ONE-DIMENSIONAL SYMMETRIC STABLE
FEYNMAN–KAC SEMIGROUPS

BY
H. BYCZKOWSKA* AND T. BYCZKOWSKI* (WROCŁAW)

Abstract. We investigate here one-dimensional Feynman–Kac semigroups based on symmetric α-stable processes. We begin with establishing the properties of Green operators of intervals and halflines on functions from the Kato class. Then we provide a sufficient condition for gaugeability of the halfline (−∞, b) and evaluate the critical value β.

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1. INTRODUCTION

In the present paper we continue studying, initiated in Section 7 of [4], the one-dimensional Feynman–Kac semigroups based on symmetric α-stable stochastic Lévy processes. In connection with this topic and the conditional gauge theorem we also mention [3], [5] and [6].

Let note that if $\alpha \geq 1 = d$, we deal with recurrent processes; thus many previously known objects take on different meanings or have different properties. Therefore, in Section 3, we check that the well-known characterization of functions in Kato class $J^\alpha$ (see [10]) remains valid also for the recurrent case. Although the transient case ($\alpha < d$) is well known, we include it here for the sake of completeness. In Section 4 we then establish properties of Green potentials of intervals and halflines. In Section 5 we consider the gauge function $u(x, y) = \mathbb{E}^x e_q(\tau_{(-\infty, y)})$ and investigate conditions assuring that $u(x, b) < \infty$ for $x < b$ (gaugeability of the sets $(-\infty, b)$). The procedure provides us with some estimates of the critical value $\beta$, i.e. the maximal value $y_0$ for which $u(x, y) < \infty$, for $x < y < y_0$. In the end of the paper we estimate $\beta$ for $q = 1_{(-\epsilon, \epsilon)}$.

* Institute of Mathematics, Wroclaw University of Technology. Research partially supported by KBN grant 2 P03A 028 16.
Let us mention that regarding potential kernels we rely on [2]; for general facts about Markov processes the reader is referred to [1]. We also apply some (unpublished) results from [9].

2. PRELIMINARIES

We adopt here the notation and terminology from [4].

We begin with some elementary facts concerning the density function of the symmetric $\alpha$-stable process. Let $p_t$ be the density function of the symmetric $\alpha$-stable process $X_t$ in $\mathbb{R}^d$, starting from 0. The (strict) $\alpha$-stability of $X_t$ yields that the distribution of $s^{-1/\alpha} X_{ts}$ is the same as that of $X_t$. We refer to this as to the scaling property of $X_t$. As a direct consequence we obtain

$$p_t(x) = s^{d/\alpha} p_{ts}(s^{1/\alpha} x).$$

Putting $s = t^{-1}$ in (1), we get

$$p_t(x) \leq t^{-d/\alpha} \sup_{x \in \mathbb{R}^d} p_t(x) = Ct^{-d/\alpha}.$$

Using the inverse Fourier transform we obtain $C = \Gamma(d) \omega_d/(2\pi)^d$.

Moreover, it is well known that

$$p_t(x) \leq C_d |x|^{-\alpha-d}.$$

Thus, we have

$$p_t(x) \leq C_d t |x|^{-\alpha-d}.$$

Applying these estimates we obtain

**LEMMA.** For all $t > 0$ the following holds:

$$\int_0^t p_s(x) ds = |x|^{\alpha-d} \int_0^t p_u(x/|x|) du \leq C_d t^2 |x|^{-\alpha-d}.$$

For $d = 1 < \alpha$ we obtain

$$\int_0^t p_s(x) ds \leq \frac{\alpha}{\pi (\alpha - 1)} t^{1-1/\alpha}.$$

We now put for $0 < \alpha < 2$ and $d \neq \alpha$:

$$K_\alpha(x) = \frac{\omega_d (d, \alpha)}{|x|^{d-\alpha}};$$

if $d = 1$ we get

$$K_\alpha(x) = \frac{|x|^\alpha - 1}{2 \Gamma(\alpha) \cos (\pi \alpha/2)}.$$
For $\alpha = 1 = d$ we put

\begin{equation}
K_1(x) = \frac{1}{\pi} \ln \frac{1}{|x|}.
\end{equation}

Let us mention that if $\alpha \geq d = 1$, then the $\alpha$-stable process is no longer transient, so its (free) Green function on the whole real line is no longer properly defined; in that case we refer to [2] for an appropriate interpretation of (4) and (5).

3. KATO CLASS $\mathcal{F}^\alpha$

**Definitions.** We say that a Borel function $q$ on $\mathbb{R}^d$ belongs to the Kato class $\mathcal{F}^\alpha$ if

\begin{equation}
\limsup_{|x| \to \infty} \sup_{|y| \leq \gamma} |K_\alpha(x - y)q(y)|dy = 0;
\end{equation}

we write $q \in \mathcal{F}_{\text{loc}}^\alpha$ if for every bounded Borel set $B$ we have $1_Bq \in \mathcal{F}^\alpha$.

**Properties of the class $\mathcal{F}^\alpha$.**

(i) We have $L^\alpha(\mathbb{R}^d) \subseteq \mathcal{F}^\alpha$. If $f \in L^\alpha(\mathbb{R}^d)$ and $q \in \mathcal{F}^\alpha$, then $fq \in \mathcal{F}^\alpha$.

(ii) If $q \in \mathcal{F}^\alpha$, then $\sup_{x \in \mathbb{R}^d} \int_{|x - y| \leq 1} |q(y)|dy < \infty$. Hence, if $q \in \mathcal{F}_{\text{loc}}^\alpha$, then $q \in L_{\text{loc}}^\alpha(\mathbb{R}^d)$.

(iii) If $q \in \mathcal{F}^\alpha$, then $\sup_{x \in \mathbb{R}^d} \int_{|x - y| \leq 1} |x - y|^{-a-d}|q(y)|dy < \infty$.

**Proof.** Condition (i) follows directly by the definition. We verify (ii). For $0 < \alpha \leq d$ and $0 < \gamma \leq 1$ we obtain

\begin{equation}
\int_{|x - y| \leq \gamma} |K_\alpha(x - y)q(y)|dy \leq |K_\alpha(\gamma)| \int_{|x - y| \leq \gamma} |q(y)|dy.
\end{equation}

There exists $N = N(d, \gamma)$ such that for every $x \in \mathbb{R}^d$ we can find $x_1, \ldots, x_N \in \mathbb{R}^d$ such that $B(x, 1) \subseteq \bigcup_{i=1}^N B(x_i, \gamma/4)$. Thus, we obtain

\begin{equation}
\int_{|x - y| \leq 1} |q(y)|dy \leq |K_\alpha(\gamma)|^{-1} \sum_{i=1}^N \int_{|x_i - y| \leq \gamma} |K_\alpha(x_i - y)q(y)|dy.
\end{equation}

For $d = 1 < \alpha < 2$ we choose $N = N(\gamma/4)$ such that for every $x \in \mathbb{R}^d$ there exist $x_1, \ldots, x_N \in \mathbb{R}^d$ such that $B(x, 1) \subseteq \bigcup_{i=1}^N B(x_i, \gamma/4)$. Put $z_i = x_i + \gamma/2$. For $y$ satisfying $|y - x_i| \leq \gamma/4$ we obtain $|y - z_i| \geq \gamma/2 - \gamma/4 = \gamma/4$. Thus, we get

\begin{equation}
\int_{|x - y| \leq 1} |q(y)|dy \leq \sum_{i=1}^N (4/\gamma)^{a-1} \int_{|x - y| \leq \gamma/4} |z_i - y|^a |q(y)|dy.
\end{equation}

Hence, we have

\begin{equation}
\int_{|x - y| \leq 1} |q(y)|dy \leq \sum_{i=1}^N (4/\gamma)^{a-1} \int_{|x - y| \leq \gamma/4} |z_i - y|^a |q(y)|dy.
\end{equation}
Remark. For \( d = 1 < \alpha < 2 \) we have for \( 0 < \gamma \leq 1 \)
\[
\sup_{x \in \mathbb{R}^d} \int_{|x - y| \leq \gamma} |y - x|^{|\alpha| - 1} |q(y)| \, dy \leq \gamma^{\alpha - 1} \sup_{x \in \mathbb{R}^d} \int_{|x - y| \leq 1} |q(y)| \, dy.
\]
Thus, \( q \in G^\alpha \) for \( d = 1 < \alpha < 2 \) if and only if
\[
(7) \quad \sup_{x \in \mathbb{R}^d} \int_{|x - y| \leq 1} |q(y)| \, dy < \infty.
\]
We now prove (iii). As in [9] we put
\[
S_y = \sup_{x \in \mathbb{R}^d} \int_{|x - y| \geq \gamma} |x - y|^{-\alpha - d} |q(y)| \, dy.
\]
There exists a constant \( C(d, \gamma) \) such that \( \{ y; \gamma n \leq |x - y| \leq (n + 1) \gamma \} \) can be covered by at most \( C(d, \gamma) n^{d-1} \) balls with radius \( \gamma \), for every \( n \). Thus, we have
\[
S_y \leq \sup \left[ \sum_{n = 1}^{\infty} \int_{\gamma n \leq |x - y| \leq (n + 1) \gamma} |x - y|^{-\alpha - d} |q(y)| \, dy \right] \\
\leq C(d, \gamma) \left( \sum_{n = 1}^{\infty} n^{-d-\alpha} n^{d-1} \right) \sup_{x \in \mathbb{R}^d} \int_{|x - y| \leq 1} |q(y)| \, dy < \infty.
\]
Now we obtain

**Theorem 1.** We have

\[
(8) \quad q \in G^\alpha \quad \text{if and only if} \quad \lim \sup_{t \to 0} \sup_{x \in \mathbb{R}^d} \int_0^t |P_s |q|| (x) \, ds = 0.
\]

**Proof.** We first show that the condition on the right-hand side of (8) implies that \( q \in G^\alpha \). With the exception of the case \( 1 = d = \alpha \) we have

\[
\int_0^t |P_s |q|| (x) \, ds = \int_0^t \left( \int_0^t P_s (x - y) \, ds \right) |q(y)| \, dy \\
= \mathcal{A} (d, \alpha)^{-1} \int_{\mathbb{R}^d} |K_\alpha (x - y)| \int_0^{t/|x - y|^\alpha} p_u \left( \frac{x - y}{|x - y|} \right) \, du |q(y)| \, dy \\
\geq \mathcal{A} (d, \alpha)^{-1} \int_{|x - y| \leq t^{1/\alpha}} |K_\alpha (x - y)| \int_0^{1} p_u \left( \frac{x - y}{|x - y|} \right) \, du |q(y)| \, dy \\
= C \mathcal{A} (d, \alpha)^{-1} \int_{|x - y| \leq t^{1/\alpha}} |K_\alpha (x - y) q(y)| \, dy,
\]
since
\[
\int_0^1 p_u \left( \frac{x - y}{|x - y|} \right) \, du = C > 0,
\]
by the rotational invariance of \( p_u \).
When $1 = d = \alpha$, by Fubini's theorem we obtain
\[
\int_0^t \int_0^t \left( \int_0^1 p_s(x-y) \, ds \right) q(y) \, dy = \int_0^t \left( \int_0^t \frac{s ds}{\pi s^2 + (x-y)^2} \right) |q(y)| \, dy = \frac{1}{2\pi} \int_0^t \ln \frac{t^2 + (x-y)^2}{(x-y)^2} |q(y)| \, dy
\]
because
\[
\frac{t^2 + (x-y)^2}{(x-y)^2} \geq \frac{t^2}{(x-y)^2} \geq \frac{1}{|x-y|^2}
\]
whenever $|x-y| \leq t^2$.

We now prove that the condition on the right-hand side of (8) holds whenever $q \in \mathcal{F}_\alpha$.

By the previous case we have
\[
\int_0^t \int_0^t \left( \int_0^1 p_s(x-y) \, ds \right) q(y) \, dy = \int_0^t \left( \int_0^y \frac{s ds}{|x-y|^{\alpha}} \right) p_u \left( \frac{x-y}{|x-y|} \right) |q(y)| \, dy
\]
\[
= \int_{|x-y| < \gamma} + \int_{|x-y| > \gamma} = I_1 + I_2.
\]

We first consider the case $\alpha < d$. We then obtain
\[
I_1 \leq \mathcal{A}(d, \alpha)^{-1} \int_{|x-y| < \gamma} |K_{\alpha}(x-y) q(y)| \int_0^\infty p_u \left( \frac{x-y}{|x-y|} \right) du \, dy
\]
\[
\leq \int_{|x-y| < \gamma} |K_{\alpha}(x-y) q(y)| \, dy,
\]
because
\[
\int_0^\infty p_u \left( \frac{x-y}{|x-y|} \right) du = K_{\alpha} \left( \frac{x-y}{|x-y|} \right) = \mathcal{A}(d, \alpha).
\]

When $1 = d = \alpha$, we get
\[
I_1 = \frac{1}{2\pi} \int_{|x-y| \leq \gamma} \ln \frac{t^2 + (x-y)^2}{(x-y)^2} |q(y)| \, dy
\]
\[
\leq \frac{1}{2\pi} \int_{|x-y| \leq \gamma} \ln \frac{t^2 + \gamma^2}{(x-y)^2} |q(y)| \, dy \leq \frac{1}{\pi} \int_{|x-y| \leq \gamma} \ln \frac{1}{|x-y|} |q(y)| \, dy
\]
whenever $t^2 + \gamma^2 < 1$. 
When $\alpha > 1 = d$, we apply the estimate (3) to obtain

$$I_1 = \int_{|x-y| \leq \gamma} \int \frac{p_u(x-y)}{|x-y|^{\alpha-d}} t^{1-1/\alpha} \int |q(y)| dy.$$

The Remark below the proof of the properties of the Kato class shows that this estimate is sufficient for our purposes.

What remains now is to estimate $I_2$. We have

$$I_2 = \int_{|x-y| > \gamma} |x-y|^{\alpha-d} \int_0^t p_u\left(\frac{x-y}{|x-y|}\right) du |q(y)| dy$$

$$\leq \frac{C t^2}{2} \int_{|x-y| > \gamma} |x-y|^{\alpha-d} |x-y|^{-4\alpha} |q(y)| dy \leq \frac{C t^2}{2} S_{\gamma}.$$

The above estimate completes the proof of the theorem.

4. GREEN OPERATORS

We now examine properties of the Green operator $G_D$ when $D$ is either an interval or halfline. We always assume that $\alpha \in (0, 2)$. We begin with the case of intervals. For this purpose, following [4], we state some useful estimates for the Green function of the interval. We put

$$I_\alpha(t) = \int_0^t \frac{u^{\alpha/2} - 1}{(u+1)^{1/2}} du, \quad t \geq 0,$$

and

$$w(x, y) = \frac{(1-x^2)(1-y^2)}{(x-y)^2}, \quad |x|, |y| \leq 1.$$

Let us recall [2] that the Green function $G(x, y)$ of the interval $(-1, 1)$ can be represented as

$$G(x, y) = \mathcal{B}_\alpha |x-y|^{\alpha-1} I_\alpha(w(x, y)), \quad |x|, |y| \leq 1,$$

where $\mathcal{B}_\alpha = 2^{-\alpha} \Gamma(\alpha/2)^{-2}$.

The behavior of the function $G$ is determined by the asymptotic properties of the integral $I_\alpha(t)$, which are summarized in the following lemma:

**Lemma 2.** There are constants $C_i = C_i(\alpha), i = 1, 2$, such that for all $t > 0$

(i) $C_1 \leq I_\alpha(t) / [t^{\alpha/2} \wedge 1] \leq C_2$ if $\alpha < 1$,

(ii) $C_1 \leq I_\alpha(t) / [t^{\alpha/2} \wedge t^{(\alpha-1)/2}] \leq C_2$ if $\alpha > 1$,

(iii) $C_1 \leq I_\alpha(t) / t^{\alpha/2} \leq C_2$ if $t \leq 1$, $\alpha \in (0, 2)$,
and for \( \alpha = 1 \)

(iv) \[ C_1 \leq I_1(t)/\ln(t^{1/2} + 1) \leq C_2. \]

A calculation allows for the choice of \( C_2 = 2/(\alpha - 1) \) in (ii) and \( C_1 = 2^{1/2}/\alpha \).

Using Lemma 2 we describe the behavior of the function \( G \).

**Corollary 3.** For all \( x, y \in (-1, 1) \) \((x \neq y)\) we have

\[
C_1 \leq G(x, y)/\left[ \frac{\partial}{\partial t} |x-y|^{\alpha-1} \left[ w(x, y)^{\alpha/2} \wedge 1 \right] \right] \leq C_2 \quad \text{if } \alpha < 1,
\]
\[
C_1 \leq G(x, y)/\left[ \frac{\partial}{\partial t} \ln \left( w(x, y)^{1/2} + 1 \right) \right] \leq C_2 \quad \text{if } \alpha = 1,
\]
\[
C_1 \leq G(x, y)/\left[ \frac{\partial}{\partial t} |x-y|^{\alpha-1} \left[ w(x, y)^{\alpha/2} \wedge w(x, y)^{1/(\alpha-1/2)} \right] \right] \leq C_2 \quad \text{if } \alpha > 1.
\]

We then obtain

**Theorem 4.** Let \( a, b \in \mathbb{R}^1 \), \( a < b \) and \( D = (a, b) \). Assume that \( q \in \mathcal{F}^\alpha \), \( \alpha \geq 1 \).

Then

\[ G_D q \in C_0(D). \]

**Proof.** We assume that \( D = (-1, 1) \). First consider the case \( \alpha > 1 \). Then, by Lemma 2 and the remark regarding the choice of the constants, we have

\[
G_D(x, y) \leq \frac{2((1-x^2)(1-y^2))^{(\alpha-1)/2}}{(\alpha-1)2^\alpha \Gamma(\alpha/2)^2}.
\]

Since \( D \) is bounded, we obtain \( q \in L^1(D) \), so \( G_D q \) is a continuous and bounded function of \( x \), continuously vanishing at \( \partial D \).

When \( \alpha = 1 \), we obtain

\[
G_D(x, y) = \frac{1}{\pi} \ln \frac{1-xy+\sqrt{(1-x^2)(1-y^2)}}{|x-y|} \leq \frac{1}{\pi} \ln \frac{3}{|x-y|}.
\]

The proof of the theorem in this case \((\alpha = 1)\) is similar to the one for the (standard) case \( \alpha < 1 \); it is included here for the sake of completeness. We put

\[ G_D^{(0)}(x, y) = G_D(x, y) \wedge n. \]

Then for fixed \( y < 1 \) and \( |x-y| > \gamma \) we obtain

\[ G_D(x, y) \leq (1/\pi) \ln(3/\gamma). \]

Therefore, if \( n_0 > (1/\pi) \ln(3/\gamma) \), then for \( n \geq n_0 \) we have \( |G_D(x, y) - G_D^{(0)}(x, y)| > 0 \) only on the set \( |x-y| < \gamma \). Thus, we obtain for \( n \geq n_0 \)

\[
|G_D q(x) - G_D^{(0)} q(x)| \leq \int_{|x-y| < \gamma} |G_D(x, y) - G_D^{(0)}(x, y)||q(y)||dy
\]

\[
\leq \sup_{x \in \mathbb{R}^1} \int_{|x-y| < \gamma} \frac{1}{\pi} \ln \frac{3}{|x-y|} |q(y)||dy.
\]

Since \( q \in \mathcal{F}^\alpha \), the above inequality yields that \( G_D^{(0)} q \to G_D q \) uniformly. Obviously, \( G_D^{(0)} q \in C_0(D) \), which completes the proof.
Next, we consider the case of the halflines. As in the case of intervals, we establish first some elementary properties of the Green function of the set \( D = (-\infty, b) \subset \mathbb{R}^1 \).

We put
\[
J_a(t) = \int_0^t \frac{u^{a/2} - 1}{(u + 1)^{1-a/2}} \, du, \quad t \geq 0,
\]
and
\[
v_b(x, y) = \frac{(b-x) \wedge (b-y)}{|x-y|}, \quad x \leq b, \ y \leq b.
\]
Let us recall [8] that the Green function \( G_D(x, y) \) of \((-\infty, b)\) can be represented as
\[
G_D(x, y) = \frac{|x-y|^{a-1}}{\Gamma(\alpha/2)^2} J_a(v_b(x, y)), \quad x \leq b, \ y \leq b.
\]

As before, we first summarize the asymptotic properties of the integral \( J_a \), which determine the behavior of the function \( G_D \).

**Lemma 5.** There are constants \( C_i = C_i(\alpha), \ i = 1, 2, \) such that for all \( t > 0 \)

(i) \( \quad C_1 \leq J_a(t)/(t^{a/2} \wedge 1) \leq C_2 \) if \( \alpha < 1 \),

(ii) \( \quad C_1 \leq J_a(t)/(t^{a/2} \wedge t^{-1}) \leq C_2 \) if \( \alpha > 1 \),

(iii) \( \quad C_1 \leq J_a(t)/t^{a/2} \leq C_2 \) if \( t \leq 1, \ \alpha \in (0, 2) \).

For \( \alpha > 1 \) and \( t \geq 1 \)

(iv) \( \quad C_1 \leq J_a(t)/t^{a-1} \leq C_2 \),

and for \( \alpha = 1 \)

(v) \( \quad C_1 \leq J_1(t)/\ln(t^{1/2} + 1) \leq C_2 \).

The proof of Lemma 5 is omitted.

We note that the upper estimates in (iii) and (iv) hold for all \( t > 0 \); in the latter case with the restriction \( \alpha > 1 \). Following calculations, we can take \( C_1 = 2^{a/2}/\alpha \) and \( C_2 = 2/\alpha \) in (iii), and \( C_1 = (2^{a-1} - 1)/(\alpha - 1) \) and \( C_2 = 1/(\alpha - 1) \) in (iv).

The following corollary describes the asymptotic behavior of the function \( G_D \) for \( D = (-\infty, b) \).

**Corollary 6.** For all \( x, y \in (-\infty, b) \) \( (x \neq y) \) we have

\[
C_1 \leq \Gamma(\alpha/2)^2 G_D(x, y)/(|x-y|^{a-1} [v_b(x, y)^{a/2} \wedge 1]) \leq C_2 \quad \text{if } \alpha < 1,
\]
\[
C_1 \leq \pi G_D(x, y)/\ln(v_b(x, y)^{1/2} + 1) \leq C_2 \quad \text{if } \alpha = 1,
\]
\[
C_1 \leq \Gamma(\alpha/2)^2 G_D(x, y)/(|x-y|^{a-1} [v_b(x, y)^{a/2} \wedge v_b(x, y)^{a-1}]) \leq C_2 \quad \text{if } \alpha > 1.
\]
Whenever $\alpha > 1$ we easily obtain from Lemma 5, the remark below and Corollary 6

$$2^{\alpha - 1} - 1 \leq \frac{(x - 1) I(x/2)^2 G_D(x, y)}{(b - x) \land (b - y))^{\alpha - 1}} \leq 1,$$

where the estimate from below holds true if $((b - x) \land (b - y))/|x - y| \geq 1$.

Note that $X_t$ is transient for $\alpha < 1$, so we obtain in this case, as usual,

$$\Gamma_D(x, y) \leq K_\alpha(x - y).$$

For $\alpha = 1$ we obtain from Corollary 6

$$\Gamma_D(x, y) \leq C \ln \left( \sqrt{\frac{(b - x) \land (b - y)}{|x - y|}} + 1 \right).$$

We now state and prove a version of Theorem 4 for a halfline.

**Theorem 7.** Let $D = (-\infty, b)$ and let $q \in \mathcal{F}^\alpha$. Then $\Gamma_D|q|(x) < \infty$ whenever $(1 + |y|)^{\alpha - 1} q(y) \in L^1(D)$ for $\alpha < 1$, or if $q \in L^1(D)$ for $\alpha \geq 1$. If, additionally,

(i) $q \in L^1(R^1)$ if $\alpha < 1$,

(ii) $\ln(|y|^{1/2} + 1) q(y) \in L^1(R^1)$ if $\alpha = 1$,

(iii) $(1 + |y|)^{\alpha - 1} q(y) \in L^1(R^1)$ if $\alpha > 1$,

then $\Gamma_D q(x)$ is a continuous function of $x$ and

$$\lim_{b \to -\infty} \sup_{x \in \mathbb{R}} \Gamma_D(-\infty, b) |q|(x) = 0.$$

**Proof.** We prove the first part of the theorem for $\alpha < 1$. For a fixed $\gamma$, $0 < \gamma < 1$, we have

$$\Gamma_D q(x) = \int_{-\infty}^b \Gamma_D(x, y) q(y) dy$$

$$= \int_{y \leq b, |x - y| < \gamma} \Gamma_D(x, y) q(y) dy + \int_{y \leq b, |x - y| > \gamma} \Gamma_D(x, y) q(y) dy.$$
Therefore, the integrand in the second term is continuous in $x$, vanishes at $b$, and is bounded by
\[ K_a(x-y)|q(y)| \leq C [1 + \gamma^{-1} (|x| + 1)]^{1-a} \frac{|q(y)|}{(1+|y|)^{1-a}} \in L^1(D). \]

If we assume that $q \in L^1(R^1)$, then we obtain
\[ \sup_{x \leq b} G_{[-\infty,b]} |q|(x) \leq \varepsilon + \mathscr{A}(1, \alpha) \gamma^{\alpha-1} \int_{-\infty}^{b} |q(y)| dy, \]

where $\varepsilon$ and $\gamma$ are as above. This completes the proof of the case $\alpha < 1$.

When $\alpha = 1$, we apply the inequality (11). Under the assumption $q \in L^1(D)$, we obtain, for $0 < \gamma < 1$, 
\[ \int_{-\infty}^{b} G_{D}(x, y)|q(y)| dy \leq C \int_{-\infty}^{b} \ln \left( \sqrt{\frac{(b-x) \wedge (b-y)}{|x-y|}} + 1 \right) |q(y)| dy \]
\[ \leq C \int_{y \leq b, |x-y| \leq \gamma} \ln \left( \frac{(b-x)^{1/2} + \gamma^{1/2}}{|x-y|^{1/2}} \right) |q(y)| dy + C \int_{y \leq b, |x-y| > \gamma} \ln \left( \sqrt{(b-x)/\gamma} + 1 \right) |q(y)| dy \]
\[ \leq C \int_{|x-y| \leq \gamma} \frac{1}{|x-y|} |q(y)| dy + C \ln \left( \sqrt{(b-x)/\gamma} + 1 \right) ||q||_{L^1(D)}. \]

Taking into account $q \in \mathscr{G}^a \cap L^1(D)$ we obtain the first conclusion in the case $\alpha = 1$. The continuity follows from similar arguments as in the first part of the proof.

When additionally $\ln(|y|^{1/2} + 1) q(y) \in L^1(R^1)$, then, arguing as before, we obtain
\[ G_{[-\infty,b]} |q|(x) \leq C \int_{|x-y| \leq \gamma} \frac{1}{|x-y|} |q(y)| dy + C \int_{-\infty}^{b} \ln \left( \sqrt{(b-y)/\gamma} + 1 \right) |q(y)| dy. \]

Since for $b < 0$ we have $b-y < -y = |y|$, the proof of this part of the theorem is complete.

Finally, if $\alpha > 1 = d$, we apply the inequality (9). Assume first that $q \in L^1(D)$. We then obtain
\[ |G_{D} q(x)| \leq \frac{(b-x)^{\alpha-1}}{(\alpha-1) \Gamma(\alpha/2)^2} ||q||_{L^1(D)}. \]
When $(1+|y|)^{a-1} q(y) \in L^1 (R^1)$, then we obtain
\[
G_{(-\infty,b)} |q| (x) \leq \frac{1}{(a-1) \Gamma (a/2)^2} \int_{-\infty}^b (b-y)^{a-1} |q(y)| dy.
\]
For $b < 0$ we obtain, as before, $(b-y)^{a-1} \leq |y|^{a-1} < (1+|y|)^{a-1}$ and this completes the proof of the last part of the theorem.

5. FUNDAMENTAL EXPECTATION

Let $q \in \mathcal{F}^a$. We recall that
\[
u(x, y) = E^x e_{q}(\tau_y), \quad \text{where} \quad \tau_y = \tau_{(-\infty, y)}.
\]
Obviously, $u(x, y) = 1$ for $x \geq y$. In [4] it is shown (see Section 7) that if $u(x, y) < \infty$ for a single $x < y$, then $u(\cdot, y)$ is a continuous regular $q$-harmonic function on $(-\infty, y)$. In particular, $u(w, y) < \infty$ for every $w < y$.

We put
\[
\beta = \sup \{y \in R^1; u(x, y) < \infty \text{ for all } x < y\},
\]
with the usual convention that $\sup \emptyset = -\infty$.

We now establish a condition under which $\beta > -\infty$.

THEOREM 8. Under the assumptions (i), (ii) or (iii), respectively, from Theorem 7, we have
\[
\beta \geq \sup \{b \in R^1; \sup_{x \leq b} G_{(-\infty,b)} |q| (x) < 1\} > -\infty.
\]

Proof. Let $q \neq 0$. Define
\[
M_n (x, b) = \frac{1}{n!} E^x \left( \int_0^b |q (X_t)| dt \right)^n \quad \text{and} \quad M_n (b) = \sup_{x \leq b} M_n (x, b).
\]

Obviously,
\[
M_1 (b) = \sup_{x \leq b} G_{(-\infty,b)} |q| (x).
\]

By the proof of the Khasminski lemma (see [7]) we obtain
\[
M_n (b) \leq M_1 (b)^n.
\]

Observe that if $b_1 \leq b_2$, then $M_1 (x, b_1) \leq M_1 (x, b_2)$ for all $x \leq b_1$. Consequently,
\[
M_1 (b_1) = \sup_{x \leq b_1} M_1 (x, b_1) \leq \sup_{x \leq b_1} M_1 (x, b_2) \leq \sup_{x \leq b_2} M_1 (x, b_2) = M_1 (b_2).
\]
Moreover, $M_1(b)$ is a continuous function of $b$. To prove this, we show that $G_{(-\infty, b_2)} |q| (\cdot)$ is continuous, with respect to the supremum norm, as a function of $b$. For this purpose, assume that $b_1 < b_2$. Then for $b_1 \leq x \leq b_2$ we have

$$G_{(-\infty, b_2)} |q| (x) = G_{(-\infty, b_1)} |q| (x) = G_{(-\infty, b_2)} |q| (x).$$

We thus estimate first $G_{(-\infty, b_2)} |q| (x)$ for $b_1 \leq x \leq b_2$.

Assume that $\alpha > 1$. Using the estimate (9) we obtain

$$G_{(-\infty, b_2)} |q| (x) \leq \frac{(b_2 - b_1)^{\alpha - 1}}{(\alpha - 1) \Gamma (\alpha/2)^2} \|q\|_{L^1(\mathbb{R}^1)}.$$

If now $\alpha = 1$, then for a given $\varepsilon > 0$ there exists a $\gamma$, $0 < \gamma < 1$, such that

$$\sup_{x \in \mathbb{R}^1} \frac{C}{2} \int_{|x - y| \leq \gamma} \ln \frac{1}{|x - y|} |q(y)| dy \leq \epsilon,$$

where $C$ is the constant from the estimate (11). We then obtain

$$G_{(-\infty, b_2)} |q|(x) = \int_{-\infty}^{b_2} G_{(-\infty, b_2)} (x, y) |q(y)| dy$$

$$\leq C \int_{-\infty}^{b_2} \ln \left( \frac{b_2 - x}{|x - y|} + 1 \right) |q(y)| dy$$

$$\leq C \ln \left( \frac{\sqrt{b_2 - b_1}}{\gamma} + 1 \right) \int_{y \leq b_2, |x - y| \leq \gamma} |q(y)| dy$$

$$+ \frac{C}{2} \int_{|x - y| \leq \gamma} \ln \frac{1}{|x - y|} |q(y)| dy$$

$$+ C \ln \left( \frac{\sqrt{b_2 - b_1}}{\gamma} + 1 \right) \|q\|_{L^1(\mathbb{R}^1)} + \varepsilon.$$

We now consider the case $\alpha < 1$. As in the previous case, for a given $\varepsilon > 0$ we choose $\gamma$, $0 < \gamma < 1$, such that

$$\sup_{x \in \mathbb{R}^1} \int_{|x - y| \leq \gamma} K_\alpha (x, y) |q(y)| dy \leq \varepsilon.$$

For $|x - y| > \gamma$ we estimate $G_{(-\infty, b_2)}$ by $2\alpha^{-1} \Gamma (\alpha/2)^{-2} (b_2 - x)^{\alpha/2} |x - y|^{\alpha/2 - 1}$. Thus, we obtain

$$G_{(-\infty, b_2)} |q| (x) \leq \varepsilon + \frac{2(b_2 - b_1)^{\alpha/2}}{\alpha \Gamma (\alpha/2)^2 \gamma^{1-\alpha/2}} \|q\|_{L^1(\mathbb{R}^1)}.$$

For $|x - y| > \gamma$ we estimate $G_{(-\infty, b_2)}$ by $2\alpha^{-1} \Gamma (\alpha/2)^{-2} (b_2 - x)^{\alpha/2} |x - y|^{\alpha/2 - 1}$. Thus, we obtain

$$G_{(-\infty, b_2)} |q| (x) \leq \varepsilon + \frac{2(b_2 - b_1)^{\alpha/2}}{\alpha \Gamma (\alpha/2)^2 \gamma^{1-\alpha/2}} \|q\|_{L^1(\mathbb{R}^1)}.$$
For $x \leq b_1$ we proceed as follows:

$$G_{(-\infty,b_2)}[g_1(x) - G_{(-\infty,b_1)}[g_1(x)] = E^x[\tau_{b_1} < \tau_{b_2}; \int_0^{\tau_{b_2}} |q(X_t)| dt]$$

$$= E^x[\tau_{b_1} < \tau_{b_2}; \left( \int_0^{\tau_{b_2}} |q(X_t)| dt \circ \theta_{t_{b_2}} \right) = E^x[\tau_{b_1} < \tau_{b_2}; G_{(-\infty,b_2)}[g_1(X_{t_{b_2}})]]$$

$$= E^x[G_{(-\infty,b_2)}[g_1(X_{t_{b_2}})] \leq \sup_{b_1 \leq x \leq b_2} G_{(-\infty,b_2)}[g_1(x)].$$

The result follows by application of the first part.

Define

$$b_0 = \sup \{b \in R^1; M_1(b) < 1 \}.$$ 

By continuity of $M_1$ and Theorem 7 we have $b_0 > -\infty$ and $M_1(b_0) = 1$ if $b_0 < \infty$. Also, for $b < b_0$ we obtain $M(b) < 1$. Therefore, for $b < b_0$ and all $x \leq b$ we have

$$u(x, b) = 1 + \sum_{n=1}^{\infty} M_n(x, b) \leq 1 + \sum_{n=1}^{\infty} M_n(b)$$

$$\leq 1 + \sum_{n=1}^{\infty} M_1(b)^n = \frac{1}{1 - M_1(b)} < \infty.$$ 

This proves that $\beta \geq b_0 > -\infty$. The proof is now complete.

We now establish additional properties of the function $u(x, y)$, under the assumptions of Theorem 7, unless stated otherwise. First, observe that by the final part of the above proof we get

$$\sup_{x \leq b} |u(x, b) - 1| \leq \frac{M_1(b)}{1 - M_1(b)}.$$ 

Since we have shown in Theorem 7 that $\lim_{b \to -\infty} M_1(b) = 0$, it follows that

$$\lim_{b \to -\infty} \sup_{x \leq b} |u(x, b) - 1| = 0.$$ 

We also note that if $\alpha < 1$, then $\lim_{b \to -\infty} M_1(b) < \infty$. Indeed, since $q \in \mathcal{C}^\alpha \cap L^1(R^1)$, we obtain

$$\sup_{x \in R^1} \int_{|x-y| \leq 1} K_{\alpha}(x, y) |q(y)| dy \leq C_1.$$ 

Thus, by the estimate $G_{(-\infty,b)} \leq K_{\alpha}$, we obtain

$$G_{(-\infty,b)}[g_1(x)] \leq \int_{|x-y| \leq 1} K_{\alpha}(x, y) |q(y)| dy + \mathcal{C}(1, \alpha) \int_{|x-y| > 1} |q(y)| dy$$

$$\leq C_1 + \mathcal{C}(1, \alpha) \|q(y)\|_{L^1(R^1)}.$$
On the other hand, for $\alpha \geq 1$ we obtain
\[
\lim_{b \to \infty} M_1(b) = \infty.
\]
We show this first for $\alpha > 1$. Observe that for $b > 0$ and $-b/2 < y < b/4$ we have $(b-y)/(y+b/2) > 1$. Hence using the lower estimate (9) we obtain
\[
M_1(b) \geq M_1(-b/2, b)
\]
\[
\geq \frac{1}{\Gamma(\alpha/2)^2} \int_{-b/2}^{b/4} (y+b/2)^{x-1} \int_0^{u^{\alpha/2-1}} (u+1)^{1-\alpha/2} |q(y)| dy du
\]
\[
\geq \frac{2^{x-1}-1}{(x-1) \Gamma(\alpha/2)^2} \int_{-b/2}^{b/4} (y+b/2) |q(y)| dy
\]
\[
\geq \frac{2^{x-1}-1}{(x-1) \Gamma(\alpha/2)^2} \int_{-b/2}^{b/4} |q(y)| dy
\]
\[
\geq \frac{(3/4 b)^{x-1}}{(x-1) \Gamma(\alpha/2)^2} \int_{-b/2}^{b/4} |q(y)| dy.
\]

For $\alpha = 1$ we argue as follows:
\[
G_{(-\infty, b]}|q|(x) \geq C \int_{-\infty}^{x} \ln(\sqrt{b-x}+\sqrt{x-y}) |q(y)| dy - C \int_{-\infty}^{x} \ln|x-y| |q(y)| dy
\]
\[
\geq C \int_{-\infty}^{x} \ln(\sqrt{b-x}+1) |q(y)| dy - C \int_{-\infty}^{x} \ln|x-y| |q(y)| dy.
\]

Now, in the last term, for fixed $x$ we have
\[
0 \leq \int_{-\infty}^{x} \ln|x-y| |q(y)| dy \leq C_1 < \infty,
\]
while the previous term tends to $\infty$ as $b \to \infty$, whenever $\|q\|_{L^1(-\infty, x-1)} \neq 0$.

We now show that for fixed $b$ we have $\lim_{x \to -\infty} G_{(-\infty, b]} q(x) = 0$. Note that this property has no Brownian motion counterpart.

Let $\alpha > 1$. By the upper estimate (9) we have
\[
G_{(-\infty, b]}(x, y)|q(y)| \leq \frac{(b-y)^{x-1}}{(x-1) \Gamma(\alpha/2)^2} |q(y)| \in L^1(R^1),
\]
by the assumption. At the same time, when $x \to -\infty$, then $b-x > b-y$ for fixed $y$, whenever $|x|$ is big enough. Therefore,
\[
\frac{(b-x) \wedge (b-y)}{|x-y|} = \frac{b-y}{y-x} \to 0 \quad \text{as} \quad x \to -\infty.
\]
Using the same estimate we thus obtain, again when \(|x|\) is sufficiently big:
\[
G_{(-\infty, b)}(x, y) \leq \frac{2(y-x)^{\alpha-1} (b-y)^{\alpha/2}}{\alpha} \frac{2(b-y)^{\alpha/2}}{(y-x)^{\alpha/2}} = \frac{2}{\alpha} \frac{(y-x)^{1-\alpha/2}}{\alpha} \to 0 \quad \text{as } x \to -\infty.
\]

The application of the Lebesgue Dominated Convergence Theorem gives now the claim.

If \(\alpha = 1\), then we show first that given \(\varepsilon > 0\) we can choose \(A < b\) such that
\[
\sup_{x \leq b} \int_{-\infty}^{A} G_{(-\infty, b)}(x, y)|q(y)| \, dy \leq \varepsilon.
\]
For this purpose, assume that \(0 < \gamma \leq 1\) and estimate the above expression as follows:
\[
\int_{-\infty}^{A} G_{(-\infty, b)}(x, y)|q(y)| \, dy \\
\leq C \int_{-\infty}^{A} \ln \left(\sqrt{\frac{b-y}{x-y}} + 1\right) |q(y)| \, dy \\
\leq C \int_{y \leq A, |x-y| \leq \gamma} \ln \left(\sqrt{\frac{b-y}{x-y}} + 1\right) |q(y)| \, dy \\
\quad + C \int_{|x-y| \leq \gamma} \frac{1}{|x-y|} |q(y)| \, dy + C \int_{y \leq A, |x-y| > \gamma} \ln \left(\sqrt{(b-y)/\gamma} + 1\right) |q(y)| \, dy \\
\leq C \int_{|x-y| \leq \gamma} \frac{1}{|x-y|} |q(y)| \, dy + C \int_{-\infty}^{A} \ln \left(\sqrt{(b-y)/\gamma + 1}\right) |q(y)| \, dy.
\]
The assumption \(q \in \mathcal{F}^x\) and \(\ln(|y|^{1/2} + 1) q(y) \in L^1(R^1)\) yield now the above claim.

The conclusion now follows from the fact that \(G_{(-\infty, b)}(x, y)|q(y)|\) is dominated by the expression
\[
C \ln \left(\sqrt{\frac{b-y}{x-y}} + 1\right) |q(y)|,
\]
which converges to 0 as \(x \to -\infty\) and is uniformly with respect to \(x\) integrable over the interval \((A, b)\) for fixed but otherwise arbitrary \(A < b\).

The proof of the case \(\alpha < 1\) is even easier and is omitted.

We now show that if \(b < b_0 = \sup \{b \in R^1; M_1(b) < 1\}\), then
\[
\lim_{x \to -\infty} u(x, b) = 1.
\]

We claim that
\[
\lim_{x \to -\infty} E^x \left(\int_0^{\tau_b} |q(X_t)| \, dt\right)^n = 0, \quad n = 1, 2, \ldots
\]
Observe that since
\[ E^x \int_0^{tb} |q(X_t)| dt = G_{(-\infty,b)} |q|(x), \]
this has just been shown for \( n = 1 \). Next, if we define
\[ f_n(x) = \frac{1}{n!} E^x \left( \int_0^{tb} |q(X_t)| dt \right)^n, \]
then \( f_{n+1} = G_{(-\infty,b)} (|q| f_n) \) for \( n \geq 1 \). However, \( \sup_{x \leq b} |f_n(x)| \leq M_1 (b)^n \), so \( qf_n \)
satisfy all the assumptions imposed on \( q \) and this justifies the claim. Now, since \( b < b_0 \), we have \( M_1 (b) < 1 \), so for a given \( \varepsilon > 0 \) we can find \( n_0 \) such that
\[ \sum_{n \geq n_0} M_1 (b)^n \leq \varepsilon/2. \]
If we now choose \( x_0 < b \) such that for \( x \leq x_0 \)
\[ \sum_{n=0}^{n_0-1} |M_n(x, b)| \leq \varepsilon/2, \]
then for \( x < x_0 \) we obtain \( |u(x, b) - 1| \leq \varepsilon \), and the conclusion follows.

We conclude our considerations by providing a condition opposite to those described in Theorem 8.

**Proposition 9.** Suppose that \( q \in \mathcal{F}^a \) and \( q \geq 0 \). If
\[ \int_{-\infty}^{b} \frac{q(y)}{(1 + |y|)^{1-\alpha/2}} dy = \infty, \]
then \( \beta = -\infty \). If \( \alpha > 1 \) and
\[ \int_{-\infty}^{b} |y|^{a-1} q(y) dy = \infty, \]
then either \( \beta = -\infty \) or \( \lim_{x \to -\infty} u(x, b) = \infty \).

**Proof.** By virtue of Jensen’s inequality
\[ u(x, b) \geq \exp(G_{(-\infty,b)} q(x)), \]
it is enough to analyse \( G_{(-\infty,b)} q(x) \). Assume that the first condition holds true and suppose that \( y < 2x-b = x-(b-x) < x \). Then we have \( (b-x)/(x-y) < 1 \). Thus, we obtain
\[ G_{(-\infty,b)} q(x) \geq \frac{1}{\Gamma(a/2)^2} \int_{-\infty}^{2x-b} \int_0^{(b-x)/(x-y)} u^{a/2-1} du \frac{u^{a/2-1}}{(u+1)^{1-\alpha/2}} q(y) dy \]
\[ \geq \frac{2a/2}{\alpha \Gamma(a/2)^2} \int_{-\infty}^{2x-b} (x-y)^{a/2-1} (b-x)^{a/2} q(y) dy. \]
Since obviously $x - y \leq b - y$, the last expression is not less than
\[
\frac{2^{x/2}}{\alpha \Gamma(\alpha/2)^2} (b - x)^{\alpha/2} \int_{-\infty}^{x} (b - y)^{1 - \alpha/2} dy
\]
\[
\geq \frac{2^{x/2}}{\alpha \Gamma(\alpha/2)^2} \int_{-\infty}^{x} (1 + |y|)^{1 - \alpha/2} dy
\]
if only $b > 0$ and $y < 0$. Indeed, we then have $b - y < b - by$. This proves the first part of the proposition, since the last integral is infinite.

Let now $\alpha > 1$. Observe that for $2y > b + x$ we obtain $b - y < y - x$, so $(b - y)/(y - x) < 1$. Thus, we have

\[
G_{(\infty,b)} q(x) \geq \frac{1}{\Gamma(\alpha/2)^2} \int_{(b + x)/2}^{b} (y - x)^{\alpha/2 - 1} \int_{0}^{b - y/b} u^{\alpha/2 - 1} du \left(\frac{y}{y - x}\right) q(y) dy
\]
\[
\geq \frac{2^{\alpha/2}}{\alpha \Gamma(\alpha/2)^2} \int_{(b + x)/2}^{b} (y - x)^{\alpha/2 - 1} \left(\frac{b - y}{y - x}\right) q(y) dy
\]
\[
\geq \frac{2^{\alpha/2}}{\alpha \Gamma(\alpha/2)^2} \int_{(b + x)/2}^{b} (b - y)^{\alpha/2 - 1} q(y) dy \geq \frac{2^{\alpha/2}}{\alpha \Gamma(\alpha/2)^2} \int_{0}^{b} |y|^{\alpha/2 - 1} q(y) dy
\]

because for $y < 0$ and $b > 0$ we obtain $b - y = -y = |y|$. When $x \to -\infty$, the conclusion follows, completing the proof of the proposition.

We now indicate how to compute $b_0$, given by the formula (12) for $q = 1_{(-\infty,c)}$ with $c > 0$. We start with transforming the Green operator $G_{(\infty,b)}$ into a form which is more suitable for computation. We always assume that $q \in g^* \cap L^1(R^1)$. Then Theorem 7 applies, so for every $x < b$ we obtain $G_{(\infty,b)} |q|(x) < \infty$, and this justifies changing the order of integration in the following calculations:

\[
G_{(\infty,b)} q(x) = \frac{1}{\Gamma(\alpha/2)^2} \int_{-\infty}^{x} \left(\frac{b - x}{u + |x - y|}\right)^{1 - \alpha/2} q(y) dy
\]
\[
= \frac{1}{\Gamma(\alpha/2)^2} \int_{-\infty}^{x} \left(\frac{b - x}{u + (u + x - y)}\right)^{1 - \alpha/2} q(y) dy
\]
\[
+ \frac{1}{\Gamma(\alpha/2)^2} \int_{x}^{b} \left(\frac{b - y}{u + (u + y - x)}\right)^{1 - \alpha/2} q(y) dy
\]
\[
= \frac{1}{\Gamma(\alpha/2)^2} \int_{-\infty}^{x} \left(\frac{b}{x((u - x)(u - y))^{1 - \alpha/2}}\right) q(y) dy
\]
\[
+ \frac{1}{\Gamma(\alpha/2)^2} \int_{x}^{b} \left(\frac{b}{y((u - x)(u - y))^{1 - \alpha/2}}\right) q(y) dy
\]
\[
= \frac{1}{\Gamma(\alpha/2)^2} \int_{-\infty}^{x} \left(\frac{x}{\infty((u - x)(u - y))^{1 - \alpha/2}}\right) q(y) dy
\]
We also define for \( 0 \leq u < 1 \)

\[
P_1^u(u) = \int_0^1 \frac{t^{u/2-1}}{(1-t)^{u+1}} \, dt \quad \text{and} \quad P_2^u(u) = \int_0^1 \frac{t^{u/2}}{(1-t)^{u+1}} \, dt.
\]

We are now able to examine the above-mentioned example. As usual, we assume here that \( \alpha \in (0, 2) \).

**Example.** Let \( q = 1_{(-c,c)}, \, c > 0 \). For \( b > -c \) we obtain

\[
\sup_{x \leq b} G_{(-\infty, b)}(x) = \max_{-c < x < c} G_{(-\infty, b)}(x).
\]

For \(-c < b \leq c\) the greatest value of \( G_{(-\infty, b)}(x) \) is attained at the point \( x_0 \) determined by the unique solution \( u_0 \) of the equation

\[
\alpha(1-u)^a P_1^u(u) = u^{a/2-1}, \quad 0 < u < 1,
\]

with \( u = (b-x)/(b+c) \). The corresponding maximal value \( G_{(-\infty, b)}(x_0) \) is given as

\[
\sup_{x \leq b} G_{(-\infty, b)}(x) = \frac{(b+c)^a}{2 \Gamma(1+x/2)^2} u_0^{a/2-1}.
\]

For \( b \geq c \) the corresponding value \( u_1 \) is the unique solution of the equation

\[
(1-u)^a-1 P_1^u(u)+(u-d)^a-1 P_2^u(d/u) = u^{a/2-1} \left( \frac{1}{1-u} + \frac{d^{a/2+1}}{u-d} \right),
\]

\( 0 < u < 1 \), where \( d = (b-c)/(b+c) \) and the maximal value of the Green operator is given by

\[
2(b+c)^a \left[ (1-d)(1-u_1)^{a-1} P_1^u(u_1)-\frac{u_1^{a/2-1}}{\alpha} \left( \frac{u_1-d}{1-u_1} + d^{a/2+1} \right) \right].
\]

We first note that since \( G_{(-\infty, b)}(x, y) = 0 \) for \( y \geq b \), we obtain

\[
G_{(-\infty, b)} 1_{(-c,c)}(x) = \int_{-\infty}^b G_{(-\infty, b)}(x, y) 1_{(-c,c)}(y) \, dy = 0
\]

whenever \( b \leq -c \). We therefore assume throughout the remainder that \( b > -c \).
By the form of the Green operator we obtain

\[(18) \quad G_{(-\infty,b)} q(x) = \frac{1}{\Gamma(\alpha/2)^2} \int_{x}^{b} \frac{du}{(u-x)^{1-\alpha/2}} \left\{ \int_{-\infty}^{u} \frac{1}{(u-y)^{1-\alpha/2}} dy \right\} \]

\[= \frac{1}{\Gamma(\alpha/2)^2} \int_{x}^{b} \frac{du}{(u-x)^{1-\alpha/2}} \left\{ \int_{-\infty}^{u} \frac{d\zeta}{(\zeta-y)^{1-\alpha/2}} \right\} \]

\[= \frac{c^\alpha}{\Gamma(\alpha/2)^2} \int_{x}^{b} \frac{du}{(u-x)^{1-\alpha/2}} \left\{ \int_{-\infty}^{u} \frac{d\zeta}{(\zeta-y)^{1-\alpha/2}} \right\} , \]

where \( x' = x/c \) and \( b' = b/c \).

Assume first that \( x \leq -c \). By (18) we then obtain

\[(19) \quad G_{(-\infty,b)} 1_{(-c,c)}(x) = \frac{c^\alpha}{\Gamma(\alpha/2)^2} \int_{-\infty}^{x} \frac{du}{(u-x')^{1-\alpha/2}} \left\{ \int_{-\infty}^{u} \frac{d\zeta}{(\zeta-y)^{1-\alpha/2}} \right\} \]

\[= \frac{2c^\alpha}{\alpha \Gamma(\alpha/2)^2} \int_{-\infty}^{x} \frac{(u+1)^{\alpha/2} du}{(u-x')^{1-\alpha/2}} . \]

Since the above expression is a nondecreasing function of \( x \), we obtain

\[G_{(-\infty,b)} 1_{(-c,c)}(x) \leq G_{(-\infty,b)} 1_{(-c,c)}(-c), \quad x \leq -c.\]

Direct calculations provide the value of the last quantity for \(|b| \leq c\) as

\[2(b+c)^\alpha/\alpha^2 \Gamma(\alpha/2)^2 \text{ or} \]

\[\frac{2}{\alpha \Gamma(\alpha/2)^2} \left[ \frac{(b+c)^\alpha}{\alpha} - (2c)^\alpha P_2^2 \left( \frac{b-c}{b+c} \right) \right] \]

whenever \( b \geq c \). We now write (19) in a form which is more suitable for computations. We introduce a new variable \( t \) defined by the formula \( u+1 = (-1-x)/(t-1) \). We then obtain

\[u-x = t(-1-x)/(t-1) \quad \text{and} \quad du = (1+x) dt/(t-1)^2 \]

and, after substituting, we get

\[G_{(-\infty,b)} q(x) = \frac{2(b+c)^\alpha}{\alpha \Gamma(\alpha/2)^2} (u-1)^\alpha \int_{u}^{\infty} \frac{t^{\alpha/2-1}}{(t-1)^{\alpha+1}} dt, \]

where \( u = (b-x)/(b+c) \). Substituting once again \( v = 1/t, \quad t-1 = (1-v)/v \), \( dt = -dv/v^2 \) we finally obtain

\[(20) \quad G_{(-\infty,b)} q(x) = \frac{2(b+c)^\alpha}{\alpha \Gamma(\alpha/2)^2} (u-1)^\alpha P_2^2(1/u). \]

Assume now that \( c \leq x \leq b \). By (18) we obtain

\[G_{(-\infty,b)} 1_{(-c,c)}(x) = \frac{c^\alpha}{\Gamma(\alpha/2)^2} \int_{x}^{b} \frac{du}{(u-x')^{1-\alpha/2}} \left\{ \int_{-\infty}^{u} \frac{d\zeta}{(\zeta-y)^{1-\alpha/2}} \right\} . \]
Computing the inner integral and using integration by parts we further obtain
\[
\frac{c^x}{\Gamma((x/2)^2)} \left[ \frac{b}{(u+1)^{q/2}} - \frac{(u-1)^{q/2}}{u-x} \right] du
\]
\[
= \frac{4(b-x)^{q/2}}{\alpha^2 \Gamma((x/2)^2)} \left[ (b+c)^{q/2} - (b-c)^{q/2} \right]
\]
\[
+ \frac{2}{\alpha \Gamma((x/2)^2)} \int_x^b (u-x)^{q/2} \left[ \frac{1}{(u-c)^{1-a/2}} - \frac{1}{(u+c)^{1-a/2}} \right] du.
\]
Since the above expression is a nonincreasing function of \(x\), we obtain
\[
G_{(-\infty,b)} 1_{(-c,c)}(x) \leq G_{(-\infty,b)} 1_{(-c,c)}(c), \quad c \leq x < b.
\]
Direct calculations again provide the last value as
\[
\frac{2}{\alpha^2 \Gamma((x/2)^2)} \left[ 2(b^2 - c^2)^{q/2} - (b-c)^{q/2} - \alpha(2c)^q P_2^q \left( \frac{b-c}{b+c} \right) \right].
\]
It is easy to see that \(G_{(-\infty,b)} q(-c) > G_{(-\infty,b)} q(c)\).

We now consider the case \(-c < x < c\). We may and do assume here that \(c = 1\). The expression (18) takes then, up to constants, the form
\[
\int_x^b \frac{du}{(u-x)^{1-a/2}} = \frac{2}{\alpha} \int_x^b \frac{(u+1)^{q/2}}{(u-y)^{1-a/2}} dy.
\]
Integrating by parts the right-hand side, we obtain
\[
\left( \frac{2}{\alpha} \right)^2 (b+1)^{q/2} (b-x)^{q/2} - \frac{2}{\alpha} \int_x^b (u-x)^{q/2} du
\]
Taking the derivative of the above expression with respect to \(x\) we obtain
\[
-\frac{2}{\alpha} (b+1)^{q/2} (b-x)^{q/2} - 1 \int_x^b \frac{du}{(u+1)(u-x)}^{1-a/2}.
\]
Observe that for \(\alpha \leq 1\) the integral above tends to \(\infty\) when \(x \downarrow -1\), so the derivative also tends to \(\infty\). For \(\alpha > 1\) it follows that it converges to
\[
(b+1)^{q-1} (1/(x-1) - 1/\alpha) > 0.
\]
On the other hand, when \(x \uparrow b\), the integral above tends to 0 while the first term converges to \(-\infty\). Consequently, the derivative tends to \(-\infty\). Therefore, there exists \(x_0 \in (-1, b)\) where the expression (21) attains the maximum. Integrating by parts once again in (22) we see that \(x_0\) is a solution of the equation
\[
\frac{2-\alpha}{\alpha} \int_x^b \frac{(u-x)^{q/2}}{(u+1)^{2-a/2}} du = \frac{2(1+x)}{\alpha((b+1)(b-x))^{1-a/2}}.
\]
Since the left-hand side of (23) is decreasing and the right-hand side is increasing in \( x \), the solution \( x_0 \) is unique.

As before, we again transform the expression (21). We introduce a new variable \( t \) defined by the formula \( u + 1 = (1 + x)/(1 - t) \). We then obtain
\[
    u - x = t(1 + x)/(1 - t) \quad \text{and} \quad du = (1 + x) dt/(1 - t)^2.
\]

Taking into account the appropriate constants we obtain from (21)
(24)
\[
    G_{(-\infty, 0)} q(x) = \frac{2(b + c)^{\alpha}}{\alpha \Gamma(\alpha/2)^2} (1 - u)^{\alpha} P_1^a(u),
\]
where, as before, \( u = (b - x)/(b + c) \), \( 0 < u < 1 \).

The maximal value of the expression (24) is determined now by the root of its derivative:
\[
    \alpha(1 - u)^{\alpha - 1} P_1^a(u) = (1 - u)^{\alpha} \frac{u^{\alpha/2} - 1}{(1 - u)^{\alpha + 1}},
\]
and this equation is equivalent to (14). Taking this into account we obtain (15).

We now consider the case when \( b \geq c \) and \( -c < x < c \). Again, we assume for time being that \( c = 1 \). By the formula (18) we then obtain
\[
    \Gamma(\alpha/2)^2 G_{(-\infty, b)} q(x) = \frac{b}{\alpha} \int_{x}^{1} \frac{du}{(u - x)^{1 - \alpha/2}} \int_{1}^{b} \frac{dy}{(u - y)^{1 - \alpha/2}} + \frac{b}{\alpha} \int_{1}^{b} \frac{du}{(u - x)^{1 - \alpha/2}} \int_{1}^{b} \frac{dy}{(u - y)^{1 - \alpha/2}}
\]
\[
    = \frac{2b}{\alpha} \int_{x}^{1} \frac{(u + 1)^{\alpha/2}}{(u - x)^{1 - \alpha/2}} du - \frac{2b}{\alpha} \int_{1}^{b} \frac{(u - 1)^{\alpha/2}}{(u - x)^{1 - \alpha/2}} du.
\]

We introduce a new variable in the first integral in the expression above by the formula \( u + 1 = (1 + x)/(1 - t) \). We then obtain
\[
    u - x = t(1 + x)/(1 - t), \quad du = (1 + x) dt/(1 - t)^2
\]
and
\[
    - \frac{2b}{\alpha} \int_{x}^{1} \frac{(u + 1)^{\alpha/2}}{(u - x)^{1 - \alpha/2}} du = \frac{2}{\alpha} (1 + x)^{\alpha} P_1^a \left( \frac{b - x}{b + 1} \right).
\]
Analogously, substituting \( u - x = (1 - x)/(1 - t) \) in the second integral we obtain
\[
    u - 1 = t(1 - x)/(1 - t), \quad du = (1 - x) dt/(1 - t)^2
\]
and
\[
    \frac{2b}{\alpha} \int_{1}^{b} \frac{(u - 1)^{\alpha/2}}{(u - x)^{1 - \alpha/2}} du = \frac{2}{\alpha} (1 - x)^{\alpha} P_2^a \left( \frac{b - 1}{b - x} \right).
\]

Denoting \((b - x)/(b + c)\) by \( u \) and \((b - c)/(b + c)\) by \( d \) and taking into account
constants, we transform the expression $\Gamma(\alpha/2)^2 G_{(-\alpha,b)} q(x)$ into the following form:

\begin{equation}
2 \frac{b+c}{\alpha} [(1-u)^{\alpha} P_1^\alpha(u)-(u-d)^{\alpha} P_2^\alpha(d/u)].
\end{equation}

On the other hand, by integration by parts we obtain

$$P_1^\alpha(u) + P_2^\alpha(u) = \frac{2}{\alpha (1-u)^{\alpha}},$$

which allows us to transform (25) into the following:

\begin{equation}
2 \frac{b+c}{\alpha} \left[(1-u)^{\alpha} P_1^\alpha(u)+(u-d)^{\alpha} P_1^\alpha(d/u)-\frac{2}{\alpha} u^{\alpha/2} d^{\alpha/2}\right].
\end{equation}

The derivative of (26) again leads to the equation

\begin{equation}
(1-u)^{-1} P_1^\alpha(u)-(u-d)^{-1} P_1^\alpha(d/u) = \frac{u^{\alpha/2}-1}{\alpha} \left[d^{\alpha/2} + \frac{d^{\alpha/2} u}{u-d} - \frac{1}{1-u}\right].
\end{equation}

It is not difficult to observe that (27) is equivalent to (16). The justification of the existence of the unique solution $u_s$ satisfying (27) is similar to that of (14) and is omitted. Taking into account (26) and (27) we obtain (17).

Although we do not need the expressions for $G_{(-\alpha,b)} q(x)$ for $x < -c$ or for $c < x < b$, we provide them for the sake of completeness. We begin with the case $x < -c$:

\begin{equation}
G_{(-\alpha,b)} q(x) = \frac{2(b+c)^{\alpha}}{\alpha \Gamma(\alpha/2)^2} [(u-1)^{\alpha} P_2^\alpha(1/u)-(u-d)^{\alpha} P_2^\alpha(d/u)],
\end{equation}

where $u = (b-x)/(b+c) > 1$. For $c < x < b$ we obtain, analogously,

\begin{equation}
G_{(-\alpha,b)} q(x) = \frac{2(b+c)^{\alpha}}{\alpha \Gamma(\alpha/2)^2} [(1-u)^{\alpha} P_1^\alpha(u)-(d-u)^{\alpha} P_1^\alpha(u/d)]
\end{equation}

with $u = (b-x)/(b+c) < d$.

We now provide a more detailed analysis for the case $\alpha = 1$. Note that in this case we compute directly the integrals (13) as follows:

\begin{equation}
P_1^1(u) = \frac{\sqrt{u}}{1-u} + \frac{1}{2} \ln \frac{1+\sqrt{u}}{1-\sqrt{u}},
\end{equation}

\begin{equation}
P_2^1(u) = \frac{\sqrt{u}}{1-u} - \frac{1}{2} \ln \frac{1+\sqrt{u}}{1-\sqrt{u}}.
\end{equation}

Substituting (30) into (14) we obtain the equation

$$\sqrt{u} (1-u) \left(\frac{\sqrt{u}}{1-u} + \frac{1}{2} \ln \frac{1+\sqrt{u}}{1-\sqrt{u}}\right) = 1$$
or, equivalently,

\[
\frac{\sqrt{u}}{2} \ln \frac{1 + \sqrt{u}}{1 - \sqrt{u}} = 1, \quad 0 < u < 1.
\]

Substituting \(\exp 2t = \frac{1 + \sqrt{u}}{1 - \sqrt{u}}, \quad 0 < t < \infty\), we obtain \(u = \tanh t\), and so (32) takes the form

\[
t\tanh t = 1, \quad 0 < t < \infty.
\]

Let \(t_0\) be the unique root of the equation (33). Then we obtain

\[
\sup_{x \leq b} G_{t_0} (x) = 2 \pi^{-1} \frac{(b + c) t_0}{t_0}.
\]

Assume that \(c > \pi/4 t_0\). We get \(b_0 = \pi/2t_0 - c\).

Let now \(b \geq c\) and \(-c < x < c\). From (16) we have

\[
P_1(u) + P_2(d/u) = \frac{1}{\sqrt{u}} \left( \frac{1}{1 - u} + \frac{d^{3/2}}{u - d} \right).
\]

By (30) and (31) we then obtain equivalently

\[
\ln \left( \frac{1 + \sqrt{u} \sqrt{u - d}}{1 - \sqrt{u} \sqrt{u + d}} \right) = 2 \frac{1 - \sqrt{d}}{\sqrt{u}}.
\]

If we introduce a new variable by the formula

\[
\exp 2t = \frac{1 + \sqrt{u} \sqrt{u - d}}{1 - \sqrt{u} \sqrt{u + d}}, \quad 0 < t < \infty,
\]

then we obtain

\[
\tanh t = \frac{u - \sqrt{d}}{\sqrt{u(1 - \sqrt{d})}},
\]

so the equation (34) is being transformed into

\[
\sqrt{u} t = 1 - \sqrt{d}.
\]

The equations (35) and (36) provide then the following equivalent version of (16):

\[
\tanh t = \frac{1 + \sqrt{d}}{(1 - \sqrt{d})^2} t.
\]

Let \(t_1\) be the unique root of (37) and \(u_1\) the corresponding value of the original variable. According to (17) we then obtain

\[
\sup_{x \leq b} G_{t_0} (x) = \frac{2(b + c)}{\pi} \left[ \frac{1 - d}{2} \ln \frac{1 + \sqrt{u_1}}{1 - \sqrt{u_1}} + \frac{d}{\sqrt{u_1}(1 - \sqrt{d})} \right].
\]
If we put $\exp Zw = \left(1 - \frac{c}{I - J}\right)$, then $\tanh w = \frac{2(b + c)}{\pi} [(1 - d)w + dt]$. (39)

Remark. In the Brownian motion case the Green function equals $G_{(-\infty, b)}(x, y) = (b - x) \wedge (b - y)$, $x, y \leq b$, and the formula (18) is also valid. In this case the greatest value of $G_{(-\infty, b)}(x)$ is attained at $x = -c$, for $q$ as before, and is equal to $(b + c)^2/2$ if $-c < b < c$ and to $2bc$ if $c < b$. Thus, $b_0 = \sqrt{2 - c}$ if $c > \sqrt{2}/2$ and $b_0 = 1/(2c)$ if $c \leq \sqrt{2}/2$. However, even in that case $b_0 < \beta$ (see the Example in Section 9 of [7]) and to determine $\beta$ the more advanced methods are required.

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Institute of Mathematics
Wrocław University of Technology
Wybrzeże Wyspińskiego 27
50-370 Wrocław, Poland

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