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MULTIPLY c-DECOMPOSABLE PROBABILITY MEASURES ON R AND THEIR CHARACTERISTIC FUNCTIONS

BY

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Abstract. We obtain the characteristic functions of distributions in $L_{c,\alpha}$, i.e. α-times c-decomposable distributions in the class of infinitely divisible distributions, where $0 < \alpha \leqslant \infty$, $0 < c < 1$. The characteristic functions of α-times selfdecomposable laws (i.e. α-times c-decomposable for each $c \in (0, 1)$) are well known (see [3], [5], [9], [13]).

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1. INTRODUCTION AND NOTATION

Given a probability measure P on R, $|c| \leqslant 1$, we say that P is *c-decomposable* if $P = T_c P * P_c$ for some probability measure P_c, where $T_c x = cx$ $(x \in R)$, $T_c P(B) = P(T_c^{-1} B)$ for any non-zero c and Borel set B, and $T_0 P = \delta_0$. For a probability measure P, $\tilde{P}$ is defined to be the probability measure given by $\tilde{P}(A) = P(-A)$. We must mention the name of Loève as a pioneer of the decomposable problem [6] (this history can be also found in Bunge [1]). Loève showed in [6] that (if $0 < |c| < 1$) P is c-decomposable if and only if P is of the form $P = *_{k=0}^{\infty} T_{c^k} P_c$ for some probability measure P_c. He denoted the set of all c-decomposable laws by L_c. The class L, or the set of self-decomposable laws, is defined as $L = \bigcap_{c\in(0,1)} L_c$. A generalization of c-decomposable laws to the multiple case is given in [8]. Namely, for a given $n \in N$ we say [8] that a probability measure P is *n-times c-decomposable* if there exist probability measures $P_{c,(1)}, \ldots, P_{c,(n)}$ such that

$$(1.1) \qquad P = T_c P * P_{c,(1)} * \ldots * P_{c,(n-1)} = T_c P_{c,(n-1)} * P_{c,(n)}.$$

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Then, by (1.1), P is n-times c-decomposable if and only if P is of the form

$$P = \mathop{*}_{k=0}^{\infty} T_{c^k} P_{c,(n)}^{r(k,n)}, \tag{1.2}$$

$$r(k, n) = n(n+1)\ldots(n+k-1)/k! = \Gamma(k+n)/\Gamma(n)\Gamma(k+1),$$

where the power is taken in the convolution sense. The formula (1.2) suggests to generalize the concept of n-times c-decomposable probability measures to the non-integer case (see [8]).

Let Id denote the class of all infinitely divisible measures on $\mathbf{R}$. For $\alpha > 0$, $r(k, \alpha)$ is given by (1.2) with α in place of n. A probability measure $P \in Id$ is said to be *α-times c-decomposable* (in Id, $0 < c < 1$, $\alpha > 0$) if there exists $P_{c,(\alpha)} \in Id$ such that

$$P = \mathop{*}_{k=0}^{\infty} T_{c^k} P_{c,(\alpha)}^{r(k,\alpha)}. \tag{1.3}$$

Let $L_{c,\alpha}$ denote the subclass of Id consisting of probability measures P such that (1.3) holds for some $P_{c,(\alpha)} \in Id$. It is well known [8] that the infinite convolution (1.3) is convergent if and only if $P_{c,(\alpha)}$ has the finite $\log^{\alpha}$-moment, i.e.

$$\int_{-\infty}^{\infty} \log^{\alpha}(1+|x|) P_{c,(\alpha)}(dx) < \infty. \tag{1.4}$$

We define the class of completely c-decomposable measures by the formula $L_{c,\infty} = \bigcap_{\alpha>0} L_{c,\alpha}$ (see [1] and [7]). We note that P is completely c-decomposable if and only if it is n-times c-decomposable for every $n \in \mathbf{N}$. The probability measures in $L_{\alpha} = \bigcap_{0<c<1} L_{c,\alpha}$ are called *α-times self-decomposable* for $0 < \alpha < \infty$, and *completely self-decomposable* for $\alpha = \infty$. The measures in the classes L_{α} $(0 < \alpha \leqslant \infty)$ of multiply self-decomposable measures were also investigated on multidimensional spaces. In particular, their characteristic functionals are well known (Kumar and Schreiber [5], Sato [13], Jurek [3], Nguyen van Thu [9]).

In this paper we give characteristic functions of multiply c-decomposable distributions, i.e. distributions in $L_{c,\alpha}$ $(0 < \alpha \leqslant \infty)$.

2. MEASURES IN $L_{c,\alpha}$

Let $\varphi(t)$ be the characteristic function of $P \in Id$,

$$\varphi(t) = \exp\left\{ibt + \int_{-\infty}^{\infty} g_t(u) \frac{1+u^2}{u^2} \mu(du)\right\}, \tag{2.1}$$

where $g_t(u) = e^{itu} - 1 - itu/(1+u^2)$, b is a real constant, and μ is a finite Borel measure on $\mathbf{R}$. The function φ determines uniquely b and μ. Then for the measure $\nu = \nu(\mu)$ given by

$$\nu = (1+u^2)/u^2 \, \mu|_{(-\infty,0)\cup(0,\infty)} \tag{2.2}$$

we have

$$\Big(\int_{-\infty}^{0} + \int_{0}^{\infty}\Big) u^2/(1+u^2)\,\nu(du) < \infty \quad \text{or, equivalently,} \quad \Big(\int_{-1}^{0} + \int_{0}^{1}\Big) u^2\,\nu(du) < \infty. \tag{2.3}$$

We shall call μ and ν the *Khintchine* and the *Lévy* (spectral) *measure*, respectively, corresponding to P.

Let $\nu_{c,(\alpha)}$ be the Lévy measure corresponding to $P_{c,(\alpha)}$ satisfying (1.3). It is well known [8] that the following conditions are equivalent:

(a) the infinite convolution (1.3) is convergent;

(b) $P_{c,(\alpha)}$ has a finite $\log^{\alpha}$-moment, i.e. (1.4) holds;

(c) $\nu_{c,(\alpha)}$ satisfies the following condition:

$$\Big(\int_{-\infty}^{-1} + \int_{1}^{\infty}\Big) \log^{\alpha}(|x|)\,\nu_{c,(\alpha)}(dx) < \infty; \tag{2.4}$$

(d) the following series is convergent:

$$\nu = \sum_{k=0}^{\infty} r(k,\alpha)\, T_{c^k}\,\nu_{c,(\alpha)}. \tag{2.5}$$

Note that the Lévy measures corresponding to distributions from $L_{c,\alpha}$ are of the form (2.5). We now apply (2.5) to obtain the characteristic function of α-times c-decomposable distributions $(0 < \alpha < \infty)$.

THEOREM 2.1. *The function φ is the characteristic function of $P \in L_{c,\alpha}$ $(0 < \alpha < \infty)$ if and only if φ is of the form*

$$\varphi(t) = \exp\Big\{ibt - Gt^2/2 + \Big(\int_{-\infty}^{0} + \int_{0}^{\infty}\Big) \sum_{k=0}^{\infty} r(k,\alpha)\, g_t(c^k u)\, \nu_{c,(\alpha)}(du)\Big\}, \tag{2.6}$$

where $b \in \mathbf{R}$, $G \geqslant 0$, and $\nu_{c,(\alpha)}$ is a Borel measure on $(-\infty, 0)\cup(0, \infty)$ such that $\big(\int_{-1}^{0} + \int_{0}^{1}\big) u^2\, \nu_{c,(\alpha)}(du) < \infty$ and the condition (2.4) *is satisfied. The function φ determines uniquely b, G and $\nu_{c,(\alpha)}$.*

We shall find the relations of the representations (2.5) and (2.6) of Lévy measure and the characteristic function, respectively, corresponding to α-times c-decomposable distributions with the representations of Lévy measure and the characteristic function corresponding to α-times self-decomposable distributions. We start the study with the following lemma.

LEMMA 2.2. *If $\alpha > 0$, $h > 0$, and $x, y \in \mathbf{R}$, then*

$$\Gamma(\alpha+1)\, h^{\alpha} \sum_{j=0}^{\infty} r(j,\alpha)\, \chi_{(-\infty, y-jh]}(x) \to \big((y-x)_+\big)^{\alpha} \quad \text{as } h \to 0. \tag{2.7}$$

Proof. Let g be a non-negative non-increasing function on $\boldsymbol{R}$. It is not difficult to prove that

$$h\sum_{j=1}^{\infty} g(x+jh) \leqslant \int_{x}^{\infty} g(u)\,du \leqslant h\sum_{j=0}^{\infty} g(x+jh), \tag{2.8}$$

which implies that the series $\sum_{j=0}^{\infty} g(x+jh)$ is convergent if and only if $\int_x^{\infty} g(u)\,du < \infty$. From (2.8) we obtain $h\sum_{j=0}^{\infty} g(x+jh) \to \int_x^{\infty} g(u)\,du$ as $h \to 0$. We can prove inductively that for each $n \in N$

$$h^n \sum_{j=0}^{\infty} r(j,n)\, g(x_n+jh) \to \int_{x_n}^{\infty}\int_{x_{n-1}}^{\infty} \ldots \int_{x_1}^{\infty} g(x_0)\,dx_0\,dx_1 \ldots dx_{n-1} \quad \text{as } h \to 0. \tag{2.9}$$

Applying (2.9) to $g(x) = \chi_{(-\infty,0]}(x)$ we have

$$n!\,h^n \sum_{j=0}^{\infty} r(j,n)\,\chi_{(-\infty,-jh]}(x) \to \big((-x)_+\big)^n \quad \text{as } h \to 0. \tag{2.10}$$

Let $\alpha > 0$. It is not difficult to prove that $\sum_{j=0}^{k} r(j,\alpha) = r(k,\alpha+1)$, $k \in N$, which gives

$$\sum_{j=0}^{\infty} r(j,\alpha)\,\chi_{(-\infty,-jh]}(x) = \sum_{j=0}^{\infty} r(j,\alpha+1)\,\chi_{(-(j+1)h,-jh]}(x).$$

Recall that (cf. [8]) for $0 \leqslant \alpha \leqslant 1$ the following inequalities hold:

$$k^{1-\alpha} \leqslant \Gamma(k+1)/\Gamma(k+\alpha) \leqslant (k+1)^{1-\alpha}. \tag{2.11}$$

In the case $0 \leqslant \alpha \leqslant 1$, by (1.2) and (2.11) we obtain

$$\Gamma(\alpha+1)\,h^{\alpha} \sum_{j=0}^{\infty} r(j,\alpha)\,\chi_{(-\infty,-jh]}(x) \to \big((-x)_+\big)^{\alpha} \quad \text{as } h \to 0. \tag{2.12}$$

In the case $\alpha > 1$ we can write α in the form $\alpha = n+\beta$, where $n \in N$, $0 \leqslant \beta < 1$. Assume that $\beta > 0$. Putting

$$I_{h,\alpha}(g)(x) = \sum_{j=0}^{\infty} r(j,\alpha)\,g(x+jh),$$

we have $I_{h,\alpha_1+\alpha_2}(g) = I_{h,\alpha_1}\big(I_{h,\alpha_2}(g)\big)$. Then $I_{h,\alpha}(g) = I_{h,n}\big(I_{h,\beta}(g)\big)$. Applying (2.8) with $I_{h,\beta}(g)$ in place of g and putting $g = \chi_{(-\infty,0]}$, similarly to the above, we infer that (2.12) holds for $\alpha > 1$.

Thus we have shown that (2.12) holds for all $\alpha > 0$, which gives (2.7) and the lemma is proved.

THEOREM 2.3. *A probability measure P is α-times self-decomposable $(0 < \alpha < \infty)$ if and only if its Lévy spectral measure ν is of the form*

$$(2.13)\qquad \nu(dx) = \Big[\int_0^\infty \frac{\alpha}{x}\Big(\ln\frac{v}{x}\Big)_+^{\alpha-1}\gamma_\alpha(dv)\,\chi_{(0,\infty)}(x) + \int_{-\infty}^0 \frac{\alpha}{|x|}\Big(\ln\frac{v}{x}\Big)_+^{\alpha-1}\gamma_\alpha(dv)\,\chi_{(-\infty,0)}(x)\Big]dx$$

or, equivalently, its characteristic function φ is of the form

$$(2.14)\qquad \varphi(t) = \exp\{ibt - Gt^2/2 + (\int_{-\infty}^0 + \int_0^\infty)\int_0^\infty g_t(ve^{-y})\alpha y^{\alpha-1}\,dy\,\gamma_\alpha(dv)\},$$

where γ_α is a Borel measure on $(-\infty, 0)\cup(0, \infty)$ such that

$$(\int_{-\infty}^0 + \int_0^\infty)\int_0^\infty (ve^{-y})^2(1+(ve^{-y})^2)^{-1}y^{\alpha-1}\,dy\,\gamma_\alpha(dv) < \infty.$$

Proof. Let ν be the Lévy spectral measure corresponding to α-times self--decomposable probability measure P, i.e. $P\in L_{c,\alpha}$ for each $c\in(0, 1)$. Then so is $\tilde\nu$ defined by $\tilde\nu(B) = \nu(-B)$, $B\subset(-\infty, 0)\cup(0, \infty)$.

Thus it is sufficient to assume that ν is concentrated on $(0, \infty)$. Let $\bar\nu$ be the measure on $\boldsymbol{R}$ defined by $\bar\nu(\ln B) = \nu(B)$, $B\subset(0, \infty)$. Let f and $\bar f$ be the distribution functions of ν and $\bar\nu$, respectively, i.e. $f(x) = \nu((x, \infty))$, $x > 0$, and $\bar f(y) = \bar\nu((y, \infty))$, $y\in\boldsymbol{R}$. Then $\bar f(\ln x) = f(x)$, $x > 0$. By (2.5) we see that $\bar\nu$ is of the form

$$(2.15)\qquad \bar\nu = \sum_{j=0}^\infty r(j, \alpha)\,U_{-jh}\,\bar\nu_{c,\alpha},$$

where $h = -\ln c$, $U_a(x) = x+a$, $x, a\in\boldsymbol{R}$. Defining $\bar\nu_{c,\alpha}$ as

$$\bar\nu_{c,\alpha}(dx) = \int_{-\infty}^\infty \delta_u(x)\,\bar\nu_{c,\alpha}(du)$$

we can rewrite (2.15) in the form

$$(2.16)\qquad \bar\nu(dy) = \int_{-\infty}^\infty \sum_{j=0}^\infty r(j, \alpha)\,U_{-jh}\,\delta_u(y)\,\bar\nu_{c,\alpha}(du).$$

Then the distribution function of $\bar\nu$ is of the form

$$\bar f(y) = \int_{-\infty}^\infty \sum_{j=0}^\infty r(j, \alpha)\,\chi_{(-\infty,u-jh]}(y)\,\bar\nu_{c,\alpha}(du)$$

or, equivalently,

$$(2.17)\qquad \bar f(y) = \int_{-\infty}^\infty \Gamma(\alpha+1)h^\alpha \sum_{j=0}^\infty r(j, \alpha)\,\chi_{(-\infty,u-jh]}(y)\,\bar\gamma_{c,\alpha}(du),$$

where $\bar{\gamma}_{c,\alpha}(du) = \left(\Gamma(\alpha+1)h^{\alpha}\right)^{-1}\bar{v}_{c,\alpha}(du)$. Now it is very easy to obtain the well--known Lévy spectral measure representation of α-times self-decomposable laws. From the relation (2.17), by letting h tend to zero and making use of Lemma 2.2 and Helly's theorem, we conclude that

$$\bar{f}(y) = \int_{-\infty}^{\infty} (u-y)_+^{\alpha}\, \bar{\gamma}_{c,\alpha}(du), \qquad y \in \boldsymbol{R}.$$

Consequently, $f(x) = \int_0^{\infty} (\ln vx^{-1})_+^{\alpha}\, \bar{\gamma}_{c,\alpha}(dv)$, $x > 0$, and

$$(2.18) \qquad v(dx) = \left(\int_0^{\infty} \frac{\alpha}{x}\left(\ln\frac{v}{x}\right)_+^{\alpha-1} \bar{\gamma}_{c,\alpha}(dv)\right)dx, \qquad x > 0.$$

The formula (2.18) gives a representation of $v|_{(0,\infty)}$. For the general case note that v is given by (2.13). An easy computation shows that the characteristic function φ is given by the formula (2.14) and the theorem is proved.

Now we are going to describe the classes $L_{c,\infty}$.

3. EXTREME POINTS

Let $P \in Id$ and let v be the Lévy measure corresponding to P. By (1.1) and (2.5), the following conditions are equivalent:

(a) $P \in L_{c,\infty}$;

(b) for every $k = 1, 2, \ldots$ the measure v is of the form

$$(3.1) \qquad v = \sum_{j=0}^{\infty} r(j, k)\, T_{c^j}\, v_{c,(k)},$$

where $v_{c,(k)}$ is a Borel measure on $(-\infty, 0) \cup (0, \infty)$ satisfying the condition (2.3) (with k in place of α);

(c) for every $k = 1, 2, \ldots$ the measure $v_{c,(k)}$ satisfies the following inequalities:

$$(3.2) \qquad v_{c,(k)} - T_c\, v_{c,(k)} \geqslant 0,$$

where $P_{c,(0)} = P$, $P_{c,(k)}$ $(k = 1, 2, \ldots)$ are measures given by (1.1) and $v_{c,(k)}$ $(k = 0, 1, 2, \ldots)$ are Lévy measures corresponding to $P_{c,(k)}$.

By (1.1) we have the following equalities:

$$(3.3) \qquad v_{c,(k+1)} = v_{c,(k)} - T_c\, v_{c,(k)} \qquad (k = 0, 1, 2, \ldots).$$

Put $M^0 = \{\mu$: μ is a finite Borel measure on $\boldsymbol{R}$ such that $v = v(\mu)$ given by (2.2) is of the form (3.1) for each $k \in N\}$. Then the set of Khintchine measures corresponding to $P \in L_{c,\infty}$ coincides with the set M^0. We put $M = \{\mu$: μ is a finite Borel measure on $[-\infty, \infty]$ such that $\mu|_{\boldsymbol{R}} \in M^0\}$. Let K be the subset of M consisting of probability measures and $K^0 = K \cap M^0$. The convexity of

K follows easily from the definition. The space of all probability measures on $[-\infty, \infty]$ with weak convergence is a metrizable compact space. We consider the induced topology on K. We shall prove that K is closed and, consequently, compact. First we shall find the extreme points of the set K. Let us denote by $e(K)$ the set of extreme points of K. Put $Y_c = \{y\colon 1 \leqslant |y| < 1/c\}$. The following lemma is obvious.

LEMMA 3.1. *If $\mu \in e(K)$, then μ is concentrated on one of the following sets*: $\{-\infty\}$, $\{\infty\}$, $(0, \infty)$, $(-\infty, 0)$, *and* $\{0\}$.

LEMMA 3.2. *Let $\mu \in e(K)$. If μ is concentrated on the set $(-\infty, 0) \cup (0, \infty)$, then μ is concentrated on the set of the form*

$$\{y_0 c^{-k}\}_{k=-\infty}^{\infty}, \tag{3.4}$$

where $y_0 \in Y_c$.

Proof. Let $\mu \in e(K)$. Since $\mu \in e(K)$ if and only if $\tilde{\mu} \in e(K)$, it is sufficient to assume that μ is concentrated on $(0, \infty)$. Suppose that there exists $1 < \varepsilon < 1/c$ such that $\mu(A_1) > 0$ and $\mu(A_2) > 0$, where $A_1 = A_1(\varepsilon) = \bigcup_{k=-\infty}^{\infty} c^{-k}[1, \varepsilon)$ and $A_2 = A_2(\varepsilon) = \bigcup_{k=-\infty}^{\infty} c^{-k}[\varepsilon, 1/c)$. Then we have the equality

$$\mu = \alpha_1 \mu_1 + \alpha_2 \mu_2, \tag{3.5}$$

where $\alpha_i = \mu(A_i)$, $\mu_i = \alpha_i^{-1} \mu|_{A_i}$. Since $A_1 \cap A_2 = \emptyset$, there is no $C > 0$ such that $\mu_1 = C\mu$. By (3.4) we see that $\nu = \alpha_1 \nu_1 + \alpha_2 \nu_2$, where ν, ν_1, ν_2 are the Lévy measures corresponding to μ, μ_1, μ_2, respectively. Let $k \in N$. Since ν is of the form (3.1), we obtain

$$\nu|_{A_i} = \sum_{j=0}^{\infty} r(j, k)\, T_{c^j}(\nu_{c,(k)}|_{A_i}), \qquad i = 1, 2. \tag{3.6}$$

Obviously, for every $i = 1, 2$, $\alpha_i \nu_i$ is the Lévy measure corresponding to the Khintchine measure $\alpha_i \mu_i$ and $\alpha_i \nu_i = \nu|_{A_i}$. Thus by (3.5) we obtain $\alpha_i \mu_i \in M$ and consequently, $\mu_i \in K$. By (3.4) this contradicts that μ is the extreme point.

Thus we infer that for every $1 < \varepsilon < 1/c$ either $\mu(A_1(\varepsilon)) = 0$ or $\mu(A_2(\varepsilon)) = 0$. This implies that there exists $y_0 \in Y_c \cap (0, \infty)$ such that μ is concentrated on the set $\{y_0 c^{-k}\}_{k=-\infty}^{\infty}$. Thus the lemma is proved.

We use the following notation given in [14] and [2]. We say that a function f is *completely monotone* if it has derivatives of any finite order such that $(-1)^n f^{(n)}(x) \geqslant 0$ $(n = 0, 1, 2, \ldots)$. Completely monotone functions on $(0, \infty)$ are characterized by the Bernstein representation (see [14], Theorem 12a, p. 160).

PROPOSITION 3.3. *A function f on $(0, \infty)$ is completely monotone if and only if*

$$f(t) = \int_0^{\infty} e^{-tx} F(dx), \tag{3.7}$$

where F is a Borel measure on $[0, \infty]$. Moreover, F is uniquely determined by f.

Remark 3.4. A completely monotone function f on (a, ∞), where $-\infty \leqslant a < 0$, is also of the form (3.7). Moreover, if $-\infty < a < 0$, then $f(a+0) < \infty$ if and only if $e^{-ax} F(dx)$ is a finite measure. If $a = -\infty$, then $e^{bx} F(dx)$ is a finite measure for all $b > 0$.

Consider a sequence of real numbers $\{a_n\}_{n=0}^{\infty}$. We define the sequence given by the formula $\Delta a_n = a_{n+1} - a_n$, $n = 0, 1, 2, \ldots$ Further, we define inductively sequences as follows: $\Delta^1 a_n = \Delta a_n$, $\Delta^k a_n = \Delta(\Delta^{k-1} a_n)$, $k = 2, 3, \ldots$ The sequence $\{a_n\}_{n=0}^{\infty}$ will be called *k-times monotone*, $k = 1, 2, \ldots$, if (see [4])

$$(3.8) \qquad \Delta^j a_n \geqslant 0, \quad j = 1, 2, \ldots, k, \; n = 0, 1, 2, \ldots$$

It is well known [4] that

$$\Delta^k a_n = \binom{k}{0} a_n - \binom{k}{1} a_{n+1} + \ldots + (-1)^k \binom{k}{k} a_{n+k}.$$

The sequence $\{a_n\}_{n=0}^{\infty}$ will be called *completely monotone* if it is k-monotone for $k = 1, 2, \ldots$ We say that a sequence $\{a_n\}_{n=0}^{\infty}$ is *minimal* if decreasing a_0 makes of it a sequence which is no longer completely monotone (see [2], [14]). We say that a sequence $\{a_n\}_{n=-\infty}^{\infty}$ is *completely monotone* if each of the sequences $\{a_n\}_{n=-k}^{\infty}$, $k = 1, 2, \ldots$, is completely monotone. If $\{a_n\}_{n=-\infty}^{\infty}$ is completely monotone, then each of the sequences $\{a_n\}_{n=-k}^{\infty}$, $k = 1, 2, \ldots$, is minimal.

PROPOSITION 3.5 (Feller [2]). *Let $\{a_n\}_{n=-\infty}^{\infty}$ be a completely monotone sequence, $x_n = nh$, $h > 0$, $n \in \boldsymbol{Z}$ ($\boldsymbol{Z} = \{0, \pm 1, \pm 2, \ldots\}$). Then there is a uniquely determined function $f(x)$ on $\boldsymbol{R}$ such that $f(x)$ is completely monotone and*

$$(3.9) \qquad f(x_n) = a_n, \quad n \in \boldsymbol{Z}.$$

Remark 3.6. In [2] the function $f(x)$ is defined such that it is completely monotone for $x \geqslant x_0$ and $f(x_n) = a_n$, $n = 1, 2, \ldots, f(x_0) \leqslant a_0$, where $a_0, a_1, \ldots$ is a completely monotone sequence corresponding to a sequence of real numbers $0 \leqslant x_0 < x_1 < \ldots$ such that the series $\sum 1/x_n$ diverges. But if $a_0, a_1, \ldots$ is minimal, then we have the equality $f(x_0) = a_0$.

LEMMA 3.7. *Let $\mu \in e(K)$ and assume that μ is concentrated on the set $(-\infty, 0) \cup (0, \infty)$. Then $\nu = \nu(\mu)$ is of the form*

$$(3.10) \qquad \nu = C_0 \sum_{k=-\infty}^{\infty} (c^k)^{-\alpha_0} \delta_{y_0 c^k},$$

where $0 < \alpha_0 < 2$, $y_0 \in Y_c$, and

$$(3.11) \qquad C_0 = \Big\{ \sum_{k=-\infty}^{\infty} [(y_0 c^k)^2 (c^k)^{-\alpha_0}]/[1 + (y_0 c^k)^2] \Big\}^{-1}.$$

Proof. Let $\mu \in e(K)$ and let μ be concentrated on the set $(-\infty, 0) \cup (0, \infty)$. As in Lemma 3.2 it is sufficient to assume that μ is concentrated on $(0, \infty)$. By

Lemma 3.2, μ is concentrated on the set given by (3.6). Put

$$a_n = \nu(\{y_0 c^n\}), \quad n \in \boldsymbol{Z}. \tag{3.12}$$

By (3.2) and (3.3), the sequence $\{a_n\}_{n=-\infty}^{\infty}$ is completely monotone. Put $x_n = n\ln(c^{-1})$, $n \in \boldsymbol{Z}$. By Proposition 3.5 there exists a uniquely determined completely monotone function f on $\boldsymbol{R}$ such that

$$f(x_n) = a_n, \quad n \in \boldsymbol{Z}. \tag{3.13}$$

By Proposition 3.3, f is of the form

$$f(x) = \int_0^\infty e^{-zx} F(dz), \tag{3.14}$$

where F is a Borel measure on $(0, \infty)$ (F is without atoms on the set $\{0, \infty\}$ since $\lim a_n = 0$ as $n \to \infty$ and $a_n > 0$, $n \in \boldsymbol{Z}$.)

Now we shall prove that F is a degenerate measure. Contrary to our statement suppose that there exists $b > 0$ such that $F\big((0, b]\big) > 0$ and $F\big((b, \infty)\big) > 0$. Then we can write f as the sum of completely monotone functions: $f = f_1 + f_2$, where $f_i(x) = \int_0^\infty e^{-zx} F_i(dz)$, $i = 1, 2$, $F_1 = F|_{(0,b]}$, $F_2 = F - F_1$. In particular, we have $f(x_n) = f_1(x_n) + f_2(x_n)$, $n \in \boldsymbol{Z}$. We define the measures ν_1, ν_2 as follows: $\nu_i(\{y_0 c^{-n}\}) = f_i(x_n)$, $i = 1, 2$, $n \in \boldsymbol{Z}$. Then $\nu_i = \nu_i(\mu_i)$, $i = 1, 2$, are Lévy measures of infinitely divisible distributions such that $\mu_i \in M$ and $\mu = \mu_1 + \mu_2$. Let us put $\beta = \mu_1(\{y_0 c^{-n}\}_{n=-\infty}^{\infty})$. Then we have

$$\mu = \beta\frac{\mu_1}{\beta} + (1-\beta)\frac{\mu_2}{1-\beta}, \quad \text{where } \frac{\mu_1}{\beta}, \frac{\mu_2}{1-\beta} \in K,$$

which contradicts the fact that μ is an extreme point.

Thus, F is degenerate. Then, for example, let $F = C_0\,\delta_{z_0}$, where z_0, C_0 are positive constants. This implies by (3.14) that f is of the form

$$f(x) = C_0 \exp(-z_0 x), \quad x \in \boldsymbol{R}. \tag{3.15}$$

We introduce the function

$$\hat{f}(u) = f(x), \tag{3.16}$$

where $x = \ln(u/y_0)$, $u > 0$, $x \in \boldsymbol{R}$. By (3.12) and (3.13) we have $f(n\ln c^{-1}) = f(x_n) = \nu(\{y_0 c^{-n}\})$. Thus

$$\hat{f}(y_0 c^{-n}) = \nu(\{y_0 c^{-n}\}), \quad n \in \boldsymbol{Z}. \tag{3.17}$$

By (3.15) and (3.16) we have $\hat{f}(u) = C_0(u/y_0)^{-z_0}$. Taking into account (3.17) we see that this implies

$$\nu(\{y_0 c^{-n}\}) = C_0(c^{-n})^{-z_0}, \quad n \in \boldsymbol{Z}. \tag{3.18}$$

Denote by $\nu_{(y_0,z_0)}$ the measure given by (3.18),

$$\mu_{(y_0,z_0)}(du) = u^2/(1+u^2)\,\nu_{(y_0,z_0)}(du).$$

Since $\mu_{(y_0,z_0)}$ is a probability measure, C_0 is given by (3.11), where $0 < z_0 < 2$, $y_0 \in Y_c$. Thus the lemma is proved.

Let us put $G_c = \{\mu_{(y_0,z_0)}\colon 0 < z_0 < 2,\ y_0 \in Y_c\}$. Directly from Lemmas 3.1 and 3.7 we obtain the following lemma:

LEMMA 3.8. $e(K) \subset G_c \cup \{\delta_0, \delta_{-\infty}, \delta_\infty\}$.

LEMMA 3.9. $G_c \cup \{\delta_0, \delta_{-\infty}, \delta_\infty\} \subset e(K)$.

Proof. Once again it is sufficient to consider $\mu_{(y_0,z_0)}$ for $y_0 \in Y_c \cap (0, \infty)$, $0 < z_0 < 2$. Suppose that $\mu_{(y_0,z_0)}$ is not an extreme point. Then there exist $0 < \beta < 1$, $\mu_1, \mu_2 \in K$ such that $\mu_1 \neq \mu_2$ and

$$\mu_{(y_0,z_0)} = \beta\mu_1 + (1-\beta)\,\mu_2. \tag{3.19}$$

Clearly, both measures μ_1, μ_2 are concentrated on the set $\{u_n\}_{n=-\infty}^{\infty}$, where $u_n = y_0 c^{-n}$, $n \in \mathbf{Z}$. By (3.19) we have

$$\nu_{(y_0,z_0)} = \beta\nu_1 + (1-\beta)\,\nu_2, \tag{3.20}$$

where $\nu_i = \nu_i(\mu_i)$, $i = 1, 2$. Let f, f_1, f_2 be completely monotone functions such that $f(x_n) = \nu_{(y_0,z_0)}(u_n)$, $f_i(x_n) = \nu_i(u_n)$, $x_n = n\ln(c^{-1})$, $i = 1, 2$, $n \in \mathbf{Z}$. We note that the function $g = \beta f_1 + (1-\beta) f_2$ is a completely monotone function. By (3.20) we have $f(x_n) = g(x_n)$, $n \in \mathbf{Z}$. Since a completely monotone function is uniquely determined by its value at points x_n, $n \in \mathbf{Z}$, we have

$$f(x) = \beta f_1(x) + (1-\beta) f_2(x), \qquad x \in \mathbf{R}. \tag{3.21}$$

Since $f(x) = C_0 \exp(-z_0 x)$ and it is the extreme point in the set of completely monotone functions (see [10]), it follows by (3.21) that $f_1(x) = f_2(x) = C_0\exp(-z_0 x)$. This implies that $\nu_1 = \nu_2 = \nu_{(y_0,z_0)}$ and, consequently, $\mu_1 = \mu_2 = \mu_{(y_0,z_0)}$. This contradiction implies that $\mu_{(y_0,z_0)}$ must be an extreme point. This completes the proof of the lemma.

By Lemmas 3.8 and 3.9 we have

THEOREM 3.10. $e(K) = G_c \cup \{\delta_0, \delta_{-\infty}, \delta_\infty\}$.

By $\mu_n \Rightarrow \mu$ we denote the weak convergence of measures. Observe that $e(K)$ is closed. In particular, we have:

$$\mu_{(y_n,z)} \Rightarrow \mu_{(1,z)} \quad \text{as } y_n \to c^{-1}-0 \ (0 < z < 2),$$
$$\mu_{(y,z_n)} \Rightarrow \delta_0 \quad \text{as } z_n \to 2-0 \ (y \in Y_c),$$
$$\mu_{(y,z_n)} \Rightarrow \delta_\infty \quad \text{as } z_n \to 0+0 \ (y \in Y_c \cap (0, \infty)),$$
$$\mu_{(y,z_n)} \Rightarrow \delta_{-\infty} \quad \text{as } z_n \to 0+0 \ (y \in Y_c \cap (-\infty, 0)).$$

Thus $e(K)$ is compact and, consequently, K is compact.

LEMMA 3.11. *K is compact.*

4. THE CHARACTERISTIC FUNCTIONS OF MEASURES FROM $L_{c,\infty}$

Now we will apply Choquet's theorem on representation of the points of a compact convex set as barycenters of the extreme points ([10], p. 19). Then, taking into account Theorem 3.10, we infer that μ is in K if and only if

$$\mu = \int\limits_{G_c \cup \{\delta_0, \delta_{-\infty}, \delta_\infty\}} \theta \gamma(d\theta), \tag{4.1}$$

where γ is a probability measure on $G_c \cup \{\delta_0, \delta_{-\infty}, \delta_\infty\}$. Moreover, $\mu \in K^0$ if and only if the measure γ assigns zero mass to the set $\{\delta_{-\infty}, \delta_\infty\}$. We note that the representation of $\mu|_{(-\infty,0)\cup(0,\infty)}$, where $\mu \in K^0$, is given by (4.1) with $G_c \cup \{\delta_0, \delta_{-\infty}, \delta_\infty\}$ replaced by G_c. It is not difficult to prove that the mapping $\mu_{(y,z)} \to (y, z)$ is a homeomorphism of $Y_c \times (0, 2)$ onto G_c. Thus we can write μ from K^0 and concentrated on $(-\infty, 0) \cup (0, \infty)$ in the form

$$\mu = \int\limits_{Y_c \times (0,2)} \mu_{(y,z)} \lambda(d(y, z)), \tag{4.2}$$

where λ is a probability measure on $Y_c \times (0, 2)$. It is not difficult to prove that K is a simplex (analogously as in [11]); then from Choquet's uniqueness theorem for a metrizable space ([10], p. 70) we infer that λ is determined uniquely (see [11]). Obviously, the measure $\mu \in M^0$ concentrated on $(-\infty, 0) \cup (0, \infty)$ is given by (4.2), where λ is a finite measure on $Y_c \times (0, 2)$. Finally, by (4.2) we obtain the representation of $\nu = \nu(\mu)$, where $\mu \in M^0$. Thus the following lemma is proved:

LEMMA 4.1. *The measure ν is the Lévy measure corresponding to $P \in L_{c,\infty}$ if and only if ν takes one of the following forms*:

$$\nu(du) = \int\limits_{Y_c \times (0,2)} \Big\{ \sum_{j=-\infty}^{\infty} (c^j)^{-z} (yc^j)^2/[1+(yc^j)^2] \Big\}^{-1} \times \sum_{n=-\infty}^{\infty} (c^n)^{-z} \delta_{yc^n}(u) \lambda(d(y, z)), \tag{4.3}$$

where λ is a finite Borel measure on $Y_c \times (0, 2)$, or, equivalently,

$$\nu(du) = \int\limits_{Y_c \times (0,2)} \sum_{n=-\infty}^{\infty} (yc^n)^{-z} \delta_{yc^n}(u) \tau(d(y, z)), \tag{4.4}$$

where τ is a Borel measure on $Y_c \times (0, 2)$ $(\tau = y^z C_0(y, z)\lambda)$ such that

$$\sum_{n=-\infty}^{\infty} |yc^n|^{2-z}/(1+yc^n)^2 \tau(d(y, z)) \tag{4.5}$$

is a finite Borel measure on $Y_c \times (0, 2)$, or, equivalently,

$$\nu(du) = \int\limits_{Y_c \times (0,2)} \sum_{n=-\infty}^{\infty} (c^n)^{-z} \delta_{yc^n}(u) \xi(d(y, z)), \tag{4.6}$$

where ξ is a Borel measure on $Y_c \times (0, 2)$ such that

$$(4.7) \qquad \sum_{n=-\infty}^{\infty} (c^n)^2/(1+yc^n)^2 \, \xi(d(y, z))$$

is a finite Borel measure on $Y_c \times (0, 2)$.

Moreover, the measure ν determines uniquely the measures λ, τ, and ξ.

Applying (4.3) and (4.4) we obtain the characteristic functions of completely c-decomposable distributions.

THEOREM 4.2. *The function φ is the characteristic function of $P \in L_{c,\infty}$ if and only if φ is of the form*

$$(4.8) \qquad \varphi(t) = \exp\{ibt - Gt^2/2$$

$$+ \int_{Y_c \times (0,2)} [\sum_{j=-\infty}^{\infty} (c^j)^{-z} (yc^j)^2/[1+(yc^j)^2]]^{-1} \sum_{n=-\infty}^{\infty} (c^n)^{-z} g_t(yc^n) \lambda(d(y, z))\},$$

where λ is a finite Borel measure on $Y_c \times (0, 2)$, $b \in \mathbf{R}$, $G \geqslant 0$, or, equivalently,

$$(4.9) \qquad \varphi(t) = \exp\{ibt - Gt^2/2 + \int_{Y_c \times (0,2)} \sum_{n=-\infty}^{\infty} |yc^n|^{-z} g_t(yc^n) \tau(d(y, z))\},$$

where τ is a Borel measure on $Y_c \times (0, 2)$ such that the condition (4.5) is satisfied, $b \in \mathbf{R}$, $G \geqslant 0$.

Moreover, b, G, λ, and τ are uniquely determined.

Remark 4.3. In the particular case $\tau = \tau_1 \times \tau_2$ we can rewrite the formula (4.9) in the form

$$(4.10) \qquad \varphi(t) = \exp\{ibt - Gt^2/2 + \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} g_t(yc^n) \chi_{Y_c}(y) \int_0^2 |yc^n|^{-z} \tau_2(dz) \tau_1(dy)\}.$$

Putting $h(u) = \int_0^2 |u|^{-z} \tau_2(dz)$ and $\bar{h}(x) = h(u)$, where $u > 0$ and $x = \ln u$, we obtain

$$(4.11) \qquad \varphi(t) = \exp\{ibt - Gt^2/2 + \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} g_t(yc^n) \chi_{Y_c}(y)\, h(yc^n) \tau_2(dz) \tau_1(dy)\},$$

where $h(u)$ is a function on $(-\infty, 0) \cup (0, \infty)$ for which $h(-u) = h(u)$ and $\bar{h}(x)$ is a completely monotone function (on $\mathbf{R}$) such that the measure F determined by the Bernstein representation (3.7) is concentrated on $(0, 2)$, i.e. $\bar{h}(x) = \int_0^2 e^{-zx} \tau_2(dz)$.

We shall derive from Lemma 4.1 the well-known representation of completely self-decomposable distribution.

THEOREM 4.4. *A probability measure P is completely self-decomposable if and only if its Lévy spectral measure ν is of the form*

$$(4.12) \qquad \nu(dx) = \int_0^2 z|x|^{-1-z} \gamma(dz)$$

or, equivalently, its characteristic function φ *is of the form*

$$\varphi(t) = \exp\{ibt - Gt^2/2 + \int_0^2 (\int_{-\infty}^0 + \int_0^\infty) g_t(u) z |u|^{-1-z} du\, \gamma(dz)\}, \tag{4.13}$$

where γ *is a finite Borel measure on* $(0, 2)$.

Proof. Let a probability measure P be completely self-decomposable, i.e. $P \in L_{c,\infty}$ for every $c \in (0, 1)$. Let ν be the Lévy spectral measure corresponding to P. By Lemma 4.1, ν is of the form (4.4). Assume that ν is concentrated on $(0, \infty)$. Then we can write $\bar{\nu}$ and its distribution function $\bar{f}$ in the form

$$\bar{\nu}(du) = \int_{[0,h)\times(0,2)} \sum_{j=-\infty}^{\infty} e^{-z(y-jh)} \delta_{y-jh}(u) \gamma_h(d(y, z)), \tag{4.14}$$

$$\bar{f}(u) = \int_{[0,h)\times(0,2)} \sum_{j=-\infty}^{\infty} e^{-z(y-jh)} \chi_{(y-(j+1)h, y-jh]}(u) \times [(1-e^{-zh})^{-1} \gamma_h](d(y, z)), \tag{4.15}$$

respectively. From (4.15), by letting h tend to zero and making use of Helly's theorem, we conclude that $\bar{f}(u) = \int_{\{0\}\times(0,2)} e^{-zu} \gamma(d(y, z))$. Taking the measure $\gamma(dz)$ in place of the measure $\gamma(d(y, z))$, respectively, we obtain

$$\bar{f}(u) = \int_0^2 e^{-zu} \gamma(dz), \qquad f(x) = \int_0^2 x^{-z} \gamma(dz) \tag{4.16}$$

and

$$\nu(dx) = \int_0^2 z x^{-z-1} \gamma(dz). \tag{4.17}$$

The above formula gives a representation of $\nu|_{(0,\infty)}$. Consequently, ν is determined by (4.12). Easy computations show that the characteristic function is given by (4.13). Thus the theorem is proved.

REFERENCES

[1] J. Bunge, *Nested classes of C-decomposable laws*, Ann. Probab. 25 (1997), pp. 215–229.

[2] W. Feller, *Completely monotone functions and sequences*, Duke Math. J. 5 (1939), pp. 661–674.

[3] Z. Jurek, *The classes $L_m(Q)$ of probability measures on Banach spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 31 (1983), pp. 51–62.

[4] K. Knopp, *Mehr fach monotone Zahlenfolgen*, Math. Z. 22 (1925), pp. 75–85.

[5] A. Kumar and B. M. Schreiber, *Characterization of subclasses of class L probability distributions*, Ann. Probab. 6 (1978), pp. 279–293.

[6] M. Loève, *Nouvelles classes de lois limites*, Bull. Soc. Math. France 73 (1945), pp. 107–126.

[7] M. Maejima and Y. Naito, *Semi-selfdecomposable distributions and a new class of limit theorems*, Research Report, Keio University, Japan, 1997.
[8] Nguyen van Thu, *Multiply c-decomposable probability measures on Banach spaces*, Probab. Math. Statist. 5 (1985), pp. 251–263.
[9] Nguyen van Thu, *Multiply self-decomposable probability measures on Banach spaces*, Studia Math. 66 (1979), pp. 160–175.
[10] R. P. Phelps, *Lectures on Choquet's Theorem*, New York 1966.
[11] T. Rajba, *A representation of distributions from certain classes* L_S^{id}, Probab. Math. Statist. 4 (1984), pp. 67–78.
[12] T. Rajba, *On multiple decomposability of probability measures on* $\boldsymbol{R}$, Demonstratio Math. 2 (2001), pp. 275–294.
[13] K. Sato, *Class L of multivariate distributions and its subclasses*, J. Multivariate Anal. 10 (1980), pp. 207–232.
[14] D. V. Widder, *The Laplace Transform*, University Press, Princeton, N. J., 1941.

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