A CHARACTERIZATION OF SIGN-SYMMETRIC LIOUVILLE-TYPE DISTRIBUTIONS

BY

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Abstract. Sign-symmetric Liouville-type distributions on $n$-dimensional space are characterized by certain $(n-1)$-dimensional distribution of quotients in a special form.

1. Introduction. In 1996 Gupta et al. [1] introduced a family of multivariate distributions which is an important generalization of many classes of distributions. It may be obtained by the following construction. For $\alpha, \beta > 0$ let $\mathcal{I}(\alpha, \beta)$ denote a distribution with probability density function

$$f(z) := \frac{\alpha}{2^{\beta/(\alpha)}} |z|^{\beta-1} \exp(-|z|^\alpha), \quad z \in \mathbb{R}.$$ 

When $Z_1, \ldots, Z_n$ are mutually independent, real-valued random variables and $Z_i \sim \mathcal{I}(\alpha_i, \beta_i)$ for some positive parameters $\alpha_i$ and $\beta_i$ ($i = 1, \ldots, n$), then the distribution of the vector

$$(1.1) \quad (X_1, \ldots, X_n) := \left(\frac{Z_1 \cdot \Theta^{1/\alpha_1}}{\left(\sum_{j=1}^{n} |Z_j|^\alpha_j\right)^{1/\alpha_1}}, \ldots, \frac{Z_n \cdot \Theta^{1/\alpha_n}}{\left(\sum_{j=1}^{n} |Z_j|^\alpha_j\right)^{1/\alpha_n}}\right),$$

where $\Theta$ is a positive random variable independent of

$$(1.2) \quad (U_1, \ldots, U_n) := \left(\frac{Z_1}{\left(\sum_{j=1}^{n} |Z_j|^\alpha_j\right)^{1/\alpha_1}}, \ldots, \frac{Z_n}{\left(\sum_{j=1}^{n} |Z_j|^\alpha_j\right)^{1/\alpha_n}}\right).$$

is called the sign-symmetric Liouville-type distribution and denoted by $\mathcal{IL}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta)$. (Besides, the distribution of the vector (1.2) is called the sign-symmetric Dirichlet-type distribution and denoted by $\mathcal{ID}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n).$)

The problem of explaining relations between distributions of random vectors $X = (X_1, \ldots, X_n)$ and quotients $(X_1/X_n, \ldots, X_{n-1}/X_n)$ has a long history. In one of the recent investigations in this field Wesołowski [3] proved a theorem characterizing symmetrically invariant two-dimensional distributions by the Cauchy distribution of quotients. This result was generalized to finite-dimensional $\alpha$-symmetrically invariant distributions by Szabłowski [2], who con-
sidered a certain Cauchy-like (so-called α-Cauchy) distribution of the quotients. On the other hand, Wesolowski [4] proved that a distribution of an α-spherical random vector is uniquely determined by a distribution of quotients. In this paper, methods adapted from [2] and [4] are applied to obtain a characterization of the sign-symmetric Liouville-type distribution that generalizes previous results.

We will use the following notation: If \( x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \), then

\[
\|x\|_\alpha := \left( \sum_{i=1}^{n} |x_i|^\alpha \right)^{1/\alpha}.
\]

For \( x \in \mathbb{R} \) and \( q > 0 \), \( x^{(q)} := \text{sign}(x) \cdot |x|^q \). Throughout the paper \( \alpha, \ldots, \alpha \) and \( \beta_1, \ldots, \beta_n \) are positive parameters and \( p_i := \sum_{j=1}^{n} \beta_j / \alpha_j \) for \( i = 1, \ldots, n \).

2. Characterization.

**Definition 1.** Let \( a_1, \ldots, a_n, b_1, \ldots, b_{n+1} \geq 0 \). We say that a random vector \( X = (X_1, \ldots, X_n) \) has a distribution \( \mathcal{D}(a_1, \ldots, a_n; b_1, \ldots, b_{n+1}) \) if its joint density function is

\[
\frac{\prod_{i=1}^{n} a_i \Gamma \left( \sum_{i=1}^{n+1} b_i \right)}{\sum_{i=1}^{n+1} \Gamma (b_i) \prod_{i=1}^{n} |x|^{a_i b_i - 1} \left( \sum_{i=1}^{n} |x|^a_i + 1 \right)^{-\sum_{i=1}^{n+1} b_i}}.
\]

The distribution \( \mathcal{D} \) is a generalization of α-Cauchy distribution defined by Szabowski [2]. More specifically, we have the following

**Remark 1.** A random vector \( (X_1, \ldots, X_n) \sim \mathcal{D}(\alpha, \ldots, \alpha; 1/\alpha, \ldots, 1/\alpha) \) has the \( n \)-dimensional α-Cauchy distribution (\( \alpha > 0 \)).

On the other hand, the distribution introduced by Definition 1 is a special case of the sign-symmetric Liouville distribution. Applying Proposition 3.2 from [1] we get

**Corollary 1.** A random vector \( (X_1, \ldots, X_n) \sim \mathcal{D}(a_1, \ldots, a_n; b_1, \ldots, b_{n+1}) \) has the distribution \( \mathcal{L}(a_1, \ldots, a_n; b_1, \ldots, b_{n+1}; \Theta) \), where \( \Theta \) has the inverted beta-distribution \( \mathcal{B} \left( \sum_{i=1}^{n} b_i, b_{n+1}, 1 \right) \), i.e. the probability density function of \( \Theta \) is

\[
f(r) = \frac{1}{B \left( \sum_{i=1}^{n} b_i, b_{n+1} \right) \prod_{i=1}^{n} \beta_i + 1 \left( \frac{1}{r + 1} \right)^{\sum_{i=1}^{n} b_i}}, \quad 0 < r < \infty,
\]

\( B \) denoting the Euler beta-function.

The main result of this paper is

**Theorem 1.** A random vector \( X \) without an atom at 0 has the sign-symmetric Liouville-type distribution \( \mathcal{L}(a_1, \ldots, a_n; \beta_1, \ldots, \beta_n; \Theta) \) if and only if the following three conditions hold:

(i) \( X \sim -X \) (here the symbol \( \sim \) denotes the equidistribution);
(ii) for some \( j \in \{1, \ldots, n \} \) and for some \( \alpha > 0 \) the random vector
\[
Y_j := \left( \frac{X_1^{(x_1/\alpha)}}{X_j^{(x_j/\alpha)}}, \ldots, \frac{X_j^{(x_j-1/\alpha)}}{X_j^{(x_j/\alpha)}}, \frac{X_j^{(x_j+1/\alpha)}}{X_j^{(x_j/\alpha)}}, \ldots, \frac{X_n^{(x_n/\alpha)}}{X_j^{(x_j/\alpha)}} \right)
\]
has the \((n-1)\)-dimensional distribution \( \mathcal{D} (\alpha, \ldots, \alpha; \beta_1/\alpha_1, \ldots, \beta_n/\alpha_n) \);

(iii) \( Y_j \) and \( \sum_{i=1}^{n} |X_i|^{\alpha_i} \) are independent.

It is easily seen that sign-symmetric Liouville distributions contain \( \alpha \)-symmetrically invariant distributions as a subclass. Hence and in view of Remark 1, Theorem 1 is an important generalization of Szablowski's result [2].

3. Auxiliary results and proofs. We begin with three technical lemmas (an easy proof of the first one is left to the reader):

**Lemma 1.** If \( q > 0 \), then \( Z \sim \mathcal{F} (\alpha, \beta) \) if and only if \( Z^{(q)} \sim \mathcal{F} (\alpha/q, \beta/q) \).

**Lemma 2.** Let \( Z_1, \ldots, Z_n \) be mutually independent, real-valued random variables and \( Z_i \sim \mathcal{F} (\alpha_i, \beta_i) \) for some positive parameters \( \alpha_i \) and \( \beta_i \) (\( i = 1, \ldots, n \)). Fix \( j \in \{1, \ldots, n \} \) and let \( q_i > 0 \) for \( i \neq j \). Then the joint density function of the random vector
\[
Z = \left( \frac{Z_j^{(q_j/q_1)}}{Z_j^{(q_j/q_1)}}, \ldots, \frac{Z_j^{(q_j-1/q_j)}}{Z_j^{(q_j/q_1)}}, \frac{Z_j^{(q_j+1/q_j)}}{Z_j^{(q_j/q_1)}}, \ldots, \frac{Z_j^{(q_j/q_1)}}{Z_j^{(q_j/q_1)}} \right)
\]
is
\[
(3.1) \quad \prod_{i \neq j} g_i (z_i) \prod_{i \neq j} \frac{\alpha_j}{2 \Gamma (\beta_j/\alpha_j)} \left| z \right|^{\beta_j - 1} \exp ( - \left| z \right|^{\beta_j} ) \, dz_i,
\]
where \( g_i \) stands for the probability density function of \( Z_i^{(q_j/q_i)} \).

**Proof.** Let \( f : \mathbb{R}^{n-1} \to \mathbb{R} \) be a bounded function. Since \( Z_i \) are independent, we see that
\[
\mathbb{E} f (\mathbf{Z}) = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-2}} f \left( \frac{z_1}{z_i^{(q_j/q_i)}}, \ldots, \frac{z_j-1}{z_j^{(q_j/q_j)}}, \frac{z_j+1}{z_j^{(q_j/q_j)}}, \ldots, \frac{z_n}{z_j^{(q_j/q_j)}} \right)
\]
\[
\times \prod_{i \neq j} g_i (z_i) \, dz_1 \ldots dz_{j-1} \, dz_{j+1} \ldots dz_n \cdot g (z) \, dz,
\]
where \( g \) is the probability density function of \( Z_j \). Let (with \( z \) fixed)
\[
x_i := z_i^{(q_j/q_i)} \quad \text{for} \ i \neq j.
\]
The Jacobian of this transformation is \( \prod_{i \neq j} z_i^{(q_j/q_i)} \). Formula (3.1) is now easily seen. \( \square \)

**Lemma 3.** Let \( X = (X_1, \ldots, X_n) \sim \mathcal{F}^\mathcal{D} (\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta) \). Fix \( j \in \{1, \ldots, n \} \) and let \( q_i > 0 \) for \( i \neq j \). The random vector
\[
\tilde{X} = \left( \frac{X_1^{(q_1/q_1)}}{X_j^{(q_j/q_1)}}, \ldots, \frac{X_j^{(q_j-1/q_j)}}{X_j^{(q_j/q_1)}}, \frac{X_j^{(q_j+1/q_j)}}{X_j^{(q_j/q_1)}}, \ldots, \frac{X_n^{(q_n/q_1)}}{X_j^{(q_j/q_1)}} \right)
\]
has the distribution \( \mathcal{D} (q_1, \ldots, q_j-1, q_j+1, \ldots, q_n; \beta_1/\alpha_1, \ldots, \beta_n/\alpha_n) \).
Proof. Since for every \( k \in \{1, \ldots, j-1, j+1, \ldots, n \} \)
\[
\frac{X_k^{(a_k/q_k)}}{X_j^{(a_j/q_j)}} = \frac{Z_k^{(a_k/q_k)}}{Z_j^{(a_j/q_j)}},
\]
the probability density function of \( X_k \) is defined by (3.1). Consequently,
\[
g(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) = \int \prod_{i \neq j} x_i^{(a_i/q_i)} \left| \prod_{i \neq j} \frac{q_i}{2} \Gamma \left( \beta_i/\alpha_i \right) \right| x_i z^{(a_i/q_i)}(\beta_i/\alpha_i) \gamma_i \alpha_i^{-1} \exp \left( -z x_i^{(a_i/q_i)} \right) \frac{\alpha_j}{2} \Gamma \left( \beta_j/\alpha_j \right) [z]^{\beta_j-1} \exp \left( -[z]^\gamma \right) dz.
\]
Using symmetry and the integral formula \( \mu, \nu > 0 \)
\[
\int_0^{\infty} x^{\nu-1} \exp \left( -\mu x^\nu \right) dx = \frac{1}{|p|} \Gamma \left( \frac{\nu}{\mu} \right),
\]
we get
\[
\int_R |z|^{\alpha_j \gamma_j - 1} \exp \left[ -|z|^{\alpha_j} (\sum_{i \neq j} |x_i|^{q_i} + 1) \right] dz = \frac{2}{\alpha_j} (\sum_{i \neq j} |x_i|^{q_i} + 1)^{-\gamma_j} \Gamma \left( \gamma_j \right).
\]
Therefore
\[
g(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)
\]
\[
= \frac{\alpha_j}{2^n} \prod_{i \neq j} q_i \prod_{i \neq j} \Gamma \left( \beta_i/\alpha_i \right) \left( \sum_{i \neq j} |x_i|^{q_i} + 1 \right)^{-\gamma_j} \Gamma \left( \gamma_j \right).
\]
Now we are in a position to prove Theorem 1.

Proof of Theorem 1. Let \( \tilde{X} := (X_1^{(a_1/q_1)}, \ldots, X_n^{(a_n/q_n)}) \) for some \( \alpha > 0 \). Then \( \tilde{X} = U^{\alpha^2} \cdot \Theta^{1/\alpha} \), where
\[
U^{\alpha} := \left( Z_1^{(a_1/q_1)}, \ldots, Z_n^{(a_n/q_n)} \right) / \left( \sum_{i=1}^n \left| Z_i^{(a_1/q_1)} \right|^{\alpha} \right)^{1/\alpha},
\]
\( U_\alpha \) and \( \Theta \) are independent. From Lemma 1 we get
\[
U_\alpha \sim \mathcal{G} \left( \alpha, \ldots, \frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_n}{\alpha_n} \right)
\]
and
\[ \bar{X} \sim \mathcal{I}(\alpha, \ldots, \alpha; \frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_n}{\alpha_n}; \Theta). \]

Furthermore,
\[ X \sim \mathcal{I}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta) \Leftrightarrow \bar{X} \sim \mathcal{I}(\alpha, \ldots, \alpha; \frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_n}{\alpha_n}; \Theta). \]

Wesolowski [4] showed the following fact. Let the distributions of random vectors
\[ ((-1)^{e_1} Y_1, \ldots, (-1)^{e_n} Y_n), \]
which have been constructed from a random vector \( Y = (Y_1, \ldots, Y_n) \) concentrated on the unit \( \alpha \)-sphere \( S_\alpha := \{ x \in \mathbb{R}^n : \| x \|_\alpha = 1 \} \), coincide for any \( (e_1, \ldots, e_n) \in \{0, 1\}^n \) and let a random vector \( P = (P_1, \ldots, P_n) \) take the form \( P = R Y \) (for some positive random variable \( R \) independent of \( Y \)). Then the distribution of the vector \( P \) is uniquely determined by the distribution of the quotients \( (P_1/P_j, \ldots, P_{j-1}/P_j, P_{j+1}/P_j, \ldots, P_n/P_j), j = 1, \ldots, n \). Since \( \bar{X} \) obviously satisfies the assumptions of the above statement, applying Lemma 3 we obtain the result of Theorem 1.

Acknowledgement. The author is indebted to P. J. Szablowski for valuable suggestions and remarks and to J. Wesolowski for showing the contents of paper [4] prior to publication.

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Received on 4.6.1998; revised version on 10.12.1999