THE MAXIMAL $\mathcal{F}$-REGULAR PART
OF A $q$-VARIATE WEAKLY STATIONARY PROCESS

BY

LUTZ KLOTZ (LEIPZIG)

Abstract. Let $x$ be a $q$-variate (weakly) stationary process over a locally compact Abelian group $G$, and $\mathcal{F}$ a family of subsets of $G$ invariant under translation. We show that the set of all regular non-negative Hermitian matrix-valued measures $M$ not exceeding the (non-stochastic) spectral measure of $x$ and such that the Hilbert space $L^2(M)$ is $\mathcal{F}$-regular contains a unique maximal element. Moreover, this maximal element coincides with the spectral measure of the $\mathcal{F}$-regular part of the Wold decomposition of $x$.

1991 Mathematics Subject Classification: Primary 60G25; Secondary 15A57.

1. INTRODUCTION

Let $N$ be the set of positive integers and $q \in N$. By $M_q$ we denote the algebra of $q \times q$-matrices with entries from the field of complex numbers $C$ and by $M_q^\geq$ the subset of non-negative Hermitian matrices. The symbol $I$ stands for the unit matrix of $M_q$.

Let $G$ be a locally compact Abelian group, $\Gamma$ its dual, and $\langle g, \gamma \rangle$ the value of a character $\gamma \in \Gamma$ on $g \in G$. If $J$ is a subset of $G$, then a (finite) $M_q$-linear combination of functions $\langle g, \cdot \rangle I, g \in J$, is called a trigonometric polynomial with frequencies from $J$.

Let $x$ be a $q$-variate (weakly) stationary process over $G$, and $H_x$ its time domain, i.e. the left Hilbert-$M_q$-module spanned by the values of $x$. If $\mathcal{F}$ is a family of subsets of $G$ invariant under translation, then there exists a unique Wold decomposition of $x$ into an orthogonal sum of $q$-variate stationary processes $y$ and $z$ such that $y$ is $\mathcal{F}$-regular and $z$ is $\mathcal{F}$-singular (cf. [12], Theorem 2.13). It could be expected that, in a certain sense, the process $y$ is the “maximal $\mathcal{F}$-regular part of $x$”. The aim of this note is to specify this statement. To do this it is more convenient to work with the spectral domain instead of the time domain of $x$.

Let $\mathcal{B}(\Gamma)$ be the $\sigma$-algebra of Borel sets of $\Gamma$. The (non-stochastic) spectral measure $M_x$ of $x$ (cf. [12], Definition 3.5) is a regular $M_q^\geq$-valued measure on
2. PRELIMINARIES

For any matrix $B$ with complex entries, denote by $B^*$ its adjoint and by $\mathcal{R}(B)$ its range. For $A \in M_q$, let $\ker A$, $\text{tr} A$, and $A^+$ be the kernel, trace, and Moore–Penrose inverse of $A$, respectively. Let $P_A$ be the orthoprojector in the left Hilbert-$M_q$-module $C^q$ of column vectors of length $q$ onto $\mathcal{R}(A)$. If $A \in M_q^\ge$, we denote by $A^{1/2}$ the unique non-negative Hermitian square root of $A$. We equip $M_q^\ge$ with Loewner’s partial ordering, i.e. we write $A \le B$ if and only if $B - A$ is a non-negative Hermitian, $A, B \in M_q^\ge$.

We give some more or less known results on $M_q^\ge$ and the measurability of $M_q^\ge$-valued functions, which for ease of reference will be stated as lemmas.

**Lemma 2.1.** Let $\mathcal{D}$ be a directed subset of $M_q^\ge$, which has an upper bound. Then there exists a least upper bound $C$ of $\mathcal{D}$ and we have

$$u^* Cu = \sup \{u^* Du: D \in \mathcal{D}\}, \quad u \in C^q.$$  

**Proof.** For $u \in C^q$, set $t(u) := \sup \{u^* Du: D \in \mathcal{D}\}$. Obviously, if $\lambda \in \mathcal{C}$, we have

$$t(\lambda u) = |\lambda|^2 t(u),$$

and if $u, v \in C^q$, we obtain

$$\sup \{u^* Du + v^* Dv: D \in \mathcal{D}\} \le t(u) + t(v).$$

Since $\mathcal{D}$ is directed, for $D_1, D_2 \in \mathcal{D}$ there exists $D_3 \in \mathcal{D}$ such that

$$u^* D_1 u + v^* D_2 v \le u^* D_3 u + v^* D_3 v.$$  

This yields

$$t(u) + t(v) \le \sup \{u^* Du + v^* Dv: D \in \mathcal{D}\}.$$
The parallelogram identity implies that
\[(2.4) \quad \sup \{(u+v)^* D(u+v) + (u-v)^* D(u-v) : D \in \mathfrak{D}\} = 2 \sup \{u^* Du + v^* Dv : D \in \mathfrak{D}\}.\]

Combining (2.4), (2.2), and (2.3), we get
\[(2.5) \quad t(u+v) + t(u-v) = 2t(u) + 2t(v).\]

From (2.1) and (2.5) it follows that there exists $C \in M_q^\geq$ such that $t(u) = u^* Cu$, $u \in C^O$. From the definition of $t$ it is clear that $C$ is the least upper bound of $\mathfrak{D}$.

**Lemma 2.2** (cf. [1], Theorem 1). Let $p, q \in N$. A block matrix
\[X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{pmatrix} \in M_{p+q},\]
belongs to $M_{p+q}^\geq$ if and only if
(i) $\mathcal{R}(X_{12}^*) \subseteq \mathcal{R}(X_{22})$,
(ii) $X_{22} \in M_q^\geq$,
(iii) $X_{11} - X_{12} X_{22} X_{12}^* =: (X/X_{22})$ is a non-negative Hermitian.

**Lemma 2.3** (cf. [3], p. 391). If $F$ is a (Borel) measurable $M_q^\geq$-valued function on $\Gamma$, then $P_F$ is measurable. If $W$ is a measurable $M_q^\geq$-valued function on $\Gamma$, then $W^{1/2}$ and $W^+$ are measurable.

Let $M$ be a regular $M_q^\geq$-valued measure on $\mathcal{B}(\Gamma)$ and $\tau$ a regular non-negative $\sigma$-finite measure on $\mathcal{B}(\Gamma)$ such that $M$ is absolutely continuous with respect to $\tau$. For example, one can take $\tau = \text{tr} M$. Let $W := dM/d\tau$ be the Radon–Nikodym derivative of $M$ with respect to (abbreviated to "w.r.t.") $\tau$. By definition, the left Hilbert-$M_q^\geq$-module $L^2(M)$ consists of (equivalence classes of) measurable $M_q^\geq$-valued functions $F$ on $\Gamma$ such that
\[\int F(\gamma) W(\gamma) F(\gamma)^* \tau(d\gamma) < \infty.\]

The corresponding scalar product of $L^2(M)$ is defined by
\[\int F(\gamma) W(\gamma) G(\gamma)^* \tau(d\gamma), \quad F, G \in L^2(M).\]

The definition does not depend on the choice of $\tau$ (cf. [10]).

**Lemma 2.4**. Let $F \in L^2(M)$. Then $F = 0$ in $L^2(M)$ if and only if $\mathcal{R}(W) \subseteq \ker F$ $\tau$-a.e.

**Proof.** Since $F W F^* = F W^{1/2} (F W^{1/2})^*$, we have $F = 0$ in $L^2(M)$ if and only if $\mathcal{R}(W^{1/2}) \subseteq \ker F$ $\tau$-a.e. Since $\mathcal{R}(W^{1/2}) = \mathcal{R}(W)$, the result follows.

If $M_x$ is the spectral measure of a $q$-variate stationary process $x$ over $G$, the corresponding space $L^2(M_x)$ is called the *spectral domain* of $x$. There exists an isometric and isomorphic map $V_x$ of $H_x$ onto $L^2(M_x)$ such that $V_x x_g = \langle g, \cdot \rangle I$,
The map $V_x$ is called Kolmogorov’s isomorphism. It enables us to formulate $\mathcal{F}$-regularity and $\mathcal{F}$-singularity of $x$ in terms of $L^2(M_x)$. According to this we call a space $L^2(M)$ $\mathcal{F}$-regular or $\mathcal{F}$-singular if and only if

$$\bigcap_{J \in \mathcal{F}} \bigvee \{ \langle g, \cdot \rangle I: g \in J \} = \{0\} \quad \text{or} \quad \bigvee_{M} \{ \langle g, \cdot \rangle I: g \in J \} = L^2(M)$$

for all $J \in \mathcal{F}$, respectively. The symbol $\bigvee_{M}$ stands for the closed $M_q$-linear hull in $L^2(M)$. We simply write $\bigvee$ if $M = M_x$ is the spectral measure of the process $x$.

3. THE MAXIMAL $\mathcal{F}$-REGULAR PART

Let $M_x$ be the spectral measure of a $q$-variate stationary process over $G$, $\tau_x := \text{tr} M_x$, and $W_x := dM_x/d\tau_x$. In the sequel, all relations between measurable functions on $\Gamma$ are to be understood as relations which hold true $\tau_x$-a.e.

Let $\mathcal{W}_x$ be the set of all measurable $M_q$-valued functions $W$ on $\Gamma$ such that $W \leq W_x$ and let $\mathcal{W}_x$ be the set of all $M_q$-valued measures of the form $W d\tau_x$, $W \in \mathcal{W}_x$. The partial ordering on $\mathcal{W}_x$ induces a partial ordering on $\mathcal{W}_x$: define $W_1 d\tau_x \leq W_2 d\tau_x$ if and only if $W_1 \leq W_2$, $W_1, W_2 \in \mathcal{W}_x$. Note that for $M_1, M_2 \in \mathcal{W}_x$ we have $M_1 \leq M_2$ if and only if $M_1(\Delta) \leq M_2(\Delta)$, $\Delta \in \mathcal{B}(\Gamma)$.

Lemma 3.1. For any directed subset $\mathcal{D}$ of $\mathcal{W}_x$, there exists a least upper bound.

Proof. According to the remarks preceding the lemma it is enough to show that the subset $\mathcal{D} := \{ W d\tau_x: W \in \mathcal{D} \}$ of $\mathcal{W}_x$ has the least upper bound. For $\Delta \in \mathcal{B}(\Gamma)$, let $\mathcal{D}_\Delta$ be the set of matrices of the form

$$\sum_{j=1}^{n} M_j(\Delta_j),$$

where $M_1, \ldots, M_n \in \mathcal{D}$, and $\{\Delta_1, \ldots, \Delta_n\}$ is a partition of $\Delta$, $n \in N$. The matrix $M_x(\Delta)$ is an upper bound of $\mathcal{D}_\Delta$. Moreover, $\mathcal{D}_\Delta$ is a directed set. In fact, if (3.1) and

$$\sum_{k=1}^{m} M_k(\Delta_k)$$

are two elements of $\mathcal{D}_\Delta$, consider $M_{jk} \in \mathcal{D}$ such that $M_j \leq M_{jk}$, $M_k' \leq M_{jk}$, $j = 1, \ldots, n$, $k = 1, \ldots, m$. Then $\sum_{j=1}^{n} \sum_{k=1}^{m} M_{jk}(\Delta_j \cap \Delta_k')$ belongs to $\mathcal{D}_\Delta$ and exceeds both matrices (3.1) and (3.2). From Lemma 2.1 it follows that $\mathcal{D}_\Delta$ has the least upper bound $N(\Delta)$ and that

$$u^* N(\Delta) u = \sup \{ u^* Du: D \in \mathcal{D}_\Delta \}, \quad u \in C^q.$$  

Standard measure-theoretic arguments (cf. the proof of Theorem 5 of Section III.7 of [2]) show that, for $u \in C^q$, $u^* Nu$ is an additive function on $\mathcal{B}(\Gamma)$. Hence
N is additive. Since \( N \leq M_{x} \), it even belongs to \( \mathcal{D} \). Finally, from (3.3) it follows easily that \( N \) is the least upper bound of \( \mathcal{D} \).

If \( W \in \mathcal{W}_{x} \), set \( L^{2}(W) := L^{2}(W \delta_{x}) \). Moreover, we define

\[
\mathcal{W}_{x}^{(r)} := \{ W \in \mathcal{W}_{x} : L^{2}(W) \text{ is } \mathcal{J}\text{-regular} \}.
\]

**Lemma 3.2.** The set \( \mathcal{W}_{x}^{(r)} \) is directed.

**Proof.** Let \( W_{1}, W_{2} \in \mathcal{W}_{x}^{0} \) and let \( Q(\gamma) \) be the orthogonal projection in \( C^{q} \) onto the algebraic sum \( \mathcal{R}(W_{1}(\gamma)) + \mathcal{R}(W_{2}(\gamma)), \gamma \in \Gamma. \) From von Neumann’s alternating projections theorem (cf. [4], Problem 96) we can conclude the measurability of the function \( Q \). Let

\[
W_{x} = \begin{pmatrix} W_{x,11} & W_{x,12} \\ W_{x,12} & W_{x,22} \end{pmatrix}
\]

be the block partition of \( W_{x} \) w.r.t. the orthogonal decomposition

\[
C^{q} = QC^{q} \oplus (I - Q) C^{q}.
\]

Let us set

\[
W_{3} := \begin{pmatrix} W_{x,11} - W_{x,12} W_{x,12}^{+} W_{x,12}^{*} & 0 \\ 0 & 0 \end{pmatrix}.
\]

The measurability of \( Q \) and Lemmas 2.3 and 2.2 imply that \( W_{3} \in \mathcal{W}_{x} \). Moreover, from Lemma 2.2 it follows that \( W_{1} \leq W_{3} \) and \( W_{2} \leq W_{3} \). To complete the proof it is enough to show that \( L^{2}(W_{3}) \) is \( \mathcal{J}\text{-regular}. \) Let \( F \in L^{2}(W_{3}) \) be such that for each \( J \in \mathcal{J} \) it can be approximated by trigonometric polynomials with frequencies from \( J \) in \( L^{2}(W_{3}) \). Since \( W_{1} \leq W_{3} \), an analogous approximation exists in \( L^{2}(W_{1}) \). The \( \mathcal{J}\text{-regularity of } L^{2}(W_{1}) \) yields \( F = 0 \) in \( L^{2}(W_{1}) \). Similarly, \( F = 0 \) in \( L^{2}(W_{2}) \). Using Lemma 2.4, we can conclude that \( \mathcal{R}(W_{1}) + \mathcal{R}(W_{2}) \subseteq \ker F \). Since \( \mathcal{R}(W_{3}) \subseteq \mathcal{R}(W_{1}) + \mathcal{R}(W_{2}) \), it follows that \( F = 0 \) in \( L^{2}(W_{3}) \).

**Theorem 3.3.** The set \( \mathcal{W}_{x}^{(r)} \) has a unique maximal element.

**Proof.** By Lemmas 3.1 and 3.2, the set \( \mathcal{W}_{x}^{(r)} \) has the least upper bound \( W^{(r)} \in \mathcal{W}_{x} \). Assume that \( L^{2}(W^{(r)}) \) is not \( \mathcal{J}\text{-regular}. \) Then there exists \( F \in L^{2}(W^{(r)}), F \neq 0 \), such that, for each \( J \in \mathcal{J} \), \( F \) can be approximated by trigonometric polynomials with frequencies from \( J \). Let \( W \in \mathcal{W}_{x}^{(r)} \). Then, in particular, \( W \leq W^{(r)} \), and similar arguments to those in the proof of Lemma 3.2 show that

\[
\mathcal{R}(W) \subseteq \ker F.
\]

Let

\[
W^{(r)} = \begin{pmatrix} W_{11}^{(r)} & W_{12}^{(r)} \\ W_{12}^{(r)*} & W_{22}^{(r)} \end{pmatrix}
\]
be the block partition of \( W^{(\ell)} \) w.r.t. the orthogonal decomposition \( C^\ell = \mathcal{R}(F^\ast) \oplus \ker F \). Let us set

\[
W^{(\ell)} := \begin{pmatrix}
W_{11}^{(\ell)} - W_{12}^{(\ell)} W_{22}^{(\ell)} W_{12}^{(\ast \ell)} & 0 \\
0 & 0
\end{pmatrix}.
\]

It is not hard to see (cf. the proof of Lemma 3.2) that

\[
(3.5) \quad W^{(\ell)} \in \mathcal{W}, \quad W^{(\ell)} \leq W^{(\ell)}, \quad \text{and} \quad W \leq W^{(\ell)}.
\]

On the other hand, since \( F \neq 0 \) in \( L^2(W^{(\ell)}) \), Lemma 2.4 implies that there exists \( \Lambda \in \mathcal{R}(F) \) such that \( \tau_x(\Lambda) > 0 \) and \( \mathcal{R}(W^{(\ell)}) \) is not a subspace of \( \ker F \) on \( \Lambda \). It follows that \( W_{12}^{(\ell)} \neq 0 \) on \( \Lambda \), and hence \( W^{(\ell)} - W^{(\ell)} \neq 0 \) on \( \Lambda \). Combining this with (3.5), we obtain a contradiction to the definition of a least upper bound. Thus, \( L^2(W^{(\ell)}) \) is \( \mathcal{I} \)-regular and \( W^{(\ell)} \) is a maximal element of \( \mathcal{W}^{(\ell)} \). Its uniqueness follows from Lemma 3.2.

4. CONCORDANCE OF THE MAXIMAL REGULAR PART AND THE REGULAR PART OF THE WOLD DECOMPOSITION

Let \( x \) be a \( q \)-variate stationary process over \( G \) and \( \mathcal{I} \) a family of subsets of \( G \) invariant under translation. Let \( x_g = y_g + z_g, g \in G \), be the Wold decomposition of \( x \), where \( y \) is \( \mathcal{I} \)-regular and \( z \) is \( \mathcal{I} \)-singular. If we set \( \hat{W}_y := dM_y/d\tau_x \) and \( \hat{W}_z := dM_z/d\tau_x \), we have (cf. [9], Lemmas 4.3 and 4.4)

\[
(4.1) \quad \hat{W}_y + \hat{W}_z = W_x, \quad \mathcal{R}(\hat{W}_y) \cap \mathcal{R}(\hat{W}_z) = \{0\}.
\]

Let \( V_x \) be Kolmogorov's isomorphism of \( H_x \) onto \( L^2(W_x) \) and set

\[
V_x y_0 = : F_y \quad \text{and} \quad V_x z_0 = : F_z,
\]

where 0 is the neutral element of \( G \). It is not hard to see that

\[
(4.2) \quad F_y W_x F_y^* = \hat{W}_y \quad \text{and} \quad F_z W_z F_z^* = \hat{W}_z,
\]

\[
(4.3) \quad V_x H_y = \sqrt{\{ \langle g, \cdot \rangle F_y : g \in G \}} \quad \text{and} \quad V_x H_z = \sqrt{\{ \langle g, \cdot \rangle F_z : g \in G \}},
\]

and hence

\[
(4.4) \quad \sqrt{\{ \langle g, \cdot \rangle F_y : g \in G \}} \oplus \sqrt{\{ \langle g, \cdot \rangle F_z : g \in G \}} = L^2(W_x).
\]

From the relation (4.4) it follows that

\[
\int \langle g, \gamma \rangle F_y(\gamma) W_x(\gamma) F_z(\gamma)^* \tau_x(d\gamma) = 0, \quad g \in G,
\]

which yields

\[
(4.5) \quad F_y W_x F_z^* = 0.
\]
We can assume and we will do so in the sequel that
\[ \ker W_x \subseteq (\ker F_y \cap \ker F_z). \tag{4.6} \]
Then we have
\[ P_{W_x} = F_y + F_z \tag{4.7} \]
as well as
\[ \mathcal{R}(F_y W_x) = \mathcal{R}(F_y W_x^{1/2}) = \mathcal{R}(F_y) \quad \text{and} \quad \mathcal{R}(F_z W_x) = \mathcal{R}(F_z W_x^{1/2}) = \mathcal{R}(F_z). \tag{4.8} \]
Moreover, from (4.2) it follows that \( \mathcal{R}(\tilde{W}_y^{1/2}) = \mathcal{R}(F_y W_x^{1/2}) \) and \( \mathcal{R}(\tilde{W}_z^{1/2}) = \mathcal{R}(F_z W_x^{1/2}) \). Combining this with (4.8), we obtain
\[ \mathcal{R}(\tilde{W}_y) = \mathcal{R}(F_y) \quad \text{and} \quad \mathcal{R}(\tilde{W}_z) = \mathcal{R}(F_z). \tag{4.9} \]
Let \( W^{(r)} \) be the maximal element of \( \mathcal{W}^{(r)} \). We wish to show that \( W^{(r)} \) coincides with \( \tilde{W}_y \). In order to prove this we first derive some properties of \( F_z \), which eventually lead to the conclusion that \( P_{F_z} \neq 0 \) in \( L^2(W^{(r)}) \). Then we will see that the assumption \( W^{(r)} \neq \tilde{W}_y \) would imply that \( P_{F_z} \neq 0 \) in \( L^2(W^{(r)}) \).

**Lemma 4.1.** The values of \( F_z \) are diagonalizable matrices.

**Proof.** From (4.5) and (4.7) it follows that
\[ F_z W_x F_z^* = W_x F_z^*. \tag{4.10} \]
Since \( \mathcal{R}(F_z W_x F_z^*) = \mathcal{R}(F_z W_x^{1/2}) \), from (4.8) and (4.10) we obtain
\[ \mathcal{R}(F_z) = \mathcal{R}(W_x F_z^*) \subseteq \mathcal{R}(W_x). \tag{4.11} \]
On the other hand, (4.6) gives
\[ \mathcal{R}(F_z^*) \subseteq \mathcal{R}(W_x). \tag{4.12} \]
The relations (4.11) and (4.12) show that it is enough to prove that the restrictions \( \tilde{F}_z \) of \( F_z \) to \( \mathcal{R}(W_x) \) are diagonalizable. Denoting by \( \tilde{W}_x \) the restrictions of \( W_x \) to \( \mathcal{R}(W_x) \), from (4.10)-(4.12) we get \( \tilde{F}_z \tilde{W}_x F_z^* = \tilde{W}_x F_z^* \), which yields
\[ W_x^{-1/2} F_x W_x F_z^* \tilde{W}_x^{-1/2} = W_x^{1/2} F_z^* \]
This shows that the values of \( F_z^* \), and hence of \( \tilde{F}_z \), are similar to self-adjoint matrices, which implies that they are diagonalizable.

**Lemma 4.2.** We have \( \ker P_{F_z} \cap \mathcal{R}(F_z) = \{0\} \).

**Proof.** Let \( \gamma \in \Gamma \) and \( u \in (\ker P_{F_z}(\gamma)) \cap \mathcal{R}(F_z(\gamma)) \). Then \( u \in \ker F_z(\gamma) \), \( u = F_z(\gamma)v \) for some \( v \in \mathbb{C} \), and hence \( F_z(\gamma)^2 v = 0 \). If \( u \neq 0 \) were true, this would contradict Lemma 4.1.

**Lemma 4.3.** We have \( P_{F_z} = 0 \) in \( L^2(W^{(r)}) \).
Proof. From (4.5) we get \( F, W, P_{T} = 0 \). This implies that the function \( P_{T} \) is orthogonal (in \( L^2(W_x) \)) to \( \bigvee \{ \langle g, \cdot \rangle F_z : g \in G \} \). Examining the proof of the Wold decomposition (cf. the proof of Theorem 2.13 of [12]) and taking into account Kolmogorov's isomorphism, we obtain

\[
\bigvee \{ \langle g, \cdot \rangle F_z : g \in G \} = \bigcap_{j \in J} \bigvee \{ \langle g, \cdot \rangle I : g \in J \}.
\]

It follows that, for \( J \in \mathcal{J} \), \( P_{T} \) can be approximated by trigonometric polynomials with frequencies from \( J \) in \( L^2(W_x) \). Since \( W_x \leq W_x \), an analogous result is true for \( L^2(W_x') \). But since \( L^2(W_x') \) is \( \mathcal{J} \)-regular, we conclude that \( P_{T} = 0 \) in \( L^2(W_x') \).

**Theorem 4.4.** The functions \( W^{(x)} \) and \( W^{(y)} \) coincide.

**Proof.** Since \( W_y \in W_x \), it follows that \( W_x \leq W_x \). Assume that \( W_x \neq W_x \) on a set \( \Lambda \in \mathcal{R}(I) \) such that \( \tau_x(\Lambda) > 0 \). First note that \( \mathcal{R}(W_y) = \mathcal{R}(W_x) \) on \( \Lambda \). For if \( \mathcal{R}(W_y) = \mathcal{R}(W_x) \) and \( W_y \neq W_x \) were true on a set of positive measure \( \tau_x \), we would get \( \mathcal{R}(W_x - W_y) \cap \mathcal{R}(W_y) \neq \{0\} \), and because of \( \mathcal{R}(W_2) = \mathcal{R}(W_x - W_y) \supseteq \mathcal{R}(W_x) \neq \{0\} \), which contradicts (4.1). Thus, \( \mathcal{R}(W_x) \) is a proper subspace of \( \mathcal{R}(W_x) \) on \( \Lambda \). Then from (4.1) and (4.9) it follows that \( \mathcal{R}(W_x) \cap \mathcal{R}(F_x) \neq \{0\} \) on \( \Lambda \). Combining this with Lemma 4.2, we infer that \( \mathcal{R}(W_x) \) is not a subspace of \( \ker P_{T} \) on \( \Lambda \). Applying Lemma 2.4, we conclude that \( P_{T} \neq 0 \) in \( L^2(W_x') \), which is a contradiction to Lemma 4.3.

Let us mention the following consequence of Theorem 4.4.

**Corollary 4.5.** If \( L^2(W_x) \) is \( \mathcal{J} \)-singular, then for \( W \in W_x \) so is \( L^2(W) \).

**Proof.** The \( \mathcal{J} \)-singularity of \( L^2(W_x) \) and Theorem 4.4 imply that \( W^{(y)} = \{0\} \). For \( W \in W_x \), consider the Wold decomposition of the corresponding stationary process over \( G \). Since the spectral measure of its \( \mathcal{J} \)-regular part belongs to \( W_x \), it is zero measure. Thus, \( L^2(W) \) is \( \mathcal{J} \)-singular.

**Remark 4.6.** It would be of interest to have generalizations of Theorem 4.4 to the infinite-variate case. Treil' ([13], Theorem 3.1) gave such a result if \( G \) is the group of integers and \( \mathcal{J} \) is the family of translates of the set of non-negative integers.

### 5. The Maximal \( \mathcal{J}_0 \)-Regular Part

Let \( G \) be a discrete Abelian group, \( \mathcal{J}_0 \) the family of complements of all singletons of \( G \), and \( \sigma \) the normalized Haar measure of \( G \). Let \( M_x \) be the spectral measure of a \( q \)-variate stationary process over \( G \).

**Theorem 5.1** ([7], Theorem 5.3). The space \( L^2(M_x) \) is \( \mathcal{J}_0 \)-regular if and only if

(i) \( M_x \) is absolutely continuous w.r.t. \( \sigma \),
(ii) \( \mathcal{R}(dM_x/d\sigma) = \text{const } \sigma\text{-a.e.} \)

(iii) \( (dM_x/d\sigma)^+ \) is integrable w.r.t. \( \sigma \).

It follows that the maximal \( \mathcal{J}_0 \)-regular parts of \( M_x \) and of the absolutely continuous part of \( M_x \) coincide. Thus we can assume that \( M_x \) is absolutely continuous w.r.t. \( \sigma \) and replace the measure \( \tau_x \) of the preceding sections by \( \sigma \). For simplicity, now denote by \( W_x \) the function \( W_x = dM_x/d\sigma \) and according to this notation define the corresponding objects \( W_x \) etc. of Sections 3 and 4.

Let us set

\[
L_1 := \{ u \in C^q : u^* W_x^+ u \text{ is integrable w.r.t. } \sigma \},
\]

\[
L_2 := \{ u \in C^q : u \in \mathcal{R}(W_x) \text{ } \sigma\text{-a.e.} \},
\]

\[
L := L_1 \cap L_2.
\]

Remark 5.2. Note that the space \( L \) coincides with the space \( \mathcal{M} \) which appeared in Theorem 4.5 of [6] and was identified there as the range of the Grammian interpolation error matrix. Note further that \( L \) is the orthogonal complement of the space \( H \) of Lemma 9 of [5].

Let

\[
W_x = \begin{pmatrix} W_{x,11} & W_{x,12} \\ W_{x,12}^* & W_{x,22} \end{pmatrix}
\]

be the block representation of \( W_x \) w.r.t. the orthogonal decomposition \( C^q = L \oplus L^1 \). Set

\[
W^{(v)} := \begin{pmatrix} (W_x/W_{x,22}) & 0 \\ 0 & 0 \end{pmatrix}, \quad W^{(a)} := \begin{pmatrix} W_{x,12} & W_{x,22}^+ & W_{x,12}^* & W_{x,12} \\ W_{x,12}^* & W_{x,22} & 0 & 0 \end{pmatrix}.
\]

Using Theorem 4.4 we will show that \( W^{(v)} d\sigma \) is the spectral measure of the \( \mathcal{J}_0 \)-regular part of the Wold decomposition of \( x \), and hence \( W^{(a)} = W_x - W^{(v)} \) is the spectral measure of the \( \mathcal{J}_0 \)-singular part.

Lemma 5.3. The spaces \( \mathcal{R}(W^{(v)}) \) are equal to \( L \) \( \sigma\text{-a.e.} \).

Proof. Clearly, \( \mathcal{R}(W^{(v)}) \subseteq L \). On the other hand, \( L \subseteq \mathcal{R}(W_x) = \mathcal{R}(W^{(v)}) + \mathcal{R}(W^{(a)}) \). Thus, if \( L \) were not a subspace of \( \mathcal{R}(W^{(v)}) \), we would have \( \mathcal{R}(W^{(a)}) \cap L \neq \{0\} \). However, using (i) of Lemma 2.2 we easily get \( \mathcal{R}(W^{(a)}) \cap L = \{0\} \). It follows that \( \mathcal{R}(W^{(v)}) = L \) \( \sigma\text{-a.e.} \).

Lemma 5.4. The function \( W^{(v)} + \) is integrable w.r.t. \( \sigma \).

Proof. Since \( L \subseteq \mathcal{R}(W_x) \), we have \( \ker W_x \subseteq L^1 \), and taking into account (i) of Lemma 2.2 we easily obtain \( \ker W_x = \ker W_{22} \). Thus the generalized Banachiewicz inversion formula (cf. [8], formula (3.32)) is applicable, which implies that the left upper corner of \( W_x^+ \) is equal to \( (W_x/W_{x,22})^{-1} \). From the definition of \( L \) it follows that \( (W_x/W_{x,22})^{-1} \) is integrable w.r.t. \( \sigma \) and so is \( W^{(v)} + \).
Lemma 5.5. The space $L^2(W^{(p)})$ is $\mathcal{J}_0$-regular.

Proof. The result follows immediately from Theorem 5.1 and Lemmas 5.3 and 5.4.

Lemma 5.6. Let $W \in \mathcal{W}^{(p)}$. Then $W \leq W^{(p)}$.

Proof. According to Theorem 5.1 there exists a subspace $L_0$ of $C^q$ such that $\mathcal{R}(W) = L_0$ $\sigma$-a.e. Assume that $u \in L_0 \cap L^2$, $u \neq 0$. Then $u$ can be written as $u = u_1 + u_2$ for some $u_1 \in L^1_1$, $u_2 \in L^2$. If $u_1 = 0$, there exists $\lambda \in \mathcal{R}(I)$ such that $\sigma(\lambda) > 0$ and $u = u_2 \notin \mathcal{R}(W_x(\gamma))$ for $\sigma$-a.a. $\gamma \in \lambda$. This contradicts the inclusion $\mathcal{R}(W) \subseteq \mathcal{R}(W_x) \sigma$-a.e. It follows that $u_1 \neq 0$, and hence $u \notin L_1$. From the definition of $L_1$ we infer that $u^* W_x^+ u$ is not integrable w.r.t. $\sigma$. Let $W_0$ be the restriction of $W$ to $L_0$ and let

$$W_x = \begin{pmatrix} W^{(0)}_{x,11} & W^{(0)}_{x,12} \\ W^{(0)}_{x,12} & W^{(0)}_{x,22} \end{pmatrix}$$

be the block representation of $W_x$ w.r.t. the orthogonal decomposition $C^q = L_0 \oplus L_0^\perp$. From the definition of $\mathcal{W}^{(p)}_x$ and (iii) of Lemma 2.2 we obtain $W_0 \leq \left(W_x/W^{(0)}_{x,22}\right)^{-1}$, and hence $(W_x/W^{(0)}_{x,22})^{-1} \leq W_0^{-1}$. By the generalized Banachiewicz inversion formula it follows that

$$u^* W_x^+ u \leq u^* \left(W_x/W^{(0)}_{x,22}\right)^{-1} u \leq u^* W_0^{-1} u = u^* W_0 W^+ u.$$

Thus $u^* W_x^+ u$ is not integrable w.r.t. $\sigma$, which contradicts Theorem 5.1. We conclude that $L_0 \subseteq L$. Then again the definition of $\mathcal{W}^{(p)}_x$ and (iii) of Lemma 2.2 imply that the restriction of $W$ to $L$ does not exceed $(W_x/W_{x,22})$, which yields $W \leq W^{(p)}$ $\sigma$-a.e.

Combining Lemmas 5.5 and 5.6 with Theorem 4.4 we get the following result.

Theorem 5.7. The measures $W^{(p)} \, d\sigma$ and $W^{(p)} \, d\sigma$ are the spectral measures of the $\mathcal{J}_0$-regular and $\mathcal{J}_0$-singular parts of the Wold decomposition of $x$, respectively.

References


Fakultät für Mathematik und Informatik
Universität Leipzig
04109 Leipzig, Germany

Received on 6.12.2001