PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 22, Fasc. 2 (2002), pp. 231–252

TIME-INHOMOGENEOUS DIFFUSIONS CORRESPONDING TO SYMMETRIC DIVERGENCE FORM OPERATORS

BY

ANDRZEJ ROZKOSZ* (TORUŃ)

Abstract. We consider a time-inhomogeneous Markov family $(X, P_{s,x})$ corresponding to a symmetric uniformly elliptic divergence form operator. We show that for any φ in the Sobolev space $W_p^1 \cap W_2^1$ with $p = 2$ if $d = 1$ and $p > d$ if $d > 1$ the additive functional $X^\varphi = \{\varphi(X_t) - \varphi(X_s);\ 0 \leqslant s < t\}$ admits a unique strict decomposition into a martingale additive functional of finite energy and a continuous additive functional of zero energy. Moreover, we give a stochastic representation of the zero energy part and show that in case the diffusion coefficient is regular in time the functional X^φ is a Dirichlet process for each starting point (s, x). The paper contains also rectifications of incorrectly presented or incorrectly proved statements of our earlier paper [14].

2000 AMS Subject Classification: 60J60, 60J57.

1. INTRODUCTION

Consider a Markov family $\{(X, P_{s,x});\ (s, x) \in [0, T] \times \boldsymbol{R}^d\}$ corresponding to the divergence form operator

$$L_t = \frac{1}{2} \sum_{i,j=1}^{d} D_j\big(a^{ij}(t, x) D_i\big),$$

where $a\colon [0, T] \times \boldsymbol{R}^d \to \boldsymbol{R}^d \otimes \boldsymbol{R}^d$ is a measurable, symmetric matrix-valued function satisfying the condition

$$\lambda |\xi|^2 \leqslant \sum_{i,j=1}^{d} a^{ij}(t, x) \xi_i \xi_j \leqslant \Lambda |\xi|^2, \qquad (t, x) \in [0, T] \times \boldsymbol{R}^d,\ \xi \in \boldsymbol{R}^d, \tag{1.1}$$

for some $0 < \lambda \leqslant \Lambda$ (for construction of $(X, P_{s,x})$ see, e.g., [11], [13], [17]). In [14] it is announced that for any starting point $(s, x) \in [0, T) \times \boldsymbol{R}^d$ and any continuous φ in the Sobolev space W_p^1 with $p > 2 \vee d$ the composite process

* Faculty of Mathematics and Computer Science, Nicholas Copernicus University, Toruń.
Research supported by KBN Grant 0253 P03 2000 19.

$\varphi(X)$ is a Dirichlet process on $[s, T]$ in the sense of Föllmer [3], that is, $X^{\varphi}_{s,t} = \varphi(X_t) - \varphi(X_s)$, $t \in [s, T]$, admits a unique decomposition of the form

$$(1.2)\qquad X^{\varphi}_{s,t} = M^{x,\varphi}_{s,t} + A^{x,\varphi}_{s,t}, \quad t \in [s, T], \quad P_{s,x}\text{-a.s.},$$

where $M^{x,\varphi}_{s,\cdot}$ is a continuous $P_{s,x}$-martingale on $[s, T]$ and $A^{x,\varphi}_{s,\cdot}$ is a continuous adapted process of $P_{s,x}$-zero quadratic variation on $[s, T]$. Unfortunately, the proof of the last statement in [14] (see [14], Theorem 2.1) as well as Lemma 1.2 in [14], which is used in the proof of Theorem 2.1 in [14], is incorrect (see Remark 2.6 in [16] and the remark following Lemma 2.2 of the present paper for more details). One of our purposes is to show that the quadratic variation of $A^{x,\varphi}_{s,\cdot}$ vanishes if we assume additionally that

$$(1.3)\qquad \max_{1 \leqslant i,j \leqslant d} \int_0^T \operatorname{ess\,sup}_{x \in \mathbf{R}^d} |D_t a^{ij}(t, x)|\, dt = K_1 < \infty.$$

We show also that under (1.1) for any continuous $\varphi \in W^1_p \cap W^1_2$ with p as above X^{φ} admits a unique decomposition

$$(1.4)\qquad X^{\varphi}_{s,t} = M^{[\varphi]}_{s,t} + A^{[\varphi]}_{s,t}, \quad t \in [s, T], \quad P_{s,x}\text{-a.s.},$$

into a continuous martingale additive functional $M^{[\varphi]}$ (in the strict sense) of finite energy and a continuous additive functional $A^{[\varphi]}$ of zero energy, where the energy of an additive functional $A = \{A_{s,t},\, 0 \leqslant s < t \leqslant T\}$ of $(X, P_{s,x})$ is defined by

$$e(A) = \lim_{\alpha \to \infty} \alpha^2 e_{\alpha}(A), \quad e_{\alpha}(A) = \int_0^T \int_0^T \mathbb{1}_{[0,T]}(s+t)\, e^{-\alpha t} E_{s,m} A^2_{s,s+t}\, ds\, dt, \quad \alpha > 0,$$

whenever the limit exists. Here m is the Lebesgue measure on $\mathbf{R}^d$ and $E_{s,m}$ is the expectation sign with respect to $P_{s,m}(\cdot) = \int_{\mathbf{R}^d} P_{s,x}(\cdot)\, m(dx)$. Our decomposition may be viewed as a strict version of a decomposition of X^{φ} obtained in [12] and [19]. If moreover (1.3) is satisfied, (1.4) also holds if we replace the above definition of energy by the following:

$$\bar{e}(A) = \lim_{t \downarrow 0} \frac{1}{t} \bar{e}_t(A), \quad \bar{e}_t(A) = \int_0^{T-t} E_{s,m} A^2_{s,s+t}\, ds, \quad t \in (0, T].$$

Thus we generalize results of [15] on strict decomposition of time-homogeneous diffusions corresponding to divergence form operators (general time-homogeneous diffusions are considered in [4]–[6]).

Finally, notice that similarly to [14]–[16] our methods of proofs allow us to obtain the Lyons–Zheng decomposition of $A^{x,\varphi}_{s,\cdot}$ and $A^{[\varphi]}_{s,\cdot}$ for each starting point (s, x) (results concerning time-homogeneous diffusions can be found in [9], [10], [15], [16]).

We will use the following notation.

For a process Y on $[s, T]$ we write

$$\bar{Y}_t = Y_{T+s-t}, \quad \tilde{Y}_t = \bar{Y}_t - Y_T, \quad t \in [s, T].$$

Let $\Omega = C([0, T]; \boldsymbol{R}^d)$ be the space of continuous trajectories from $[0, T]$ into $\boldsymbol{R}^d$, and X be the canonical process on Ω. Let us put

$$\mathscr{F}_t^s = \sigma(X_u, u \in [s, t]), \quad \bar{\mathscr{F}}_t^s = \sigma(\bar{X}_u, u \in [s, t]), \quad t \in [s, T].$$

By $E_{s,x}$ we denote the expectation sign with respect to $P_{s,x}$. Let $\mathscr{M}_c(P_{s,x})$ be the space of continuous square-integrable $P_{s,x}$-martingales on $[s, T]$ vanishing at s equipped with the usual norm $(E_{s,x}\langle M\rangle_T)^{1/2}$.

$D_i = \partial/\partial x^i$ is the partial derivative in the distribution sense, $\nabla = (D_1, \ldots, D_d)$. C_0^∞ is the set of all smooth functions in $\boldsymbol{R}^d$ having compact support. L_p (respectively, $L_p(s, T)$) is the classical Banach space consisting of measurable functions on $\boldsymbol{R}^d$ (respectively, $(s, T) \times \boldsymbol{R}^d$) that are p-integrable. W_p^1 is the Banach space consisting of all elements u of L_p having derivatives $D_i u$ from L_p. Let $W_2^{0,1}(s, t)$ be the Banach space consisting of all elements u of $L_2(s, t)$ having derivatives $D_i u$ from $L_2(s, T)$, and $W_2^{1,1}(s, T)$ be the Banach space consisting of all elements u of $L_2(s, T)$ having derivatives $D_i u$ and time derivatives (in the distribution sense) D_t from $L_2(s, T)$. By $\|\cdot\|_p$ we denote the norm in L_p. By $(\cdot, \cdot)$ we mean the usual scalar product in $\boldsymbol{R}^d$, $(\cdot, \cdot)_2$ the scalar product in L_2, and $(\cdot, \cdot)_{2;s,T}$ the scalar product in $L_2(s, T)$.

By Sobolev's imbedding theorem, if $p > d$, then every $\varphi \in W_p^1$ has a continuous representative. Therefore we will always assume that φ denotes the continuous representative of a given element of W_p^1 with $p > d$.

2. PRELIMINARY RESULTS

It is known (see [1] and [8]) that under (1.1) for any $s \in [0, T)$ and $\varphi \in L_2$ there is a unique weak solution $P^{s,\cdot}\varphi(\cdot) \in W_2^{0,1}(s, T)$ to the Cauchy problem

$$(D_t - L_t)u = 0 \text{ in } (s, T) \times \boldsymbol{R}^d, \quad u(s, \cdot) = \varphi.$$

In particular, if $\varphi \in W_2^1$, then

$$(\varphi - P^{s,t}\varphi, \varphi)_2 = \frac{1}{2}\int_s^t \big(a(u, \cdot)\nabla P^{s,u}\varphi, \nabla\varphi\big)_2\, du \tag{2.1}$$

for all $t \in (s, T)$. Furthermore, in [1] it is shown that there exists a weak fundamental solution $p(s, x, t, y)$, $0 \leqslant s < t$, $x, y \in \boldsymbol{R}^d$, for L_t, and that for any $\varphi \in L_2$ we have the representation

$$P^{s,t}\varphi(x) = \int_{\boldsymbol{R}^d} \varphi(y)p(s, x, t, y)\,dy, \quad 0 \leqslant s < t \leqslant T, \; x \in \boldsymbol{R}^d.$$

The family $\{P^{s,t}, 0 \leqslant s < t \leqslant T\}$ forms a Markov semigroup of contractions on L_2, which gives rise to our Markov family $(X, P_{s,x})$ (see, e.g., [13]). In particular, p is the transition density of $(X, P_{s,x})$.

THEOREM 2.1. *Assume* (1.1) *holds. Then*:

(i) *there is a constant* $K_2 > 0$ *depending only on* λ, Λ, d, T *such that*

$$p(s, x, t, y) \leqslant K_2 (t-s)^{-d/2} \exp\left(-\frac{|y-x|^2}{K_2 (t-s)}\right) \tag{2.2}$$

for all $0 \leqslant s < t \leqslant T$ *and* $x, y \in \mathbf{R}^d$;

(ii) *for each* $(s, x) \in [0, T) \times \mathbf{R}^d$, $p(s, x, \cdot, \cdot) \in L_1(s, T; W_q^1)$ *with any* q *whose Hölder conjugate is greater than* $2 \vee d$;

(iii) *if* $d = 1$, *then for any* $0 \leqslant s < T$, $\alpha > 0$ *we have*

$$\int_s^T \int_{\mathbf{R}} (t-s)^\alpha \frac{(p')^2}{p}(s, x, t, y)\, dt\, dy < \infty$$

($p'(s, x, t, y)$ *denotes the derivative of* $y \mapsto p(s, x, t, y)$ *in the sense of distributions*).

Proof. For (i) and (ii) see Theorems 5 and 7 in [1]. To prove (iii) we can proceed as in the proof of Theorem 5.1 in [15]. ∎

Let W_2^{-1} denote the dual space of W_2^1 when one uses the inner product in L_2 to define the duality relation and let $W_2^{0,-1}(s, T) = L_2(0, T; W_2^{-1})$. The space $W_2^{0,-1}(s, T)$ is dual to $W_2^{0,1}(s, T)$ and the value $\langle v, u\rangle_{s,T}$ of a functional $v \in W_2^{0,-1}(s, T)$ at $u \in W_2^{0,1}(s, T)$ is defined by $\langle v, u\rangle_{s,T} = \int_s^T \langle v(t), u(t)\rangle\, dt$, where $\langle \cdot, \cdot \rangle$ denotes the dualization between W_2^{-1} and W_2^1.

Let $\mathscr{W}(s, T)$ denote the space of functions $u \in W_2^{0,-1}(s, T)$ having derivatives $D_t u$ from $W_2^{0,-1}(s, T)$ equipped with the norm

$$\|u\|^2_{\mathscr{W}(s,T)} = \|D_t u\|^2_{W_2^{0,-1}(s,T)} + \|u\|^2_{W^{0,1}(s,T)}.$$

It is known (see, e.g., [20]) that there is a continuous embedding of $\mathscr{W}(s, T)$ in the space $C([s, T]; L_2)$ of continuous functions on $[s, T]$ with values in L_2 equipped with the supremum norm. Therefore without ambiguity we may define the subspace $\mathscr{W}_T(s, T) = \{u \in \mathscr{W}(s, T): u(T, \cdot) = 0\}$ of $\mathscr{W}(s, T)$.

For any $\alpha \geqslant 0$, $f \in L_2(s, T)$, there exists a unique weak solution $R_\alpha f \in \mathscr{W}_T(s, T)$ of the problem

$$(\alpha - D_t - L_t) u = f \text{ in } (s, T) \times \mathbf{R}^d, \quad u(T, \cdot) = 0,$$

that is

$$\alpha (R_\alpha f, g)_{2;s,T} - \langle D_t R_\alpha f, g\rangle_{s,T} + \tfrac{1}{2}(a\nabla R_\alpha f, \nabla g)_{2;s,T} = (f, g)_{2;s,T} \tag{2.3}$$

for any $g \in W_2^{0,1}(s, T)$ (see [13], [17], [20]). Moreover, $\{R_\alpha\}_{\alpha>0}$ is a strongly continuous resolvent on $L_2(s, T)$ having the representation

$$R_\alpha f(u, x) = E_{u,x} \int_0^T \mathbb{1}_{[0,T]}(u+t) e^{-\alpha t} f(u+t, X_{u+t}) dt, \qquad (u, x) \in [s, T] \times \boldsymbol{R}^d,$$

for any bounded $f \in L_2(s, T)$ (see, e.g., [13]).

LEMMA 2.2. *Assume* (1.1) *holds. If* $f \in W_2^{0,1}(s, T)$, *then*

$$\lim_{\alpha \to \infty} \|\alpha R_\alpha f - f\|_{W_2^{0,1}(s,T)} = 0. \tag{2.4}$$

Proof. This is a particular case of a more general result proved in [17], Proposition 3.7. ∎

Remark. In Lemma 1.2 of the author's paper [14], (2.4) is stated for resolvents corresponding to operators of the form

$$L_t = \tfrac{1}{2} \sum_{i,j=1}^{d} D_j\big(a^{ij}(t, x) D_i\big) + \sum_{i=1}^{d} b^i(t, x) D_i.$$

Unfortunately, the proof of Lemma 1.2 in [14] is incorrect (even in the case $b^i = 0, i = 1, \ldots, d$). The statement of Lemma 1.2 in [14] is, however, correct. It follows from the cited-above Proposition 3.7 of [17].

From Lemma 2.2 it may be concluded that

$$\limsup_{\alpha \to \infty} \alpha(\varphi, \varphi - \alpha R_\alpha \varphi)_{2;s,T} \leqslant (2^{-1} \Lambda T + 1) \|\varphi\|_{W_2^1}^2 \tag{2.5}$$

for any $\varphi \in W_2^1$. Indeed, putting $f, g = \varphi$ in (2.3) we obtain

$$(\alpha R_\alpha \varphi, \varphi)_{2;s,T} - \langle D_t R_\alpha \varphi, \varphi \rangle_{s,T} + 2^{-1} (a \nabla R_\alpha \varphi, \nabla \varphi)_{2;s,T} = (\varphi, \varphi)_{2;s,T}.$$

Hence

$$(\varphi - \alpha R_\alpha \varphi, \varphi)_{2;s,T} = 2^{-1} (a \nabla R_\alpha \varphi, \nabla \varphi)_{2;s,T} + \big(R_\alpha \varphi(s, \cdot), \varphi\big)_2, \tag{2.6}$$

because

$$\langle D_t R_\alpha \varphi, \varphi \rangle_{s,T} = \big(R_\alpha \varphi(T, \cdot), \varphi\big)_2 - \big(R_\alpha \varphi(s, \cdot), \varphi\big)_2 = -\big(R_\alpha \varphi(s, \cdot), \varphi\big)_2$$

(see, e.g., [20]). On the other hand,

$$\begin{aligned} \big|\big(\alpha R_\alpha \varphi(s, \cdot), \varphi\big)_2\big| &= \Big| \int_0^{T-s} \alpha e^{-\alpha t} (P^{s,s+t} \varphi, \varphi)_2 \, dt\Big| \\ &\leqslant \int_0^{T-s} \alpha e^{-\alpha t} \|P^{s,s+t} \varphi\|_2 \|\varphi\|_2 \, dt \leqslant \|\varphi\|_2^2 \end{aligned}$$

and, by (2.4),

$$\limsup_{\alpha\to\infty} |(a\nabla(\alpha R_\alpha \varphi), \nabla\varphi)_{2;s,T}|$$

$$\leqslant \limsup_{\alpha\to\infty} \Lambda |(\nabla(\alpha R_\alpha \varphi - \varphi), \nabla\varphi)_{2;s,T}| + \Lambda(\nabla\varphi, \nabla\varphi)_{2;s,T} \leqslant \Lambda T \|\varphi\|^2_{W_2^1},$$

which gives (2.5) when combined with (2.6).

THEOREM 2.3. *If* (1.1) *and* (1.3) *are satisfied, then for any* $\varphi \in W_2^1$, $f \in L_2(s, T)$ *the Cauchy problem*

(2.7) $$(D_t - L_t)u = f \ \text{in}\ (s, T)\times \boldsymbol{R}^d, \quad u(s, \cdot) = \varphi,$$

has a unique solution from $W_2^{1,1}(s, T)$. *Moreover, there is* $K_3 > 0$ *depending only on* λ, Λ, d *and* K_1 *such that for every* $t \in (s, T)$

(2.8) $$\|\nabla u(t, \cdot)\|_2^2 + \|D_t u\|^2_{2;s,t} \leqslant K_3(\|\nabla\varphi\|_2^2 + \|f\|^2_{2;s,t}).$$

Proof. See Theorem 6.1 and the inequality (6.6) in Chapter III in [8]. ∎

From (2.8) it follows that for any $\varphi \in W_2^1$

(2.9) $$\|\nabla P^{s,t}\varphi\|_2^2 + (t-s)^{-1}\|\varphi - P^{s,t}\varphi\|_2^2 \leqslant K_3 \|\nabla\varphi\|_2^2$$

for all $0 \leqslant s < t \leqslant T$. Moreover, since $R_\alpha(t, x) = e^{\alpha t} u(T+s-t, x)$, where u is a solution to (2.7) with f replaced by $g(t, x) = e^{-\alpha(T+s-t)} f(T+s-t, x)$, the inequality (2.8) shows that

$$\|D_t R_\alpha \varphi\|_{2;s,T} \leqslant \alpha e^{\alpha T}\|R_\alpha\varphi\|_{2;s,T} + K_3^{1/2}\alpha^{-1}(e^{\alpha(T-s)}-1)\|\nabla\varphi\|_2 < \infty$$

for every $\alpha > 0$.

3. DIRICHLET PROCESSES

Let $(s, x) \in [0, T)\times \boldsymbol{R}^d$ and let $\{\mathscr{H}_t\}_{t\in[s,T]}$ be a filtration. We will say that $X^\varphi_{s,\cdot}$ is a *continuous* $(\{\mathscr{H}_t\}, P_{s,x})$*-Dirichlet process* on $[s, T]$ if it is $\{\mathscr{H}_t\}$-adapted and admits a decomposition of the form (1.2), where $M^{x,\varphi}_{s,\cdot}$ is a continuous $(\{\mathscr{H}_t\}, P_{s,x})$-local martingale on $[s, T]$ such that $M^{x,\varphi}_{s,s} = 0$ and $A^{x,\varphi}_{s,\cdot}$ is an $\{\mathscr{H}_t\}$-adapted process of zero quadratic variation on $[s, T]$, that is, $A^{x,\varphi}_{s,s} = 0$ and

$$Q^n_{s,T}(A^{x,\varphi}) \equiv \sum_{t_i\in\Pi_n} |A^{x,\varphi}_{s,t_{i+1}} - A^{x,\varphi}_{s,t_i}|^2 \to 0 \ \text{in}\ P_{s,x} \quad \text{as}\ n\to\infty$$

for every sequence $\{\Pi_n = \{t_0, t_1, \ldots, t_{i(n)}\}\}$ of partitions of $[s, T]$ such that $s = t_0 < t_1 < \ldots < t_{i(n)} = T$ and $\|\Pi_n\| = \max_{1\leqslant i\leqslant i(n)} |t_i - t_{i-1}| \to 0$ as $n\to\infty$.

In what follows $\nabla p(s, x, t, y)$ stands for the gradient of $y \mapsto p(s, x, t, y)$.

THEOREM 3.1. *Assume* (1.1) *and* (1.3) *hold and let* $\varphi \in W_p^1$ *with* $p=2$ *if* $d=1$ *and* $p>d$ *if* $d>1$. *Then for every* $(s,x)\in[0,T)\times R^d$ *there exists a unique triple* $(M_{s,\cdot}^{x,\varphi}, N_{s,\cdot}^{x,\varphi}, V_{s,\cdot}^{x,\varphi})$ *such that*:

(i) $M_{s,\cdot}^{x,\varphi}$ *is a continuous* $(\{\mathscr{F}_t^s\}, P_{s,x})$-*square-integrable martingale on* $[s,T]$, $N_{s,\cdot}^{x,\varphi}$ *is a continuous* $(\{\bar{\mathscr{F}}_t^s\}, P_{s,x})$-*square-integrable martingale on* $[s,T]$, $V_{s,\cdot}^{x,\varphi}$ *is a continuous* $\{\mathscr{F}_t^s\}$-*adapted process of* $P_{s,x}$-*integrable variation on* $[s,T]$;

(ii) $X_{s,\cdot}^{\varphi}$ *is an* $(\{\mathscr{F}_t^s\}, P_{s,x})$-*Dirichlet process on* $[s,T]$ *admitting the decomposition* (1.2) *with*

$$A_{s,t}^{x,\varphi} = \tfrac{1}{2}(-M_{s,t}^{x,\varphi}+\tilde{N}_{s,t}^{x,\varphi}-V_{s,t}^{x,\varphi}), \qquad t\in[s,T], \qquad P_{s,x}\text{-a.s.} \tag{3.1}$$

Moreover, for every $t\in[s,T]$,

$$\langle M_{s,\cdot}^{x,\varphi}\rangle_t = \int_s^t (a\nabla\varphi, \nabla\varphi)(u, X_u)\,du, \qquad \langle N_{s,\cdot}^{x,\varphi}\rangle_t = \int_s^t (\bar{a}\nabla\varphi, \nabla\varphi)(u, \bar{X}_u)\,du \tag{3.2}$$

with $\bar{a}=\{\bar{a}^{ij}\}$, $\bar{a}^{ij}(t,x)=a^{ij}(T+s-t,x)$, *and*

$$V_{s,t}^{x,\varphi} = \int_s^t p^{-1}(a\nabla p, \nabla\varphi)(s,x,u,X_u)\,du. \tag{3.3}$$

Proof. Since the proof of uniqueness is standard, we will prove only existence of the triple. For this purpose, we first assume additionally that $\varphi\in C_0^2$ and for $k\in N$ we set $\varphi_k = kR_k\varphi$, $\psi_k=\varphi_k-\varphi$. By [14], Lemma 1.3, for each $k\in N$,

$$X_{s,t}^{\varphi_k} = \varphi_k(t,X_t)-\varphi_k(s,X_s), \qquad t\in[s,T],$$

is an $(\{\mathscr{F}_t^s\}, P_{s,x})$-semimartingale admitting the decomposition

$$X_{s,t}^{\varphi_k} = M_{s,t}^{\varphi_k}+\int_s^t k\psi_k(u,X_u)\,du = M_{s,t}^{\varphi_k}+A_{s,t}^{\varphi_k}, \qquad t\in[s,T], \tag{3.4}$$

where $M_{s,\cdot}^{\varphi_k}$ is an $(\{\mathscr{F}_t^s\}, P_{s,x})$-square-integrable martingale with the quadratic variation process

$$\langle M_{s,\cdot}^{\varphi_k}\rangle_t = \int_s^t (a\nabla\varphi_k, \nabla\varphi_k)(u,X_u)\,du. \tag{3.5}$$

For fixed $\delta\in(0,T-s)$ let $M_{s,\cdot}^{\delta,\varphi_k}$ denote the martingale $M_{s,t\vee(s+\delta)}^{\varphi_k}-M_{s,s+\delta}^{\varphi_k}$, $t\in[s,T]$. By (3.5) and uniqueness of the decomposition (3.4),

$$\begin{aligned} E_{s,x}\langle M_{s,\cdot}^{\delta,\varphi_k}-M_{s,\cdot}^{\delta,\varphi_l}\rangle_T &= E_{s,x}\langle M_{s,\cdot}^{\delta,\varphi_k-\varphi_l}\rangle_T \\ &= E_{s,x}\int_{s+\delta}^T \big(a\nabla(\varphi_k-\varphi_l), \nabla(\varphi_k-\varphi_l)\big)(u,X_u)\,du \leqslant \Lambda K_2\,\delta^{-d/2}\|\nabla(\varphi_k-\varphi_l)\|_{2;s+\delta,T}^2 \end{aligned} \tag{3.6}$$

for all $k, l\in N$, which, when combined with (2.4), shows that $\{M^{\delta,\varphi_k}\}_{k\in N}$ is a Cauchy sequence in $\mathscr{M}(P_{s,x})$. Since for each $k\in N$ all trajectories of $M_{s,\cdot}^{\delta,\varphi_k}$ are continuous, from Lemma 4.3.3 of [18] it follows that there is an $\{\mathscr{F}_t^s\}$-adapted martingale $M^{x,\delta} \in \mathscr{M}(P_{s,x})$ whose all trajectories are right-continuous such that $\{M_{s,\cdot}^{\delta,\varphi_k}\}$ converges to $M^{x,\delta}$ in $\mathscr{M}(P_{s,x})$. Furthermore, for any $0 < \delta < \sigma < T-s$,

$$(3.7)\qquad E_{s,x}\langle M^{x,\delta}-M^{x,\sigma}\rangle_T \leqslant \lim_{k\to\infty} E_{s,x}\langle M_{s,\cdot}^{\delta,\varphi_k}-M_{s,\cdot}^{\sigma,\varphi_k}\rangle_T$$

$$= \lim_{k\to\infty} E_{s,x}\int_{s+\delta}^{s+\sigma}(a\nabla\varphi_k, \nabla\varphi_k)(u, X_u)\,du \leqslant \Lambda \int_{s+\delta}^{s+\sigma}\int_{\mathbf{R}^d}|\nabla\varphi(y)|^2 p(s, x, u, y)\,ds\,dy.$$

For $d = 1$ the right-hand side of (3.7) is bounded by

$$(3.8)\qquad \Lambda K_2\|\nabla\varphi\|_2^2\int_{s+\delta}^{s+\sigma}(u-s)^{-1/2}\,du = 2\Lambda K_2(\sigma^{1/2}-\delta^{1/2})\|\nabla\varphi\|_2^2,$$

whereas for $d > 1$, by (2.2) and Hölder's inequality, it is bounded by

$$(3.9)\qquad C_1(\sigma^{(p-d)/p}-\delta^{(p-d)/p})\|\nabla\varphi\|_p^2,$$

where C_1 depends only on p, d, K_2. Hence, applying once again Lemma 4.3.3 of [18], we conclude that in both cases there is an $\{\mathscr{F}_t^s\}$-adapted martingale $M_{s,\cdot}^{x,\varphi}$ such that all its trajectories are right-continuous and $\{M^{x,\delta}\}$ converges to $M_{s,\cdot}^{x,\varphi}$ in $\mathscr{M}(P_{s,x})$ as $\delta\downarrow 0$. Define

$$A_{s,t}^{x,\varphi} = X_{s,t}^{\varphi}-M_{s,t}^{x,\varphi}, \qquad t\in[s, T].$$

The representation (3.1) can be proved by the same method as in the proof of Theorem 2.1 (ii) in [14], and therefore we omit it. Note, however, that from the construction of $N_{s,\cdot}^{x,\varphi}$ in [14] and from [18], Lemma 4.3.3, it follows that we can assume that all its trajectories are right-continuous.

Our next goal is to show that $A_{s,\cdot}^{x,\varphi}$ is a zero quadratic variation process, that is

$$(3.10)\qquad Q_{sT}^n(A_{s,\cdot}^{x,\varphi})\to 0 \text{ in } P_{s,x} \quad \text{as } n\to\infty.$$

In order to get (3.10) we will show that for fixed $\delta\in(0, T-s)$

$$(3.11)\qquad Q_{s+\delta,T}^n(A_{s,\cdot}^{x,\varphi})\to 0 \text{ in } P_{s,x} \quad \text{as } n\to\infty$$

and that

$$(3.12)\qquad \lim_{\delta\downarrow 0}\limsup_{n\to\infty} P_{s,x}\big(Q_{s,s+\delta}^n(A_{s,\cdot}^{x,\varphi}) > \varepsilon\big) = 0$$

for every $\varepsilon > 0$. Obviously,

$$Q_{s+\delta,T}^n(A_{s,\cdot}^{x,\varphi}) = Q_{s+\delta,T}^n(X_{s,\cdot}^{-\psi_k}+M_{s,\cdot}^{\varphi_k}-M_{s,\cdot}^{x,\varphi}+A_{s,\cdot}^{\varphi_k})$$
$$\leqslant 3Q_{s+\delta,T}^n(X_{s,\cdot}^{\psi_k})+3Q_{s+\delta,T}^n(M_{s,\cdot}^{\delta,\varphi_k}-M_{s,\cdot}^{\delta,\varphi})+3Q_{s+\delta,T}^n(A_{s,\cdot}^{\varphi_k}),$$

where $M^{\delta,\varphi}_{s,\cdot}$ is defined as M^{δ,φ_k}. Since $A^{\varphi_k}_{s,\cdot}$ is a process of finite variation,

(3.13) $$Q^n_{s+\delta,T}(A^{\varphi_k}_{s,\cdot}) \to 0 \text{ in } P_{s,x} \quad \text{as } n \to \infty.$$

Observe now that for any $t \in [s+\delta, T]$

$$M^{x,\varphi}_{s,t} - M^{x,\varphi}_{s,s+\delta} = \lim_{\sigma\downarrow 0} (M^{x,\sigma}_t - M^{x,\sigma}_{s+\delta})$$

$$= \lim_{\sigma\downarrow 0}\lim_{k\to\infty} \{(M^{\varphi_k}_{s,t} - M^{\varphi_k}_{s,s+\sigma}) - (M^{\varphi_k}_{s,s+\delta} - M^{\varphi_k}_{s,s+\sigma})\} = \lim_{\sigma\downarrow 0}\lim_{k\to\infty} (M^{\varphi_k}_{s,t} - M^{\varphi_k}_{s,s+\delta})$$

$$= \lim_{\sigma\downarrow 0} M^{x,\delta}_t = M^{x,\delta}_t$$

(all limits are taken in $\mathcal{M}(P_{s,x})$). Hence $M^{\delta,\varphi}_{s,\cdot} = M^{x,\delta}$ and, consequently,

(3.14) $$\lim_{k\to\infty}\limsup_{n\to\infty} E_{s,x} Q^n_{s+\delta,T}(M^{\delta,\varphi_k}_{s,\cdot} - M^{\delta,\varphi}_{s,\cdot}) \leqslant \lim_{k\to\infty} E_{s,x} \langle M^{\delta,\varphi_k}_{s,\cdot} - M^{x,\delta}\rangle_T = 0.$$

By the Markov property and (2.2),

$$E_{s,x} Q^n_{s+\delta,T}(X^{\psi_k}) = \sum_{s+\delta\leqslant t_i<T} E_{s,x} E_{s+\delta,X_{s+\delta}} |\psi_k(t_{i+1}, X_{t_{i+1}}) - \psi_k(t_i, X_{t_i})|^2$$

$$\leqslant K_2\, \delta^{-d/2} \sum_{s+\delta\leqslant t_i<T} E_{s+\delta,m} |\psi_k(t_{i+1}, X_{t_{i+1}}) - \psi_k(t_i, X_{t_i})|^2.$$

Using once again the Markov property and performing elementary computations gives

$$E_{s+\delta,m} |\psi_k(t_{i+1}, X_{t_{i+1}}) - \psi_k(t_i, X_{t_i})|^2$$

$$= E_{s+\delta,m} \{P^{t_i,t_{i+1}} \psi_k^2(t_{i+1}, X_{t_i}) - 2\psi_k(t_i, X_{t_i}) P^{t_i,t_{i+1}} \psi_k(t_{i+1}, X_{t_i}) + \psi_k^2(t_i, X_{t_i})\}$$

$$= (\psi_k^2(t_{i+1}, \cdot) + \psi_k^2(t_i, \cdot) - 2\psi_k(t_i, \cdot) P^{t_i,t_{i+1}} \psi_k(t_{i+1}, \cdot), 1)_2$$

$$= 2(\psi_k(t_{i+1}, \cdot), \psi_k(t_{i+1}, \cdot) - P^{t_i,t_{i+1}} \psi_k(t_{i+1}, \cdot))_2$$

$$+ 2(\psi_k(t_i, \cdot) - \psi_k(t_{i+1}, \cdot), \psi_k(t_{i+1}, \cdot) - P^{t_i,t_{i+1}} \psi_k(t_{i+1}, \cdot))_2$$

$$+ (\psi_k(t_{i+1}, \cdot) - \psi_k(t_i, \cdot), \psi_k(t_{i+1}, \cdot) - \psi_k(t_i, \cdot))_2$$

$$= I_i^1 + I_i^2 + I_i^3$$

for i such that $t_i \in [s+\delta, T]$. By (2.1),

$$I_i^1 = \int_{t_i}^{t_{i+1}} (a(u, \cdot) \nabla P^{t_i,u} \psi_k(t_{i+1}, \cdot), \nabla \psi_k(t_{i+1}, \cdot))_2\, du.$$

Hence, by (2.9),

$$|I_i^1| \leqslant C_2 (t_{i+1} - t_i) \|\nabla \psi_k(t_{i+1}, \cdot)\|_2^2 \quad \text{with } C_2 = d\Lambda K_3^{1/2},$$

which implies

$$\limsup_{n\to\infty} \sum_{s+\delta\leqslant t_i<T} |I_i^1| \leqslant C_2 \|\nabla\psi_k\|_{2;s+\delta,T}^2.$$

Clearly,

$$|I_i^2| \leqslant I_i^3 + \|\psi_k(t_{i+1},\cdot)-P^{t_i,t_{i+1}}\psi_k(t_{i+1},\cdot)\|_2^2.$$

By (2.9),

$$\|\psi_k(t_{i+1},\cdot)-P^{t_i,t_{i+1}}\psi_k(t_{i+1},\cdot)\|_2^2 \leqslant K_3(t_{i+1}-t_i)\|\nabla\psi_k(t_{i+1},\cdot)\|_2^2,$$

and

$$\sum_{s+\delta\leqslant t_i<T} I_i^3 \sum_{s+\delta\leqslant t_i<T} (t_{i+1}-t_i)\|D_t\varphi_k\|_{2;t_i,t_{i+1}}^2 \leqslant \|\Pi_n\|\cdot\|D_t\varphi_k\|_{2;s,T}^2.$$

Therefore

$$\limsup_{n\to\infty} \sum_{s+\delta\leqslant t_i<T} |I_i^2+I_i^3| \leqslant K_3\|\nabla\psi_k\|_{2;s+\delta,T}^2.$$

From the above, (2.2) and (2.4) we conclude that

$$\lim_{k\to\infty}\limsup_{n\to\infty} E_{s,x}Q_{s+\delta,T}^n(X^{\psi_k}) \leqslant K_2(C_2+K_3)\delta^{-d/2}\lim_{k\to\infty}\|\nabla\psi_k\|_{2;s+\delta,T}^2 = 0,$$

which together with (3.13), (3.14) gives (3.11). Finally, by (3.1), for every $\varepsilon>0$,

$$P_{s,x}\big(Q_{s,s+\delta}^n(A_{s,\cdot}^{x,\varphi})>\varepsilon\big) \leqslant P_{s,x}\big(Q_{s,s+\delta}^n(M_{s,\cdot}^{x,\varphi})>2\varepsilon/3\big)+P_{s,x}\big(Q_{s,s+\delta}^n(\tilde{N}_{s,\cdot}^{x,\varphi})>2\varepsilon/3\big)$$
$$+P_{s,x}\big(Q_{s,s+\delta}^n(V_{s,\cdot}^{x,\varphi})>2\varepsilon/3\big),$$

from which (3.12) easily follows. This completes the proof for $\varphi\in C_0^2$.

To prove the general case, we choose a sequence $\{f_k\}\subset C_0^2$ such that $f_k\to\varphi$ in W_p^1 and uniformly in compact sets in R^d. Then for each $k\in N$ the process $X_{s,\cdot}^{f_k}$ is an $(\{\mathscr{F}_t^s\}, P_{s,x})$-Dirichlet process having the representation

(3.15) $$X_{s,t}^{f_k} = M_{s,t}^{x,f_k}+A_{s,t}^{x,f_k}, \quad A_{s,t}^{x,f_k} = \tfrac{1}{2}(-M_{s,t}^{x,f_k}+\tilde{N}_{s,t}^{x,f_k}-V_{s,t}^{x,f_k}), \quad t\in[s, T].$$

Since the decomposition (3.15) is unique, we have

(3.16) $$M_{s,t}^{x,f_k}-M_{s,t}^{x,f_l} = M_{s,t}^{f_k-f_l}, \quad N_{s,t}^{x,f_k}-N_{s,t}^{x,f_l} = N_{s,t}^{x,f_k-f_l}, \quad t\in[s, T],$$

for k, $l\in N$. Hence, by (3.2) and (2.2),

(3.17) $$E_{s,x}\langle M_{s,\cdot}^{x,f_k}-M_{s,\cdot}^{x,f_l}\rangle_T+E_{s,x}\langle N_{s,\cdot}^{x,f_k}-N_{s,\cdot}^{x,f_l}\rangle_T \leqslant C_3\|\nabla(f_k-f_l)\|_p^2$$

for some C_3 depending only on λ, Λ, d, p and T. Moreover, from the first part of the proof it follows that we can assume that for each $k\in N$ all trajectories of $M_{s,\cdot}^{x,f_k}$ and $N_{s,\cdot}^{x,f_k}$ are right-continuous. Therefore there exist an $(\{\mathscr{F}_t^s\}, P_{s,x})$-square-integrable martingale $M_{s,\cdot}^{x,\varphi}$ and an $(\{\bar{\mathscr{F}}_t^s\}, P_{s,x})$-square-

-integrable martingale $N^{x,\varphi}_{s,\cdot}$ such that

$$(3.18)\qquad E_{s,x}\langle M^{x,f_k}_{s,\cdot}-M^{x,\varphi}_{s,\cdot}\rangle_T\to 0,\qquad E_{s,x}\langle N^{x,f_k}_{s,\cdot}-N^{x,\varphi}_{s,\cdot}\rangle_T\to 0.$$

Moreover, if we define $V^{x,\varphi}_{s,\cdot}$ by (3.3), then

$$(3.19)\qquad \lim_{k\to\infty} E_{s,x}\sup_{s\leqslant t\leqslant T}|V^{x,f_k}_{s,t}-V^{x,\varphi}_{s,t}| \leqslant \lim_{k\to\infty}\int_s^T |(a\nabla p(s,x,u,\cdot),\nabla(f_k-\varphi))_2|\,du = 0,$$

the last equality being a consequence of the assertion (i) of Theorem 2.1 in the case $d>1$, and the assertions (i) and (iii) for $d=1$ because, by the Schwarz inequality,

$$\Big(\int_s^T\int_{\boldsymbol{R}}|p'(s,x,t,y)(f_k-\varphi)'(y)|\,dy\Big)^2 \leqslant \int_s^T\int_{\boldsymbol{R}}(t-s)^{1/4}\frac{(p')^2}{p}(s,x,t,y)\,dt\,dy\int_s^T\int_{\boldsymbol{R}}(t-s)^{-1/4}|(f_k-\varphi)'(y)|^2p(s,x,t,y)\,dt\,dy.$$

From (3.15)–(3.19) and the continuous mapping theorem we obtain (1.2) and (3.1)–(3.3), so what is left is to show (3.10). Since $A^{x,\varphi}_{s,\cdot}$ has the representation (3.1) and $V^{x,\varphi}_{s,\cdot}$ is a process of finite variation on $[s,T]$, (3.10) will be proved once we prove that $\langle M^{x,\varphi}_{s,\cdot}-\tilde{N}^{x,\varphi}_{s,\cdot}\rangle_T=0$. It is easily seen, however, that the last assertion follows from (3.18) and the fact that $\langle M^{x,f_k}_{s,\cdot}-\tilde{N}^{x,f_k}_{s,\cdot}\rangle_T=\langle A^{x,f_k}_{s,\cdot}\rangle_T=0$. ∎

We will see in Section 4 that X^φ admits a decomposition of the form (1.2), (3.1) with $M^{x,\varphi}$, $N^{x,\varphi}$, $V^{x,\varphi}$ satisfying the assertion (i) even if we drop the assumption (1.3). We do not know, however, whether in this case $A^{x,\varphi}$ is a zero quadratic variation process, and, consequently, we neither know that X^φ is a Dirichlet process nor that the triple is unique.

4. ADDITIVE FUNCTIONALS

In this section we give conditions on a and φ under which for each $T>0$ the additive functional (AF) $X^\varphi=\{X^\varphi_{s,t},\ 0\leqslant s\leqslant t\leqslant T\}$ admits a unique decomposition into a martingale additive functional (MAF) of finite energy and a continuous additive functional (CAF) of zero energy. We begin with basic definitions.

Put $\mathscr{P}=\{P_{s,\mu}:\ \mu$ is a probability measure on $\mathscr{B}\}$, where $\mathscr{B}$ is a Borel σ-field of $\boldsymbol{R}^d$, $P_{s,\mu}(\cdot)=\int_{\boldsymbol{R}^d}P_{s,x}(\cdot)\,\mu(dx)$ and define $\mathscr{G}$ as the completion of $\mathscr{F}^s_T$ with respect to the family $\mathscr{P}$, and then $\mathscr{G}^s_t$ as the completion of $\mathscr{F}^s_t$ in $\mathscr{G}$ with respect to $\mathscr{P}$ (see [7], Section I.3, for more details).

We say that the family of random variables $A=\{A_{s,t},\ 0\leqslant s\leqslant t\leqslant T\}$ is an AF of $(X,P_{s,x})$ (on $[0,T]$) if $A_{s,t}$ is $\mathscr{G}^s_t$-measurable for every $0\leqslant s\leqslant t\leqslant T$ and

$$P_{s,x}(A_{s,t}=A_{s,u}+A_{u,t},\ s\leqslant u\leqslant t\leqslant T)=1$$

for every $(s, x) \in [0, T] \times \mathbf{R}^d$. If, in addition,

$$P_{s,x}(\{\omega \in \Omega\colon [s, T] \ni t \mapsto A_{s,t}(\omega) \text{ is continuous}\}) = 1$$

for every $(s, x) \in [0, T) \times \mathbf{R}^d$, then A is called a *CAF*.

For an AF A of $(X, P_{s,x})$ we define $e(A)$ and $\bar{e}(A)$ as in Section 1. If $e(A) < \infty$ $(e(A) = 0)$, we call A an *AF of finite energy* (*zero energy*).

Changing the variable $t \mapsto \alpha\tau$ and using Fubini's theorem yields

$$\alpha^2 e_\alpha(A) = \alpha \int_0^\infty \mathbf{1}_{[0,\alpha T]}(\tau) e^{-\tau} \bar{e}_{\tau/\alpha}(A)\, d\tau.$$

Therefore, if (for instance) $\sup_{0<t\leqslant T}(1/t)\bar{e}_t(A) < \infty$, then applying the Lebesgue dominated convergence theorem shows that if $\bar{e}(A)$ exists, then $e(A)$ exists and $e(A) = \bar{e}(A)$. In particular, if

$$A_{s,t} = \int_s^t f(u, X_u)\, du, \qquad 0 \leqslant s < t \leqslant T, \tag{4.1}$$

for some $f \in L_2(0, T)$, then using Schwarz's inequality and Fubini's theorem we obtain

$$\bar{e}_t(A) \leqslant \int_0^{T-t} t\Big(\int_s^{s+t} \|f(u, \cdot)\|_2^2\, du\Big)\, ds \leqslant t^2 \|f\|_{2;0,T}^2.$$

Hence $e(A) = \bar{e}(A) = 0$.

We say that $M = \{M_{s,t}, 0 \leqslant s \leqslant t \leqslant T\}$ is a *continuous MAF* of $(X, P_{s,x})$ if it is a CAF such that $E_{s,x} M_{s,t}^2 < \infty$, $E_{s,x} M_{s,t} = 0$ for every $0 \leqslant s \leqslant t \leqslant T$, $x \in \mathbf{R}^d$.

Let us remark that if M is an MAF of $(X, P_{s,x})$, then $M_{s,\cdot}$ is a $(\{\mathscr{G}_t^s\}, P_{s,x})$-martingale on $[s, T]$, because, by the Markov property,

$$E_{s,x}(M_{s,t} \mid \mathscr{G}_u^s) = E_{s,x}(M_{s,u} + M_{u,t} \mid \mathscr{G}_u^s) = M_{s,u} + E_{u,X_u} M_{u,t} = M_{s,u}$$

for all $s < u \leqslant t \leqslant T$ and $x \in \mathbf{R}^d$. By the Markov property we also have

$$E_{s,m} M_{s,s+u+v}^2 = E_{s,m} M_{s,s+u}^2 + E_{s+u,m} M_{s+u,s+u+v}^2$$

for $s, u, v \geqslant 0$ such that $s+u+v \leqslant T$, and hence, by an elementary computation, $\bar{e}_{u+v}(M) \leqslant \bar{e}_u(M) + \bar{e}_v(M)$ for all $u, v > 0$ such that $u+v \leqslant T$. Consequently, by the well-known properties of subadditive functions, $\bar{e}(M)$ is well defined and

$$\bar{e}(M) = \sup_{0<t\leqslant T} t^{-1} \bar{e}_t(M). \tag{4.2}$$

Notice also that in the case where $\langle M_{s,\cdot}\rangle_t = A_{s,t}$ and A is given by (4.1) with some non-negative integrable f, we have

$$\bar{e}(M) = \lim_{t\downarrow 0} \frac{1}{t} \int_0^{T-t} \Big(\int_s^{s+t} \|f(u, \cdot)\|_1\, du\Big)\, ds = \|f\|_{1;0,T}. \tag{4.3}$$

A decomposition of any CAF of $(X, P_{s,x})$ into a continuous MAF of finite energy and a CAF of zero energy is unique in the sense that we have

PROPOSITION 4.1. *Suppose ${}_1M$ and ${}_2M$ are continuous MAFs of finite energy and ${}_1A$ and ${}_2A$ are CAFs of zero energy such that ${}_1M_{s,t}+{}_1A_{s,t} = {}_2M_{s,t}+{}_2A_{s,t}$, $t\in[s, T]$, $P_{s,m}$-a.s., for a.e. $s\in[0, T]$. Then*

$$P_{s,x}({}_1M_{s,t} = {}_2M_{s,t},\ {}_1A_{s,t} = {}_2A_{s,t},\ t\in[s, T]) = 1$$

for every $(s, x)\in[0, T)\times \boldsymbol{R}^d$.

Proof. Fix $(s, x)\in[0, T)\times \boldsymbol{R}^d$ and $t\in(0, T-s)$. Since $M = {}_1M-{}_2M$ is a continuous MAF of zero energy, it follows from (4.2) that $\bar{e}_t(M) = 0$. Therefore there is $\{\delta_n\}\subset(s, T-t)$ such that $\delta_n\downarrow s$ and $E_{\delta_n,m}M^2_{\delta_n,\delta_n+t} = 0$, $n\in\boldsymbol{N}$. For each $n\in \boldsymbol{N}$ we have

$$E_{s,x}M^2_{s,s+t}\leqslant 2E_{s,x}M^2_{s,\delta_n}+2E_{s,x}M^2_{\delta_n,s+t} = 2E_{s,x}M^2_{s,\delta_n}, \tag{4.4}$$

because

$$E_{s,x}M^2_{\delta_n,s+t} = E_{s,x}\big(E_{s,x}(M^2_{\delta_n,s+t}\,|\,\mathscr{G}^s_{\delta_n})\big) = E_{s,x}(E_{\delta_n,X_{\delta_n}}M^2_{\delta_n,s+t})$$
$$\leqslant \int_{\boldsymbol{R}^d} E_{\delta_n,y}(M^2_{\delta_n,\delta_n+t})\,p(s, x, \delta_n, y)\,dy = 0.$$

Since M is continuous, letting $n\uparrow\infty$ in (4.4) we get $E_{s,x}M^2_{s,s+t} = 0$, and hence the desired conclusion ∎

THEOREM 4.2. *Assume* (1.1) *holds and let $\varphi\in W^1_p\cap W^1_2$ with $p = 2$ if $d = 1$ and $p > d$ if $d > 1$. Then there exists a unique continuous MAF $M^{[\varphi]} = \{M^{[\varphi]}_{s,t},\ 0\leqslant s\leqslant t\leqslant T\}$ of finite energy and a unique CAF $A^{[\varphi]} = \{A^{[\varphi]}_{s,t},\ 0\leqslant s\leqslant t\leqslant T\}$ of zero energy such that* (1.4) *is satisfied for every $(s, x)\in[0, T)\times\boldsymbol{R}^d$. Moreover,*

$$\langle M^{[\varphi]}_{s,\cdot}\rangle_t = \int_s^t (a\nabla\varphi, \nabla\varphi)(u, X_u)\,du,\qquad t\in[s, T],\qquad P_{s,x}\text{-a.s.}, \tag{4.5}$$

for every $(s, x)\in[0, T)\times\boldsymbol{R}^d$.

Proof. First we show that there is no loss of generality in assuming that $\varphi\in C^2_0$. To see this, choose a sequence $\{f_k\}\subset C^2_0$ such that $f_k\to\varphi$ uniformly on compact sets in $\boldsymbol{R}^d$, in W^1_p and W^1_2. If the theorem were true for functions of the class C^2_0, for each $k\in\boldsymbol{N}$ we would have

$$X^{f_k}_{s,t} = M^{[f_k]}_{s,t}+A^{[f_k]}_{s,t},\qquad t\in[s, T],\qquad P_{s,x}\text{-a.s.}, \tag{4.6}$$

for every $(s, x)\in[0, T]\times\boldsymbol{R}^d$, where $M^{[f_k]}$ is a continuous MAF of finite energy, $A^{[f_k]}$ is a CAF of zero energy and (1.4), (4.5) are satisfied with φ replaced by f_k. By Proposition 4.1 the decomposition (4.6) is unique. Consequently,

$M^{[f_k]}-M^{[f_l]} = M^{[f_k-f_l]}$ for $k, l\in N$. On the other hand, by (2.2), there is C_3 depending only on $\lambda, \Lambda, d, p, T$ such that

$$E_{s,x}\langle M_{s,\cdot}^{[f_k-f_l]}\rangle_T \leqslant C_3 \|\nabla(f_k-f_l)\|_p^2.$$

Therefore, by Doob's inequality, there is a subsequence $\{k_n\}\subset N$ such that

$$P_{s,x}\big(\sup_{s\leqslant t\leqslant T} |M_{s,t}^{[f_{k_{n+1}}]}-M_{s,t}^{[f_{k_n}]}| > 2^{-n}\big) \leqslant 2^{-n}$$

for every $(s, x)\in[0, T]\times R^d$. Hence, if we set

$$\Omega_s = \{\omega\in\Omega\colon \{M_{s,t}^{[f_{k_n}]}\}_{n\in N} \text{ converges uniformly in } t \text{ on } [s, T]\},$$

then $P_{s,x}(\Omega_s) = 1$ for every $x\in R^d$, by the Borel–Cantelli lemma. For given $s\in[0, T)$ let

$$M_{s,t}^{[\varphi]}(\omega) = \lim_{n\to\infty} M_{s,t}^{[f_{k_n}]}(\omega),\ \omega\in\Omega_s, \qquad M_{s,t}^{[\varphi]}(\omega) = 0,\ \omega\notin\Omega_s, \qquad t\in[s, T], \tag{4.7}$$

and

$$A_{s,t}^{[\varphi]}(\omega) = X_{s,t}^{\varphi}(\omega)-M_{s,t}^{[\varphi]}(\omega), \qquad \omega\in\Omega,\ t\in[s, T]. \tag{4.8}$$

From (4.7), (4.8) it follows that $M^{[\varphi]}$ and $A^{[\varphi]}$ are continuous AFs satisfying (1.4) and (4.22). Moreover, by (4.5) and (4.3), $M^{[\varphi]}$ is an MAF with

$$\bar{e}(M^{[\varphi]}) = (a\nabla\varphi, \nabla\varphi)_{2;0,T} < \infty. \tag{4.9}$$

Since $e(A^{[f_k]}) = 0$,

$$\begin{aligned} e(A^{[\varphi]}) = e(X^{\varphi}-M^{[\varphi]}) &\leqslant 3e(X^{f_k-\varphi})+3e(M^{[f_k]}-M^{[\varphi]})+3e(A^{[f_k]}) \\ &= 3e(X^{f_k-\varphi})+3e(M^{[f_k]}-M^{[\varphi]}). \end{aligned} \tag{4.10}$$

Put $g_k = f_k-\varphi$, $k\in N$. We have

$$\alpha^2 e_\alpha(X^{g_k}) = 2\alpha^2 \int_0^T ds \int_0^{T-s} e^{-\alpha t}(g_k, g_k-P^{s,s+t} g_k)_2\, dt \leqslant 2\alpha(g_k, g_k-\alpha R_\alpha g_k)_{2;0,T},$$

and so, by (2.5),

$$\lim_{k\to\infty} e(X^{f_k-\varphi}) = 0. \tag{4.11}$$

For $k, l\in N$ we also have

$$\begin{aligned} \bar{e}_t(M^{[f_k]}-M^{[f_l]}) &= \int_0^{T-t} ds \int_s^{s+t} \big(a(u, \cdot)\nabla(f_k-f_l), \nabla(f_k-f_l)\big)_2\, du \\ &\leqslant \Lambda(T-t)\, t\, \|\nabla(f_k-f_l)\|_2^2. \end{aligned}$$

Therefore

$$(4.12)\qquad \lim_{k\to\infty} \bar{e}(M^{[f_k]}-M^{[\varphi]}) \leqslant \Lambda T \lim_{k\to\infty} \liminf_{l\to\infty} \|\nabla(f_k-f_l)\|_2^2 = 0$$

by Fatou's lemma. Combining (4.10)–(4.12) gives $e(A^{[\varphi]}) = 0$.

By what has already been proved, it suffices to prove the theorem for $\varphi \in C_0^2$. For this purpose, we define φ_k, ψ_k, M^{φ_k}, M^{δ,φ_k}, $M^{x,\delta}$ as in the proof of Theorem 3.1. In view of (2.4), (3.6) and Doob's inequality we can choose a subsequence $\{k_n\} \subset N$ such that for each $\delta \in (0, T-s)$

$$P_{s,x}(\sup_{s\leqslant t\leqslant T} |M_{s,t}^{\delta,\varphi_{k_{n+1}}} - M_{s,t}^{\delta,\varphi_{k_n}}| > 2^{-n}) \leqslant 2^{-n}\delta^{-d/2}$$

for all $n \in N$. Let us set

$$\Gamma_s^1 = \{\omega \in \Omega\colon \{M_{s,t}^{\delta,\varphi_{k_n}}\}_{n\in N} \text{ converges uniformly in } t \text{ on } [s, T]\}.$$

By the Borel–Cantelli lemma, $P_{s,x}(\Gamma_s^1) = 1$ for every $x \in \boldsymbol{R}^d$. Next, for given $t \in [0, T-s)$ let

$$M_{s,t}^{[\delta]}(\omega) = \lim_{n\to\infty} M_{s,t}^{\delta,\varphi_{k_n}}(\omega),\ t \in [s, T], \quad \text{for } \omega \in \Gamma_s^1$$

and

$$M_{s,t}^{[\delta]}(\omega) = 0 \quad \text{for } \omega \notin \Gamma_s^1.$$

Since $P_{s,x}(M_t^{x,\delta} = M_{s,t}^{[\delta]},\ t \in [s, T]) = 1$, it follows from (3.7)–(3.9) that there is a subsequence $\{\delta_m\} \subset (0, T-s)$ such that $\delta_m \downarrow 0$ and $P_{s,x}(\Gamma_s^2) = 1$ for every $x \in \boldsymbol{R}^d$, where

$$\Gamma_s^2 = \{\omega \in \Omega\colon \{M_{s,t}^{[\delta_m]}\}_{n\in N} \text{ converges uniformly in } t \text{ on } [s, T]\}.$$

Define now

$$M_{s,t}^{[\varphi]}(\omega) = \lim_{m\to\infty} M_{s,t}^{[\delta_m]}(\omega),\ t \in [s, T], \quad \text{for } \omega \in \Gamma_s^1 \cap \Gamma_s^2,$$

$$M_{s,t}^{[\varphi]}(\omega) = 0 \quad \text{for } \omega \notin \Gamma_s^1 \cap \Gamma_s^2$$

and $A^{[\varphi]}$ by (4.8). Then obviously (1.4) and (4.5) are satisfied and $M^{[\varphi]}$ is continuous. Moreover, since $M_{s,t}^{\varphi_k}$ is $\mathscr{F}_t^s$-measurable for every $k \in N$, $M_{s,t}^{[\delta]}$ is $\mathscr{G}_t^s$-measurable for every $\delta \in (0, T-s)$, and hence $M_{s,t}^{[\varphi]}$ is also $\mathscr{G}_t^s$-measurable. Finally, for fixed $0 \leqslant s < u < t \leqslant T$, $n \in N$ and every $m \in N$ such that $\delta_m \leqslant \min\{u-s, t-u\}$ we have

$$(4.13)\qquad M_{s,t}^{\delta_m,\varphi_{k_n}} = M_{s,u}^{\delta_m,\varphi_{k_n}} + M_{u,t}^{\delta_m,\varphi_{k_n}} + M_{u,u+\delta_m}^{\varphi_{k_n}}.$$

By (2.2) and (2.4),

$$\lim_{m\to\infty}\limsup_{n\to\infty} E_{s,x}\langle M^{\varphi_{k_n}}_{u,\cdot}\rangle_{u+\delta_m} \leqslant \lim_{m\to\infty}\limsup_{n\to\infty} \Lambda K_2 (u-s)^{-d/2} \|\nabla\varphi_{k_n}\|^2_{2;u,u+\delta_m}$$

$$\leqslant \lim_{m\to\infty} \Lambda K_2 (u-s)^{-d/2}\delta_m \|\nabla\varphi\|_2^2 = 0.$$

Therefore letting $n\uparrow\infty$ and then $m\uparrow\infty$ in (4.13) we get $M^{[\varphi]}_{s,t} = M^{[\varphi]}_{s,u}+M^{[\varphi]}_{u,t}$ $P_{s,x}$-a.s. From this and (4.5) and (4.9) we see that $M^{[\varphi]}$ is an MAF of finite energy. By (4.8), $A^{[\varphi]}$ is a CAF, so we only need to show that $e(A^{[\varphi]}) = 0$. To see this, we first prove that

$$\limsup_{k\to\infty}\limsup_{\alpha\to\infty} \alpha(\psi_k, \psi_k-\alpha R_\alpha\psi_k)_{2;0,T} = 0. \tag{4.14}$$

By (2.3) we have

$$\alpha(R_\alpha\psi_k, \psi_k)_{2;0,T} - \langle D_t R_\alpha\psi_k, \psi_k\rangle_{0,T} + 2^{-1}(a\nabla R_\alpha\psi_k, \nabla\psi_k)_{2;0,T} = (\psi_k, \psi_k)_{2;0,T}.$$

Since $\psi_k, R_\alpha\psi_k \in \mathscr{W}_T(0, T)$, we have

$$\langle D_t R_\alpha\psi_k, \psi_k\rangle_{0,T} = -\langle D_t\psi_k, R_\alpha\psi_k\rangle_{0,T} - \big(R_\alpha\psi_k(0,\cdot), \psi_k(0,\cdot)\big)_2$$

(see, e.g., [20]) and, consequently,

$$\alpha(\psi_k, \psi_k-\alpha R_\alpha\psi_k)_{2;0,T} = 2^{-1}\big(a\nabla(\alpha R_\alpha\psi_k), \nabla\psi_k\big)_{2;0,T} + \langle D_t\psi_k, \alpha R_\alpha\psi_k\rangle_{0,T} + \big(\alpha R_\alpha\psi_k(0,\cdot), \psi_k(0,\cdot)\big)_2. \tag{4.15}$$

By (2.4),

$$\limsup_{k\to\infty}\limsup_{\alpha\to\infty} \big|\big(a\nabla(\alpha R_\alpha\psi_k), \nabla\psi_k\big)_{2;0,T}\big| \tag{4.16}$$

$$\leqslant \Lambda \limsup_{k\to\infty}\big\{\limsup_{\alpha\to\infty}\big|\big(\nabla(\alpha R_\alpha\psi_k-\psi_k), \nabla\psi_k\big)_{2;0,T}\big| + (\nabla\psi_k, \nabla\psi_k)_{2;0,T}\big\} = 0.$$

Since $2\langle D_t\psi_k, \psi_k\rangle_{0,T} = -\big(\psi_k(0,\cdot), \psi_k(0,\cdot)\big)_2$, using (2.4) we also get

$$\lim_{\alpha\to\infty}\langle D_t\psi_k, \alpha R_\alpha\psi_k\rangle_{0,T} = \langle D_t\psi_k, \psi_k\rangle_{0,T} \leqslant 0, \tag{4.17}$$

because $|\langle D_t\psi_k, \alpha R_\alpha\psi_k-\psi_k\rangle_{0,T}| \leqslant \|D_t\psi_k\|_{W_2^{0,-1}(0,T)}\|\alpha R_\alpha\psi_k-\psi_k\|_{W_2^{0,1}(0,T)}$. Observe now that

$$\big|\big(\alpha R_\alpha\psi_k(0,\cdot), \psi_k(0,\cdot)\big)_2\big| = \Big|\int_0^T \alpha e^{-\alpha t}\big(P^{0,t}\psi_k(t,\cdot), \psi_k(0,\cdot)\big)_2\,dt\Big|$$

$$\leqslant \int_0^T \alpha e^{-\alpha t}\|\psi_k(t,\cdot)\|_2\cdot\|\psi_k(0,\cdot)\|_2\,dt = \int_0^{\alpha T} e^{-t}\|\psi_k(t/\alpha,\cdot)\|_2\cdot\|\psi_k(0,\cdot)\|_2\,dt.$$

Since $\psi_k \in \mathscr{W}(0, T)$ and $\mathscr{W} \subset C([0, T]; L_2)$, $\|\psi_k(t/\alpha, \cdot)\|_2 \to \|\psi_k(0, \cdot)\|_2$ as $\alpha \to \infty$. At the same time, for any $s \in [0, T)$,

$$\|\psi_k(s, \cdot)\|_2^2 = \int_{R^d} \Big| \int_0^{T-s} ke^{-kt} E_{s,x} \varphi(X_{s+t})\, dt \Big|^2 dx \leqslant \int_0^{T-s} ke^{-kt} E_{s,m} |\varphi(X_{s+t})|^2 \leqslant \|\varphi\|_2^2,$$

so applying Fatou's lemma gives

$$(4.18) \quad \limsup_{\alpha\to\infty} \big|(\alpha R_\alpha \psi_k(0, \cdot), \psi_k(0, \cdot))_2\big| \leqslant \int_0^\infty e^{-t} \|\psi_k(0, \cdot)\|_2^2\, dt = \|\psi_k(0, \cdot)\|_2^2.$$

Moreover,

$$\int_{R^d} \Big| \int_0^T ke^{-kt} E_{0,x}(\varphi(X_t) - \varphi(x))\, dt \Big|^2 dx \leqslant \int_0^T ke^{-kt} E_{0,m} |\varphi(X_t) - \varphi(X_0)|^2\, dt$$
$$= 2\int_0^T ke^{-kt} (\varphi - P^{0,t}\varphi, \varphi)_2\, dt = 2\int_0^{kT} e^{-t} (\varphi - P^{0,t/k}\varphi, \varphi)_2\, dt.$$

Clearly, $(\varphi - P^{0,t/k}\varphi, \varphi)_2 \leqslant 2\|\varphi\|_2^2$ and, by (2.1), $(\varphi - P^{0,t/k}\varphi, \varphi)_2 \to 0$ as $k \to \infty$, so applying once again Fatou's lemma we obtain

$$(4.19) \quad \limsup_{k\to\infty} \|\psi_k(0, \cdot)\|_2^2$$
$$\leqslant \limsup_{k\to\infty} \Big(4\int_0^{kT} e^{-t} (\varphi - P^{0,t/k}\varphi, \varphi)_2\, dt + 2e^{-2kT} \|\varphi\|_2^2\Big) = 0.$$

Combining (4.15)–(4.19) gives (4.14). We have

$$\int_0^{T-s} e^{-\alpha t} E_{s,m} (X_{s,s+t}^{\psi_k})^2\, dt$$
$$\leqslant \int_0^{T-s} e^{-\alpha t} (\psi_k(s+t, \cdot), \psi_k(s+t, \cdot))_2\, dt + \int_0^T e^{-\alpha t} (\psi_k(s, \cdot), \psi_k(s, \cdot))_2\, dt$$
$$-2\int_0^{T-s} e^{-\alpha t} (\psi_k(s, \cdot), P^{s,s+t} \psi_k(s+t, \cdot))_2\, dt$$
$$= \int_0^{T-s} e^{-\alpha t} (\psi_k(s+t, \cdot), \psi_k(s+t, \cdot))_2\, dt - \int_0^T e^{-\alpha t} (\psi_k(s, \cdot), \psi_k(s, \cdot))_2\, dt$$
$$+ \frac{2}{\alpha}(1 - e^{-\alpha T}) (\psi_k(s, \cdot), \psi_k(s, \cdot))_2 - 2(\psi_k, R_\alpha \psi_k(s, \cdot))_2,$$

and hence, by (4.14),

$$(4.20) \quad \limsup_{k\to\infty} \limsup_{\alpha\to\infty} \alpha^2 e_\alpha (X^{\psi_k})$$
$$\leqslant 2\limsup_{k\to\infty} \limsup_{\alpha\to\infty} \alpha (\psi_k, \psi_k - \alpha R_\alpha \psi_k)_{2;0,T} = 0.$$

Furthermore, for any $t\in(T-s)$ and $\delta\in(0,t)$ we have

$$E_{s,m}(M^{\varphi_k}_{s,s+t}-M^{[\varphi]}_{s,s+t})^2 \leqslant 3E_{s,m}(M^{\varphi_k}_{s,s+t}-M^{\varphi_k}_{s,s+\delta}-M^{[\varphi]}_{s,s+t}+M^{[\varphi]}_{s,s+\delta})^2 + 3E_{s,m}(M^{\varphi_k}_{s,s+\delta})^2+3E_{s,m}(M^{[\varphi]}_{s,s+\delta})^2.$$

By Fatou's lemma,

$$\int_0^{T-t} E_{s,m}(M^{\varphi_k}_{s,s+t}-M^{\varphi_k}_{s,s+\delta}-M^{[\varphi]}_{s,s+t}+M^{[\varphi]}_{s,s+\delta})^2\,ds$$

$$\leqslant \liminf_{l\to\infty}\int_0^{T-t} E_{s,m}(M^{\varphi_k-\varphi_l}_{s,s+t}-M^{\varphi_k-\varphi_l}_{s,s+\delta})^2\,ds \leqslant \liminf_{l\to\infty}\int_0^{T-t} E_{s,m}(M^{\varphi_k-\varphi_l}_{s,s+t})^2\,ds.$$

On the other hand, by (3.5) and Fubini's theorem,

$$\int_0^{T-t} E_{s,m}(M^{\varphi_k-\varphi_l}_{s,s+t})^2\,ds = \int_0^{T-t} ds\int_s^{s+t}\big(a\nabla(\varphi_k-\varphi_l)(u,\cdot),\,\nabla(\varphi_k-\varphi_l)(u,\cdot)\big)_2\,du$$

$$\leqslant t\int_0^T\big(a\nabla(\varphi_k-\varphi_l)(u,\cdot),\,\nabla(\varphi_k-\varphi_l)(u,\cdot)\big)_2\,du \leqslant t\Lambda\,\|\nabla(\varphi_k-\varphi_l)\|^2_{2;0,T}$$

and

$$\int_0^{T-t} E_{s,m}\big((M^{\varphi_k}_{s,s+\delta})^2+(M^{[\varphi]}_{s,s+\delta})^2\big)\,ds \leqslant \delta\Lambda\,(\|\nabla\varphi_k\|^2_{2;0,T}+T\,\|\nabla\varphi\|^2_2).$$

From the above and (2.4) we get

$$\bar{e}(M^{\varphi_k}-M^{[\varphi]}) \leqslant 3\Lambda\liminf_{l\to\infty}\|\nabla(\varphi_k-\varphi_l)\|^2_{2;0,T} = 3\Lambda\,\|\nabla\psi_k\|^2_{2;0,T},$$

and hence

$$\lim_{k\to\infty}\bar{e}(M^{\varphi_k}-M^{[\varphi]})=0, \tag{4.21}$$

again by (2.4). Finally, since $\psi_k\in L_2(0,T)$, we have $e(A^{\varphi_k})=0$. Thus (4.10) holds with f_k replaced by φ_k and, consequently, (4.20) and (4.21) give $e(A^{[\varphi]})=0$. ∎

The zero energy part of the decomposition (1.4) admits a unique representation of the form (3.1). More precisely, we have

PROPOSITION 4.3. *Under the assumptions of Theorem* 4.2, *for every* $(s,x)\in[0,T)\times\boldsymbol{R}^d$ *there exists a unique triple* $(M^{x,\varphi}_{s,\cdot},\,N^{x,\varphi}_{s,\cdot},\,V^{x,\varphi}_{s,\cdot})$ *satisfying the assertion* (i) *of Theorem* 3.1 *and such that*

$$M^{[\varphi]}_{s,t}=M^{x,\varphi}_{s,t},\qquad A^{[\varphi]}_{s,t}=\tfrac{1}{2}(-M^{x,\varphi}_{s,t}+\tilde{N}^{x,\varphi}_{s,t}-V^{x,\varphi}_{s,t}),\qquad t\in[s,T],\qquad P_{s,x}\text{-a.s.} \tag{4.22}$$

Moreover, the triple can be chosen so that $\langle M^{x,\varphi}_{s,\cdot}\rangle$, $\langle N^{x,\varphi}_{s,\cdot}\rangle$ *are given by* (3.2) *and* $V^{x,\varphi}_{s,\cdot}$ *by* (3.3).

Proof. Uniqueness of the triple is a consequence of uniqueness of the decomposition of a continuous process into a continuous martingale and a continuous process of finite variation.

Existence of the triple can be proved by the same method as in the proof of Theorem 3.1. Indeed, for fixed $(s, x) \in [0, T) \times \boldsymbol{R}^d$, $\varphi \in C_0^2$ let $(M_{s,\cdot}^{x,\varphi}, N_{s,\cdot}^{x,\varphi}, V_{s,\cdot}^{x,\varphi})$ be the triple constructed in the first part of the proof of Theorem 3.1 (to construct the triple we did not use (1.3)). Comparing its construction with the proof of the decomposition (1.4) we see that (4.22) is satisfied. Take now a sequence $\{f_k\}$ considered at the beginning of the proof of Theorem 4.2. Since $\{f_k\} \subset C_0^2$, it follows from the above observation that for each $k \in N$ there is a triple $(M_{s,\cdot}^{x,f_k}, N_{s,\cdot}^{x,f_k}, V_{s,\cdot}^{x,f_k})$ satisfying the assertion (i) of Theorem 3.1 and such that (3.2), (3.3) and (4.22) hold with f_k in place of φ. The last property together with Proposition 4.1 implies (3.16), so letting $k \to \infty$ proves existence of the representation (4.22) for general φ. ∎

Let us set $\mathscr{Y} = [0, \infty) \times \boldsymbol{R}^d$ and suppose that $a\colon \mathscr{Y} \to \boldsymbol{R}^d \otimes \boldsymbol{R}^d$ is symmetric and satisfies (1.1) for all $(t, x) \in \mathscr{Y}$. Then one can construct a time-homogeneous family $\{(Y, \hat{P}_y);\ y \in \mathscr{Y}\}$ by putting $Y_t((s, \omega)) = (s+t, X_{s+t}(\omega))$ and $\hat{P}_{y=(s,x)}(\Gamma) = P_{s,x}(\{\omega\colon (s, \omega) \in \Gamma\})$ (for a construction of such a family by using Dirichlet forms theory see [11] and [17]). Let $Y_t^\varphi = \varphi(Y_t) - \varphi(Y_0)$, $t \geqslant 0$. Since the decomposition (4.22) is unique, without ambiguity we can set

$$\hat{M}_t^{[\varphi]}(\hat{\omega} = (s, \omega)) = M_{s,s+t}^{[\varphi],T}(\omega), \qquad \hat{A}_t^{[\varphi]}(\hat{\omega}) = (Y_t^\varphi - \hat{M}_t^{[\varphi]})(\hat{\omega}) \qquad \text{for } s, t \geqslant 0$$

such that $s+t \leqslant T$, where $M^{[\varphi],T}$ is the MAF of Theorem 4.2 on $[0, T]$. For $B := Y^\varphi, \hat{M}^{[\varphi]}, \hat{A}^{[\varphi]}$ we then have $\hat{P}_y(B_{s+t} = B_s + B_t \circ \hat{\theta}_s) = 1$ for any $s, t \geqslant 0$, $y \in \mathscr{Y}$, where $\hat{\theta}_t((s, \omega)) = (s+t, \omega)$. Moreover, if we denote by $\hat{E}_{s,m}$ the expectation sign with respect to $\hat{P}_{(s,m)}(\cdot) = \int_{\boldsymbol{R}^d} \hat{P}_{(s,x)}(\cdot)\, m(dx)$, then

$$e_\alpha(X^\varphi) = \int_0^T \int_0^T \mathbf{1}_{[0,T]}(s+t)\, e^{-\alpha t}\, \hat{E}_{s,m}(Y_t^\varphi)^2\, ds\, dt, \qquad \bar{e}_t(M^{[\varphi]}) = \int_0^{T-t} \hat{E}_{s,m}(\hat{M}_t^{[\varphi]})^2\, ds.$$

Therefore (1.4) may be viewed as a strict version (i.e. for every starting point $y \in \mathscr{Y}$) of the decomposition of the functional Y^f with $f(t, x) = \mathbf{1}_{[0,T]}(t)\, \varphi(x)$ obtained in [12] and [19]. Note, however, that the results obtained in [12] and [19] lead to a decomposition of Y^f for general time-inhomogeneous diffusions (Y, P_y) and general time-dependent f.

THEOREM 4.4. *Under the assumptions of Theorem* 4.2, *if moreover* (1.3) *is satisfied*, $\bar{e}(A^{[\varphi]}) = 0$.

Proof. We first prove that

$$\bar{e}(X^\varphi) = (a\nabla\varphi, \nabla\varphi)_{2;0,T}. \tag{4.23}$$

To this end, we write

$$\bar{e}_t(X^\varphi) = 2\int_0^{T-t} (\varphi, \varphi - P^{s,s+t}\varphi)_2\, ds = \int_0^{T-t} ds \int_s^{s+t} \big(a(u,\cdot)\nabla P^{s,u}\varphi, \nabla\varphi\big)_2\, du$$

$$= \int_0^t du \int_0^u f(s,u)\, ds + \int_t^T du \int_{u-t}^u f(s,u)\, ds + \int_0^{T-t} ds \int_s^{s+t} \big(a(u,\cdot)\nabla\varphi, \nabla\varphi\big)_2\, du$$

$$= I_1(t) + I_2(t) + I_3(t),$$

where $f(s,u) = \big(a(u,\cdot)\nabla(P^{s,u}\varphi - \varphi), \nabla\varphi\big)_2$. It is easily seen that $\lim_{t\downarrow 0} t^{-1} I_1(t) = 0$. By (2.9), $\{P^{s_n,u}\varphi - \varphi\}$ is bounded in W_2^1 and $P^{s_n,u}\varphi - \varphi \to 0$ in L_2 for any $\{s_n\} \subset (u-t, u)$ such that $s_n \uparrow u$. Therefore the function $(u-t, u] \ni s \mapsto f(s,u)$ is continuous, and hence $t^{-1}\int_{u-t}^u f(s,u)\, ds \to f(u,u) = 0$ as $t \downarrow 0$. From this and the dominated convergence theorem we conclude that $\lim_{t\downarrow 0} t^{-1} I_2(t) = 0$. Finally, since $I_3(t) = \bar{e}_t(M^{[\varphi]})$, by (4.9) we have $\lim_{t\downarrow 0} t^{-1} I_3(t) = (a\nabla\varphi, \nabla\varphi)_{2;0,T}$, and (4.23) is proved. Following the proof of Theorem 4.1 in [15] we now show that

$$\bar{e}(A^{[\varphi]}) = \bar{e}(M^{[\varphi]}) - \bar{e}(X^\varphi) \tag{4.24}$$

for any $\varphi \in C_0^2$. To this end, we first use Theorem 3.1 and (4.22) to deduce that for every $(s,x) \in [0,T) \times \mathbf{R}^d$, $t \in (0, T-s]$ there exists the integral $\int_s^{s+t} A_{s,u}^{[\varphi]}\, dA_{s,u}^{[\varphi]}$ as the limit in $P_{s,x}$ of Riemann sums, and moreover

$$(A_{s,s+t}^{[\varphi]})^2 = 2\int_s^{s+t} (A_{s,s+t}^{[\varphi]} - A_{s,u}^{[\varphi]})\, dA_{s,u}^{[\varphi]}$$

$$= 2\int_s^{s+t} \big(\varphi(X_{s+t}) - \varphi(X_u)\big)\, dA_{s,u}^{[\varphi]} - 2\int_s^{s+t} (M_{s,s+t}^{[\varphi]} - M_{s,u}^{[\varphi]})\, dA_{s,u}^{[\varphi]}$$

$$= \varphi(X_{s+t})(-M_{s,s+t}^{[\varphi]} + \tilde{N}_{s,s+t}^\varphi) - \int_s^{s+t} \varphi(X_u)\, d(-M_{s,u}^{[\varphi]} + \tilde{N}_{s,u}^\varphi)$$

$$- \int_s^{s+t} X_{u,s+t}^\varphi\, dV_{s,u}^\varphi - 2\int_s^{s+t} A_{s,u}^{[\varphi]}\, dM_{s,u}^{[\varphi]},$$

where

$$\int_s^{s+t} \varphi(X_u)\, d\tilde{N}_{s,u}^\varphi = -\int_{T-t}^T \varphi(\bar{X}_u)\, dN_{s,u}^\varphi - \langle X_{s,\cdot}^\varphi, \tilde{N}_{s,\cdot}^\varphi\rangle_{s+t}$$

and

$$\langle X_{s,\cdot}^\varphi, \tilde{N}_{s,\cdot}^\varphi\rangle_{s+t} = \langle M_{s,\cdot}^{[\varphi]}, \tilde{N}_{s,\cdot}^\varphi\rangle_{s+t} = \langle M_{s,\cdot}^{[\varphi]}\rangle_{s+t}.$$

Clearly,

$$E_{s,m}\big(\varphi(X_{s+t}) M_{s,s+t}^{[\varphi]}\big) = E_{s,m}(X_{s,s+t}^\varphi M_{s,s+t}^{[\varphi]})$$

and, since $\varphi(X_{s+t})$ is $\sigma(X_u, u\in[s+t, T]) = \mathscr{F}^s_{T-t}$-measurable, we obtain

$$E_{s,x}\big(\varphi(X_{s+t})\tilde{N}^{\varphi}_{s,s+t}\big) = E_{s,x}\{\varphi(X_{s+t})\big(E_{s,x}(N^{\varphi}_{T-t}-N^{\varphi}_{T})\,|\,\mathscr{F}^s_{T-t}\big)\} = 0.$$

We have also

$$E_{s,m}\int_s^{s+t} X^{\varphi}_{u,s+t}\,dV^{\varphi}_{s,u} = 0,$$

because, by the Markov property,

$$E_{s,x}\int_s^{s+t} p^{-1}(a\nabla p, \nabla\varphi)(s, x, u, X_u)X^{\varphi}_{u,s+t}\,du$$

$$= \int_s^{s+t} E_{s,x}\{p^{-1}(a\nabla p, \nabla\varphi)(s, x, u, X_u)E_{u,X_u}X^{\varphi}_{u,s+t}\}\,du$$

$$= \int_s^{s+t}\int_{R^d}\sum_{i,j=1}^{d} a^{ij}(u, y)D_i p(s, x, u, y)\,D_j\varphi(y)(P^{u,s+t}\varphi-\varphi)(y)\,du\,dy.$$

By the above calculations, we get

$$E_{s,m}(A^{[\varphi]}_{s,s+t})^2 = E_{s,m}(-X^{\varphi}_{s,s+t}M^{[\varphi]}_{s,s+t}+\langle M^{[\varphi]}_{s,\cdot}\rangle_{s+t})$$

$$= E_{s,m}\big(-A^{[\varphi]}_{s,s+t}M^{[\varphi]}_{s,s+t}+(M^{[\varphi]}_{s,s+t})^2+\langle M^{[\varphi]}_{s,\cdot}\rangle_{s+t}\big) = -E_{s,m}(A^{[\varphi]}_{s,s+t}M^{[\varphi]}_{s,s+t}),$$

and hence

$$\bar{e}_t(X^{\varphi}) = \bar{e}_t(M^{[\varphi]})+\bar{e}_t(A^{[\varphi]})+2\int_0^{T-t} E_{s,m}(A^{[\varphi]}_{s,s+t}M^{[\varphi]}_{s,s+t})\,ds = \bar{e}_t(M^{[\varphi]})-\bar{e}_t(A^{[\varphi]})$$

for every $t\in[0, T]$. Obviously, this gives (4.24) and proves the theorem for $\varphi\in C^2_0$ when combined with (4.23). To prove the general case, we approximate φ in W^1_2 by a sequence $\{f_k\}\subset C^2_0$ and use the estimate

$$\bar{e}(A^{[\varphi]}) \leqslant 3e(X^{f_k-\varphi})+3e(M^{[f_k-\varphi]})+3e(A^{[f_k]}) = 6\big(a\nabla(f_k-\varphi), \nabla(f_k-\varphi)\big)_{2;0,T},$$

which follows from uniqueness of the decomposition (1.4), the fact that $\bar{e}(A^{[f_k]}) = 0$ and (4.9), (4.23). ∎

Acknowledgement. I would like to thank the referee for valuable comments on the paper, particularly for pointing out an error in its first version and for calling my attention to papers [17] and [19].

REFERENCES

[1] D. G. Aronson, *Non-negative solutions of linear parabolic equations*, Ann. Scuola Norm. Sup. Pisa 22 (1968), pp. 607–693.

[2] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York 1968.

[3] H. Föllmer, *Dirichlet processes*, in: *Stochastic Integrals*, D. Williams (Ed.), Lecture Notes in Math. 851, Springer, Berlin 1981, pp. 476–478.

[4] M. Fukushima, *On a strict decomposition of additive functionals for symmetric diffusion processes*, Proc. Japan Acad. Ser. A Math. Sci. 70 (1994), pp. 277–281.

[5] M. Fukushima, *On a decomposition of additive functionals in the strict sense for a symmetric Markov process*, in: *Dirichlet Forms and Stochastic Processes*, Z. Ma, M. Röckner and J. Yan (Eds.), Walter de Gruyter, Berlin–New York 1995, pp. 155–169.

[6] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, Walter de Gruyter, Berlin–New York 1994.

[7] I. I. Gihman and A. V. Skorokhod, *The Theory of Stochastic Processes. II*, Nauka, Moscow 1973; English translation: Springer, New York–Heidelberg 1975.

[8] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and quasi-linear equations of parabolic type*, Nauka, Moscow 1967; English translation: Transl. Math. Monographs, Vol. 23, Amer. Math. Soc., Providence, R. I., 1968.

[9] T. J. Lyons and L. Stoica, *The limits of stochastic integrals of differential forms*, Ann. Probab. 27 (1999), pp. 1–49.

[10] T. J. Lyons and W. A. Zheng, *On conditional diffusion processes*, Proc. Roy. Soc. Edinburgh 115A (1990), pp. 243–255.

[11] Y. Oshima, *On a construction of Markov processes associated with time dependent Dirichlet spaces*, Forum Math. 4 (1992), pp. 395–415.

[12] Y. Oshima, *Some properties of Markov processes associated with time dependent Dirichlet forms*, Osaka J. Math. 29 (1992), pp. 103–127.

[13] A. Rozkosz, *Weak convergence of diffusions corresponding to divergence form operators*, Stochastics Stochastics Rep. 57 (1996), pp. 129–157.

[14] A. Rozkosz, *Stochastic representation of diffusions corresponding to divergence form operators*, Stochastic Process. Appl. 63 (1996), pp. 11–33.

[15] A. Rozkosz, *On Dirichlet processes associated with second order divergence form operators*, Potential Anal. 14 (2001), pp. 123–149.

[16] A. Rozkosz and L. Słomiński, *Extended convergence of Dirichlet processes*, Stochastics Stochastics Rep. 65 (1998), pp. 79–109.

[17] W. Stannat, *The Theory of Generalized Dirichlet Forms and Its Applications in Analysis and Stochastics*, Mem. Amer. Math. Soc. 142, No. 678 (1999), viii+101 pp.

[18] D. W. Stroock and S. R. S. Varadhan, *Multidimensional Diffusion Processes*, Springer, New York 1979.

[19] G. Trutnau, *Stochastic calculus of generalized Dirichlet forms and applications to stochastic differential equations in infinite dimensions*, Osaka J. Math. 37 (2000), pp. 315–343.

[20] V. V. Zhikov, S. M. Kozlov and O. A. Oleinik, *G-convergence of parabolic operators*, Uspekhi Mat. Nauk 36 (1981), pp. 11–58; English translation: Russian Math. Surveys 36 (1981), pp. 9–60.

Faculty of Mathematics and Computer Science
Nicholas Copernicus University
ul. Chopina 12/18
87-100 Toruń, Poland
E-mail: rozkosz@mat.uni.torun.pl

Received on 18.4.2001;
revised version on 20.5.2002