), under \(P(\cdot | T = a)\), conditioned to hit 0 at \(t = 1\) ("bridge of \(Y\)'). This statement is straightforward in the case of Brownian motion; for the general Bessel case see [9], Theorem 5.8, p. 324. Rewriting in terms of \((X_i)\) shows that the law of

\(\left(\frac{1}{2} X_{u/a} | 0 \leq u \leq 1\right)\) under \(P(\cdot | T = a)\)

is the same for all \(a > 1\). But \(T = d^1(Y) = g_1^{-1}\), and so the distribution of

\(\left(g_1^{-1/2} X_{u/a} | 0 \leq u \leq 1\right)\) under \(P(\cdot | T = a) = P(\cdot | g_1 = a^{-1})\)

does not depend on \(a\), which means that \(g_1\) and \(\left(g_1^{-1/2} X_{u/a} | 0 \leq u \leq 1\right)\) are independent under \(P\). This immediately proves the statement. \(\square\)

Remark 4. If \((X_i)\) is a Brownian motion, then \(\left(g_1^{-1/2} X_{u/a} | 0 \leq u \leq 1\right)\) is a Brownian bridge (see [11]). Similarly, we can construct bridges from symmetrized Bessel processes. Except the moments of \(A_t\), we can use \((*)\) and \((***)\) to calculate the moments of \(\tilde{A}_t\), which correspond to the bridge process \(\left(g_1^{-1/2} X_{u/a} | 0 \leq u \leq 1\right)\).

Remark 5. The results of Proposition 1 and Theorem 2 are also valid in the general \(H\)-self-similar case. Obviously, any \(H\)-self-similar symmetric diffusion can be constructed from the radial excursions in the same way as the symmetrized Bessel process is constructed from the ordinary Bessel process; and using Remark 2 of Section 2 and the \(H\)-self-similarity we can show that \((R_t)\) and \((t^{1/2} R_{1/t})\) are equivalent, where \((R_t)\) is an \(H\)-self-similar diffusion on \([0, \infty)\) starting at 0. Using these two facts it is easily seen that the arguments used in the proof of Proposition 1 and Theorem 2 are applicable in the general case.

Acknowledgement. The authors want to thank Paavo Salminen for reading an earlier version of this manuscript and for his useful comments.

REFERENCES


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Received on 24.11.1998;
revised version on 14.2.2000