Abstract. We consider a time-inhomogeneous Markov family 
\((X, P_{s,x})\) corresponding to a symmetric uniformly elliptic divergence form operator. We show that for any \(\varphi\) in the Sobolev space \(W^1_p \cap W^2_1\) with \(p = 2\) if \(d = 1\) and \(p > d\) if \(d > 1\) the additive functional 
\(X^\varphi = \{\varphi(X_s) - \varphi(X_t); 0 \leq s < t\}\) admits a unique strict decomposition into a martingale additive functional of finite energy and a continuous additive functional of zero energy. Moreover, we give a stochastic representation of the zero energy part and show that in case the diffusion coefficient is regular in time the functional \(X^\varphi\) is a Dirichlet process for each starting point \((s, x)\). The paper contains also rectifications of incorrectly presented or incorrectly proved statements of our earlier paper [14].

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1. INTRODUCTION

Consider a Markov family \(\{(X, P_{s,x}); (s, x) \in [0, T] \times \mathbb{R}^d\}\) corresponding to the divergence form operator

\[ L_t = \frac{1}{2} \sum_{i,j=1}^d D_j (a^{ij}(t, x) D_i), \]

where \(a: [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \otimes d}\) is a measurable, symmetric matrix-valued function satisfying the condition

\[ \lambda |\xi|^2 \leq \sum_{i,j=1}^d a^{ij}(t, x) \xi_i \xi_j \leq A |\xi|^2, \quad (t, x) \in [0, T] \times \mathbb{R}^d, \quad \xi \in \mathbb{R}^d, \]

for some \(0 < \lambda \leq A\) (for construction of \((X, P_{s,x})\) see, e.g., [11], [13], [17]). In [14] it is announced that for any starting point \((s, x) \in [0, T] \times \mathbb{R}^d\) and any continuous \(\varphi\) in the Sobolev space \(W^1_p\) with \(p > 2\) \(\forall d\) the composite process

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\(q(X)\) is a Dirichlet process on \([s, T]\) in the sense of Föllmer [3], that is, \(X_{s,t}^q = \varphi(X_t) - \varphi(X_s), \ t \in [s, T]\), admits a unique decomposition of the form

\[
X_{s,t}^q = M_{s,t}^{x,q} + A_{s,t}^{x,q}, \quad t \in [s, T], \quad P_{s,x}\text{-a.s.,}
\]

where \(M_{s,t}^{x,q}\) is a continuous \(P_{s,x}\)-martingale on \([s, T]\) and \(A_{s,t}^{x,q}\) is a continuous adapted process of \(P_{s,x}\)-zero quadratic variation on \([s, T]\). Unfortunately, the proof of the last statement in [14] (see [14], Theorem 2.1) as well as Lemma 1.2 in [14], which is used in the proof of Theorem 2.1 in [14], is incorrect (see Remark 2.6 in [16] and the remark following Lemma 2.2 of the present paper for more details). One of our purposes is to show that the quadratic variation of \(A_{s,t}^{x,q}\) vanishes if we assume additionally that

\[
\max_{1 \leq i,j \leq d} \int \text{ess sup}_{x \in \mathbb{R}^d} |D_i \alpha^{ij}(t, x)| \, dt = K_1 < \infty.
\]

We show also that under (1.1) for any continuous \(\varphi \in W^1_p \cap W^1_2\) with \(p\) as above \(X^q\) admits a unique decomposition

\[
X_{s,t}^q = M_{s,t}^{\varphi} + A_{s,t}^{\varphi}, \quad t \in [s, T], \quad P_{s,x}\text{-a.s.,}
\]

into a continuous martingale additive functional \(M^{\varphi}\) (in the strict sense) of finite energy and a continuous additive functional \(A^{\varphi}\) of zero energy, where the energy of an additive functional \(A = \{A_{s,t}, 0 \leq s < t \leq T\}\) of \((X, P_{s,x})\) is defined by

\[
e(A) = \lim_{\alpha \to \infty} \alpha^2 e_\alpha(A), \quad e_\alpha(A) = \int_0^T \int_{[0,T]} (s + t) e^{-\alpha t} E_{s,m} A_{s,t}^2 ds dt, \quad \alpha > 0,
\]

whenever the limit exists. Here \(m\) is the Lebesgue measure on \(\mathbb{R}^d\) and \(E_{s,m}\) is the expectation sign with respect to \(P_{s,m} (\cdot) = \int_{\mathbb{R}^d} P_{s,x} (\cdot) m(dx)\). Our decomposition may be viewed as a strict version of a decomposition of \(X^q\) obtained in [12] and [19]. If moreover (1.3) is satisfied, (1.4) also holds if we replace the above definition of energy by the following:

\[
\bar{e}(A) = \lim_{\alpha \to 0} \frac{1}{\alpha} \bar{e}_\alpha(A), \quad \bar{e}_\alpha(A) = \int_0^T E_{s,m} A_{s,t}^2 ds, \quad t \in (0, T].
\]

Thus we generalize results of [15] on strict decomposition of time-homogeneous diffusions corresponding to divergence form operators (general time-homogeneous diffusions are considered in [4]–[6]).

Finally, notice that similarly to [14]–[16] our methods of proofs allow us to obtain the Lyons–Zheng decomposition of \(A_{s,t}^{x,q}\) and \(A_{s,t}^{\varphi}\) for each starting point \((s, x)\) (results concerning time-homogeneous diffusions can be found in [9], [10], [15], [16]).

We will use the following notation.
For a process $Y$ on $[s, T]$ we write

$$Y_t = Y_{T+s-t}, \quad \tilde{Y}_t = Y_t - Y_T, \quad t \in [s, T].$$

Let $\Omega = C([0, T]; \mathbb{R}^d)$ be the space of continuous trajectories from $[0, T]$ into $\mathbb{R}^d$, and $X$ be the canonical process on $\Omega$. Let us put

$$\mathcal{F}_t^Z = \sigma(X_u, u \in [s, t]), \quad \mathcal{F}_t^X = \sigma(X_u, u \in [s, t]), \quad t \in [s, T].$$

By $E_{s,x}$ we denote the expectation sign with respect to $P_{s,x}$. Let $\mathcal{M}(P_{s,x})$ be the space of continuous square-integrable $P_{s,x}$-martingales on $[s, T]$ vanishing at $s$ equipped with the usual norm $(E_{s,x} \langle M \rangle_T)^{1/2}$.

$D_i = \partial/\partial x^i$ is the partial derivative in the distribution sense, $V = (D_1, \ldots, D_d)$. $C_0^\infty$ is the set of all smooth functions in $\mathbb{R}^d$ having compact support. $L_p$ (respectively, $L_p(s, T)$) is the classical Banach space consisting of measurable functions on $\mathbb{R}^d$ (respectively, $(s, T) \times \mathbb{R}^d$) that are $p$-integrable. $W^p_1$ is the Banach space consisting of all elements $u$ of $L_p$ having derivatives $D_i u$ from $L_p$. Let $W^{0,1}_2(s, t)$ be the Banach space consisting of all elements $u$ of $L_2(s, t)$ having derivatives $D_i u$ from $L_2(s, T)$, and $W^{1,1}_2(s, T)$ be the Banach space consisting of all elements $u$ of $L_2(s, T)$ having derivatives $D_i u$ and time derivatives (in the distribution sense) $D_t$ from $L_2(s, T)$. By $\| \cdot \|_p$ we denote the norm in $L_p$. By $(\cdot, \cdot)$ we mean the usual scalar product in $\mathbb{R}^d$, $(\cdot, \cdot)_2$ the scalar product in $L_2$, and $(\cdot, \cdot)_{2s, T}$ the scalar product in $L_2(s, T)$.

By Sobolev's imbedding theorem, if $p > d$, then every $\varphi \in W^1_p$ has a continuous representative. Therefore we will always assume that $\varphi$ denotes the continuous representative of a given element of $W^1_p$ with $p > d$.

2. PRELIMINARY RESULTS

It is known (see [1] and [8]) that under (1.1) for any $s \in [0, T)$ and $\varphi \in L_2$ there is a unique weak solution $P_{s,t} \varphi(\cdot) \in W^{0,1}_2(s, T)$ to the Cauchy problem

$$(D_i - L_i) u = 0 \text{ in } (s, T) \times \mathbb{R}^d, \quad u(s, \cdot) = \varphi.$$ 

In particular, if $\varphi \in W^1_2$, then

$$\int_0^t \left( a(u, \cdot) \nabla P^{s,t} \varphi, \nabla \varphi \right)_2 du$$

for all $t \in (s, T)$. Furthermore, in [1] it is shown that there exists a weak fundamental solution $p(s, x, t, y), 0 \leq s < t, x, y \in \mathbb{R}^d$, for $L_1$, and that for any $\varphi \in L_2$ we have the representation

$$P^{s,t} \varphi(x) = \int_{\mathbb{R}^d} \varphi(y) p(s, x, t, y) dy, \quad 0 \leq s < t \leq T, \ x \in \mathbb{R}^d.$$
The family \( \{P^s_t, 0 \leq s < t \leq T \} \) forms a Markov semigroup of contractions on \( L_2 \), which gives rise to our Markov family \((X, P_{s,a})\) (see, e.g., [13]). In particular, \( p \) is the transition density of \((X, P_{s,a})\).

**Theorem 2.1.** Assume (1.1) holds. Then:

(i) there is a constant \( K_2 > 0 \) depending only on \( \lambda, \Lambda, d, T \) such that

\[
p(s, x, t, y) \leq K_2 (t-s)^{-d/2} \exp \left( -\frac{|y-x|^2}{K_2(t-s)} \right)
\]

for all \( 0 \leq s < t \leq T \) and \( x, y \in \mathbb{R}^d \);

(ii) for each \((s, x) \in [0, T) \times \mathbb{R}^d \), \( p(s, x, \cdot, \cdot) \in L_1 (s, T; W^1_d) \) with any \( q \) whose Hölder conjugate is greater than \( 2 + d \);

(iii) if \( d = 1 \), then for any \( 0 \leq s < T, \alpha > 0 \) we have

\[
\int_0^T \int_{\mathbb{R}} (t-s)^{\alpha} \frac{(p')^2}{p}(s, x, t, y) \, dt \, dy < \infty
\]

\((p'(s, x, t, y))\) denotes the derivative of \( y \mapsto p(s, x, t, y) \) in the sense of distributions.

**Proof.** For (i) and (ii) see Theorems 5 and 7 in [1]. To prove (iii) we can proceed as in the proof of Theorem 5.1 in [15]. \(\square\)

Let \( W_2^{-1} \) denote the dual space of \( W_2^1 \) when one uses the inner product in \( L_2 \) to define the duality relation and let \( W_2^{0,-1}(s, T) = L_2(0, T; W_2^{-1}) \). The space \( W_2^{0,-1}(s, T) \) is dual to \( W_2^{1,1}(s, T) \) and the value \( \langle v, u \rangle_{s,T} \) of a functional \( v \in W_2^{0,-1}(s, T) \) at \( u \in W_2^{1,1}(s, T) \) is defined by \( \langle v, u \rangle_{s,T} = \int_s^T (v(t), u(t)) \, dt \), where \( \langle \cdot, \cdot \rangle \) denotes the dualization between \( W_2^{1,1} \) and \( W_2^{-1} \).

Let \( \mathcal{W}(s, T) \) denote the space of functions \( u \in W_2^{0,-1}(s, T) \) having derivatives \( D_t u \) from \( W_2^{0,-1}(s, T) \) equipped with the norm

\[
||u||_{\mathcal{W}(s, T)}^2 = ||D_t u||_{W_2^{-1}(s,T)}^2 + ||u||_{W^{0,1}(s,T)}^2
\]

It is known (see, e.g., [20]) that there is a continuous embedding of \( \mathcal{W}(s, T) \) in the space \( C([s, T]; L_2) \) of continuous functions on \([s, T]\) with \( \lambda \) values in \( L_2 \) equipped with the supremum norm. Therefore without ambiguity we may define the subspace \( \mathcal{W}_T(s, T) = \{u \in \mathcal{W}(s, T): u(T, \cdot) = 0\} \) of \( \mathcal{W}(s, T) \).

For any \( \alpha \geq 0, f \in L_2(s, T) \), there exists a unique weak solution \( R_{\alpha} f \in \mathcal{W}_T(s, T) \) of the problem

\[
(x-D_t-L_d) u = f \text{ in } (s, T) \times \mathbb{R}^d, \quad u(T, \cdot) = 0,
\]

that is

\[
\alpha (R_{\alpha} f, g)_{2;\alpha,T} - \langle D_t R_{\alpha} f, g \rangle_{s,T} + \frac{1}{2} (aVR_{\alpha} f, Vg)_{2;\alpha,T} = (f, g)_{2;\alpha,T}
\]
for any \( g \in W^{0,1}_2(s, T) \) (see [13], [17], [20]). Moreover, \( \{R_{\alpha}\}_{\alpha > 0} \) is a strongly continuous resolvent on \( L_2(s, T) \) having the representation

\[
R_{\alpha} f(u, x) = E_{u,x} \int_0^T (u + t) e^{-\alpha t} f(u + t, X_{u+t}) dt, \quad (u, x) \in [s, T] \times \mathbb{R}^d,
\]

for any bounded \( f \in L_2(s, T) \) (see, e.g., [13]).

**Lemma 2.2.** Assume (1.1) holds. If \( f \in W^{0,1}_2(s, T) \), then

\[
\lim_{\alpha \to \infty} \|\alpha R_{\alpha} f - f\|_{W^{0,1}_2(s, T)} = 0.
\]

**Proof.** This is a particular case of a more general result proved in [17], Proposition 3.7. \( \Box \)

**Remark.** In Lemma 1.2 of the author's paper [14], (2.4) is stated for resolvents corresponding to operators of the form

\[
L_\alpha = \frac{1}{2} \sum_{i,j=1}^d D_j(a^{ij}(t, x) D_i) + \sum_{i=1}^d b^i(t, x) D_i.
\]

Unfortunately, the proof of Lemma 1.2 in [14] is incorrect (even in the case \( b^i = 0, i = 1, \ldots, d \)). The statement of Lemma 1.2 in [14] is, however, correct. It follows from the cited-above Proposition 3.7 of [17].

From Lemma 2.2 it may be concluded that

\[
\limsup_{\alpha \to \infty} \mathcal{A}(\varphi, \varphi - \alpha R_{\alpha} \varphi)_{2, s, T} \leq (2^{-1} \Lambda T + 1) \|\varphi\|_{W^{1,2}_2}^2
\]

for any \( \varphi \in W^{1,2}_2 \). Indeed, putting \( f, g = \varphi \) in (2.3) we obtain

\[
(\alpha R_{\alpha} \varphi, \varphi)_{2, s, T} - \langle D_\alpha \varphi, \varphi \rangle_{2, s, T} + 2^{-1}(a VR_{\alpha} \varphi, \varphi)_{2, s, T} = (\varphi, \varphi)_{2, s, T}.
\]

Hence

\[
(\varphi - \alpha R_{\alpha} \varphi, \varphi)_{2, s, T} = 2^{-1}(a VR_{\alpha} \varphi, \varphi)_{2, s, T} = (R_{\alpha} \varphi(s, \cdot), \varphi)_{2, s, T},
\]

because

\[
\langle D_\alpha \varphi, \varphi \rangle_{s, T} = (R_{\alpha} \varphi(T, \cdot), \varphi)_{2, s, T} = (R_{\alpha} \varphi(s, \cdot), \varphi)_{2, s, T} = -(R_{\alpha} \varphi(s, \cdot), \varphi)_{2, s, T}
\]

(see, e.g., [20]). On the other hand,

\[
|\langle \alpha R_{\alpha} \varphi(s, \cdot), \varphi \rangle_{2, s, T}| = \int_0^T a e^{-\alpha t} (P^{a,t} \varphi, \varphi)_{2, s, T}dt \\ \leq \int_0^T a e^{-\alpha t} \|P^{a,t} \varphi\|_{2, s, T}dt \leq \|\varphi\|_2^2
\]
and, by (2.4),
\[
\lim_{n \to \infty} \sup_{s \to \infty} \|aV(\alpha R_s \varphi), V\varphi)_{2; s, T}\|
\leq \lim_{n \to \infty} \sup_{s \to \infty} A \|V(\alpha R_s \varphi - \varphi), V\varphi)_{2; s, T}\| + A(V\varphi, V\varphi)_{2; s, T} \leq AT \|\varphi\|_{W^2_s},
\]
which gives (2.5) when combined with (2.6).

**Theorem 2.3.** If (1.1) and (1.3) are satisfied, then for any \( \varphi \in W^1_s, f \in L^2(s, T) \) the Cauchy problem
\[
(D_t - L_t)u = f \quad \text{in} \quad (s, T) \times \mathbb{R}^d, \quad u(s, \cdot) = \varphi,
\]
has a unique solution from \( W^1_{2; s, T} \). Moreover, there is \( K_3 > 0 \) depending only on \( \lambda, A, d \) and \( K_1 \) such that for every \( t \in (s, T) \)
\[
\|V\varphi(t, \cdot)\|_2^2 + \|D_t\varphi\|_{2; s, T}^2 \leq K_3 \left( \|V\varphi\|_2^2 + \|f\|_{2; s, T}^2 \right).
\]

**Proof.** See Theorem 6.1 and the inequality (6.6) in Chapter III in [8].

From (2.8) it follows that for any \( \varphi \in W^1_s \)
\[
\|V P^{s, t} \varphi\|_2^2 + (t - s)^{-1} \|\varphi - P^{s, t} \varphi\|_{2; s, T}^2 \leq K_3 \|V\varphi\|_2^2
\]
for all \( 0 \leq s < t \leq T \). Moreover, since \( R_s(t, x) = e^{a t} u(T + s - t, x) \), where \( u \) is a solution to (2.7) with \( f \) replaced by \( g(t, x) = e^{-a(T + s - t)} f(T + s - t, x) \), the inequality (2.8) shows that
\[
\|D_t R_s \varphi\|_{2; s, T} \leq a e^{a T} \|R_s \varphi\|_{2; s, T} + K_3^{1/2} a^{-1} (e^{a(T - s)} - 1) \|V\varphi\|_2 < \infty
\]
for every \( \alpha > 0 \).

### 3. Dirichlet Processes

Let \((s, x) \in [0, T) \times \mathbb{R}^d \) and let \( \{H_t\}_{t \in [s, T]} \) be a filtration. We will say that \( X^\varphi_s \) is a continuous \((\{H_t\}, P_{s,x})\)-Dirichlet process on \([s, T] \) if it is \( \{H_t\}\)-adapted and admits a decomposition of the form (1.2), where \( M^\varphi_t \) is a continuous \((\{H_t\}, P_{s,x})\)-local martingale on \([s, T] \) such that \( M_{s,t}^\varphi = 0 \) and \( A^\varphi_{s,t} \) is an \( \{H_t\}\)-adapted process of zero quadratic variation on \([s, T] \), that is, \( A^\varphi_{s,t} = 0 \) and
\[
Q^n_{s,T}(A^\varphi_{s,t}) \equiv \sum_{t_i \in I_n} |A^\varphi_{s,t_{i+1}} - A^\varphi_{s,t_i}|^2 \to 0 \quad \text{in} \quad P_{s,x} \quad \text{as} \quad n \to \infty
\]
for every sequence \( \{I_n = \{t_0, t_1, \ldots, t_{i(n)}\}\} \) of partitions of \([s, T] \) such that \( s = t_0 < t_1 < \ldots < t_{i(n)} = T \) and \( ||I_n|| = \max_{1 \leq i \leq i(n)} |t_i - t_{i-1}| \to 0 \) as \( n \to \infty \).

In what follows \( V_p(s, x, t, y) \) stands for the gradient of \( y \mapsto p(s, x, t, y) \).
THEOREM 3.1. Assume (1.1) and (1.3) hold and let $\varphi \in W^1_p$ with $p = 2$ if $d = 1$ and $p > d$ if $d > 1$. Then for every $(s, x) \in [0, T) \times \mathbb{R}^d$ there exists a unique triple $(M^x_{s,t}, N^x_{s,t}, V^x_{s,t})$ such that:

(i) $M^x_{s,t}$ is a continuous $\mathcal{F}^x_{s,t}$-square-integrable martingale on $[s, T]$, $N^x_{s,t}$ is a continuous $\mathcal{F}^x_{s,t}$-square-integrable martingale on $[s, T]$, $V^x_{s,t}$ is a continuous $\mathcal{F}^x_{s,t}$-adapted process of $P_{s,t}$-integrable variation on $[s, T]$;

(ii) $X^x_{s,t}$ is an $\mathcal{F}^x_{s,t}$-Dirichlet process on $[s, T]$ admitting the decomposition (1.2) with

$$A^x_{s,t} = \frac{1}{2}(-M^x_{s,t} + N^x_{s,t} - V^x_{s,t}), \quad t \in [s, T], \quad P_{s,t}\text{-a.s.}$$

Moreover, for every $t \in [s, T]$,

$$\langle M^x_{s,t} \rangle_t = \int_s^t (a V \varphi, \varphi)(u, X_u) du, \quad \langle N^x_{s,t} \rangle_t = \int_s^t (a V \varphi, \varphi)(u, X_u) du$$

with $\bar{a} = \{\bar{a}^{ij}\}$, $\bar{a}^{ij}(t, x) = a^{ij}(T + s - t, x)$, and

$$V^x_{s,t} = \int_s^t p^{-1} (a V \varphi, \varphi)(s, x, u, X_u) du.$$ 

Proof. Since the proof of uniqueness is standard, we will prove only existence of the triple. For this purpose, we first assume additionally that $\varphi \in C^2$ and for $k \in N$ we set $\varphi_k = k R_k \varphi$, $\psi_k = \varphi_k - \varphi$. By [14], Lemma 1.3, for each $k \in N$,

$$X^x_{s,t} = \varphi_k(t, X_t) - \varphi_k(s, X_s), \quad t \in [s, T],$$

is an $\mathcal{F}^x_{s,t}$-semimartingale admitting the decomposition

$$X^x_{s,t} = M^x_{s,t} + \int_s^t k \psi_k(u, X_u) du = M^x_{s,t} + A^x_{s,t}, \quad t \in [s, T],$$

where $M^x_{s,t}$ is an $\mathcal{F}^x_{s,t}$-square-integrable martingale with the quadratic variation process

$$\langle M^x_{s,t} \rangle_t = \int_s^t (a V \varphi_k, \varphi_k)(u, X_u) du.$$ 

For fixed $\delta \in (0, T - s)$ let $M^x_{s,t}$ denote the martingale $M^x_{s,t} \Delta_{s,t} - M^x_{s,\delta} \Delta_{s,\delta}$, $t \in [s, T]$. By (3.5) and uniqueness of the decomposition (3.4),

$$E_{s,x} \langle M^x_{s,t} - M^x_{s,\delta} \rangle_T = E_{s,x} \langle M^x_{s,\delta} - \varphi \rangle_T$$

for all $k, l \in N$, which, when combined with (2.4), shows that $\{M^x_{s,t} \}_{t \in [s, T]}$ is a Cauchy sequence in $\mathcal{M}(P_{s,t})$. Since for each $k \in N$ all trajectories of $M^x_{s,t}$ are
continuous, from Lemma 4.3.3 of [18] it follows that there is an \( \{\mathcal{F}_t\} \)-adapted martingale \( M^{x,\delta} \in \mathcal{M}(P_{s,x}) \) whose all trajectories are right-continuous such that \( \{M^{\delta,\psi}\} \) converges to \( M^{x,\delta} \) in \( \mathcal{M}(P_{s,x}) \). Furthermore, for any \( 0 < \delta < \sigma < T-s \),

\[
(3.7) \quad E_{s,x} \langle M^{x,\delta} - M^{x,\sigma} \rangle_T \leq \lim_{k \to \infty} E_{s,x} \langle M^{\delta,\psi_k} - M^{\sigma,\psi_k} \rangle_T
\]

\[
= \lim_{k \to \infty} E_{s,x} \int_{s+\delta}^{s+\sigma} (aV\varphi_k, V\varphi_k)(u, X_u) \, du \leq \Lambda \int_{s+\delta}^{s+\sigma} |V\varphi(y)|^2 p(s, u, y) \, ds \, dy.
\]

For \( d = 1 \) the right-hand side of (3.7) is bounded by

\[
(3.8) \quad \Lambda K_2 \|V\varphi\|_2^2 \int_{s+\delta}^{s+\sigma} (u-s)^{-1/2} \, du = 2\Lambda K_2 (\sigma^{1/2} - \delta^{1/2}) \|V\varphi\|_2^2,
\]

whereas for \( d > 1 \), by (2.2) and Hölder's inequality, it is bounded by

\[
(3.9) \quad C_1 (\sigma^{(p-d)/p} - \delta^{(p-d)/p}) \|V\varphi\|_2^2,
\]

where \( C_1 \) depends only on \( p, d, K_2 \). Hence, applying once again Lemma 4.3.3 of [18], we conclude that in both cases there is an \( \{\mathcal{F}_t\} \)-adapted martingale \( M^{x,\delta}_{s,t} \) such that all its trajectories are right-continuous and \( \{M^{x,\delta}\} \) converges to \( M^{x,\delta}_{s,t} \) in \( \mathcal{M}(P_{s,x}) \) as \( \delta \downarrow 0 \). Define

\[
A_{s,t}^{x,\sigma} = X_{s,t}^{\psi} - M^{x,\sigma}_{s,t}, \quad t \in [s, T].
\]

The representation (3.1) can be proved by the same method as in the proof of Theorem 2.1 (ii) in [14], and therefore we omit it. Note, however, that from the construction of \( N_{s,t}^{x,\sigma} \) in [14] and from [18], Lemma 4.3.3, it follows that we can assume that all its trajectories are right-continuous.

Our next goal is to show that \( A_{s,t}^{x,\sigma} \) is a zero quadratic variation process, that is

\[
(3.10) \quad Q^n_{s,T}(A_{s,t}^{x,\sigma}) \to 0 \text{ in } P_{s,x} \text{ as } n \to \infty.
\]

In order to get (3.10) we will show that for fixed \( \delta \in (0, T-s) \)

\[
(3.11) \quad Q^n_{s+\delta,T}(A_{s+\delta,t}^{x,\sigma}) \to 0 \text{ in } P_{s,x} \text{ as } n \to \infty
\]

and that

\[
(3.12) \quad \lim_{\delta \downarrow 0} \limsup_{n \to \infty} P_{s,x} (Q^n_{s,s+\delta}(A_{s,t}^{x,\sigma}) > \varepsilon) = 0
\]

for every \( \varepsilon > 0 \). Obviously,

\[
Q^n_{s+\delta,T}(A_{s,t}^{x,\sigma}) = Q^n_{s+\delta,T}(X_{s,t}^{\psi} + M_{s,t}^{\delta,\psi} - M_{s,t}^{x,\sigma} + A_{s,t}^{\psi})
\]

\[
\leq 3Q^n_{s+\delta,T}(X_{s,t}^{\psi}) + 3Q^n_{s+\delta,T}(M_{s,t}^{\delta,\psi} - M_{s,t}^{x,\sigma}) + 3Q^n_{s+\delta,T}(A_{s,t}^{\psi}),
\]

where

\[
\text{for every } \varepsilon > 0.\]
where $M^{\delta, \phi}_x$ is defined as $M^{\delta, \phi}_t$. Since $A^{\phi}_t$ is a process of finite variation,
\begin{equation}
Q^a_{s+\delta, T}(A^{\phi}_t) \to 0 \text{ in } P_{s,x} \text{ as } n \to \infty.
\end{equation}
Observe now that for any $t \in [s+\delta, T]$
\begin{equation}
M^{x, \phi}_{s,t} - M^{x, \phi}_{s,s+\delta} = \lim_{\sigma \downarrow 0} (M^{x, \phi}_{s,t} - M^{x, \phi}_{s,s+\delta})
= \lim_{\sigma \downarrow 0} \lim_{k \to \infty} \{(M^{\phi}_k - M^{\phi}_{k+\delta}) - (M^{\phi}_k - M^{\phi}_{k+\delta})\}
= \lim_{\sigma \downarrow 0} (M^{x, \delta}_t - M^{x, \delta}_s)
\end{equation}
(all limits are taken in $\mathcal{M}(P_{s,x})$). Hence $M^{\delta, \phi}_x = M^{x, \delta}$ and, consequently,
\begin{equation}
l \lim_{k \to \infty} \limsup_{n \to \infty} E_{s,x} Q^a_{s+\delta, T}(M^{\delta, \phi}_x - M^{\delta, \phi}_s) \leq \lim_{k \to \infty} E_{s,x} (M^{\delta, \phi}_x - M^{x, \delta})_T = 0.
\end{equation}
By the Markov property and (2.2),
\begin{equation}
E_{s,x} Q^a_{s+\delta, T}(X^{\psi}_t) = \sum_{s+\delta \leq t_i < T} E_{s,x} E_{s+\delta, X_{s+\delta}} |\psi_k(t_{i+1}, X_{t_{i+1}}) - \psi_k(t_i, X_{t_i})|^2
\leq K_2 \delta^{-d/2} \sum_{s+\delta \leq t_i < T} E_{s+\delta, m} |\psi_k(t_{i+1}, X_{t_{i+1}}) - \psi_k(t_i, X_{t_i})|^2.
\end{equation}
Using once again the Markov property and performing elementary computations gives
\begin{align*}
E_{s+\delta, m} |\psi_k(t_{i+1}, X_{t_{i+1}}) - \psi_k(t_i, X_{t_i})|^2
&= E_{s+\delta, m} \{p^{t_i,t_{i+1}, \cdot} \psi_k^2(t_{i+1}, X_{t_i}) - 2p^{t_i,t_{i+1}, \cdot} \psi_k(t_{i+1}, X_{t_i}) \psi_k(t_i, X_{t_i}) + \psi_k^2(t_i, X_{t_i})\}
&= (\psi_k^2(t_{i+1}, \cdot) + \psi_k^2(t_i, \cdot) - 2\psi_k(t_{i+1}, \cdot) p^{t_i,t_{i+1}, \cdot} \psi_k(t_i, \cdot), 1)2
&= 2(\psi_k(t_{i+1}, \cdot) - \psi_k(t_i, \cdot) - P^{t_i,t_{i+1}, \cdot} \psi_k(t_i, \cdot))2
&\quad + 2(\psi_k(t_{i+1}, \cdot) - \psi_k(t_i, \cdot) - P^{t_i,t_{i+1}, \cdot} \psi_k(t_i, \cdot))2
&\quad + (\psi_k(t_{i+1}, \cdot) - \psi_k(t_i, \cdot) - \psi_k(t_{i+1}, \cdot) - \psi_k(t_i, \cdot))2
&= I_1^2 + I_2^2 + I_3^2
\end{align*}
for $i$ such that $t_i \in [s+\delta, T]$. By (2.1),
\begin{equation}
I_1 = \int_{t_i}^{t_{i+1}} (a(u, \cdot) P^{u,t_i, \cdot} \psi_k(t_{i+1}, \cdot), V \psi_k(t_{i+1}, \cdot)) du.
\end{equation}
Hence, by (2.9),
\begin{equation}
|I_1| \leq C_2 (t_{i+1} - t_i) \|V \psi_k(t_{i+1}, \cdot)\|^2 \text{ with } C_2 = d\Lambda K_3^{1/2},
\end{equation}
which implies

\[ \limsup_{n \to \infty} \sum_{s + \delta < t_1 < T} |I^2_1| \leq C_2 \|V\|_{\frac{\delta}{2};\sigma + \delta, T}. \]

Clearly,

\[ |I^2_1| \leq I^3_1 + \|\psi_k(t_{i+1}, \cdot) - \psi(t_{i+1}, \cdot)\|^2. \]

By (2.9),

\[ \|\psi_k(t_{i+1}, \cdot) - \psi(t_{i+1}, \cdot)\|^2 \leq K_3 (t_{i+1} - t_i) \|\psi(t_{i+1}, \cdot)\|^2, \]

and

\[ \sum_{s + \delta < t_1 < T} I^3_1 \sum_{s + \delta < t_1 < T} (t_{i+1} - t_i) \|D_t \varphi_k\|^2 \leq \|\psi(t_{i+1}, \cdot)\|^2. \]

Therefore

\[ \limsup_{n \to \infty} \sum_{s + \delta < t_1 < T} |I^2_1 + I^3_1| \leq K_3 \|\psi(t_{i+1}, \cdot)\|^2. \]

From the above, (2.2) and (2.4) we conclude that

\[ \limsup_{n \to \infty} \limsup_{k \to \infty} E_{s,x}^n Q^n_{\sigma + \delta, T} (X^{\psi_k}) \leq K_2 (C_2 + K_3) \delta^{-1/2} \lim_{k \to \infty} \|\psi(t_{i+1}, \cdot)\|^2_{\frac{\delta}{2};\sigma + \delta, T} = 0, \]

which together with (3.13), (3.14) gives (3.11). Finally, by (3.1), for every \( \varepsilon > 0 \),

\[ P_{s,x} (Q^n_{s,s + \delta} (A_{s,s}^{x,\psi}) > \varepsilon) < P_{s,x} (Q^n_{s,s + \delta} (M^{x,\psi}_{s,s}) > 2\varepsilon) + P_{s,x} (Q^n_{s,s + \delta} (\tilde{N}^{x,\psi}_{s,s}) > 2\varepsilon), \]

from which (3.12) easily follows. This completes the proof for \( \psi \in C^2_0 \).

To prove the general case, we choose a sequence \( \{f_k\} \subset C^2_0 \) such that \( f_k \to \psi \) in \( W^1_{2} \) and uniformly in compact sets in \( \mathbb{R}^d \). Then for each \( k \in \mathbb{N} \) the process \( X^{f_k}_{s,t} \) is an \( (\mathcal{F}^r_t), P_{s,x} \)-square-integrable martingale having the representation

\[ X^{f_k}_{s,t} = M^{x,f_k}_{s,t} + A_{s,t}^{x,f_k}, \quad A_{s,t}^{x,f_k} = \frac{1}{2} (-M^{x,f_k}_{s,t} + N^{x,f_k}_{s,t} + V^{x,f_k}_{s,t}), \quad t \in [s, T]. \]

Since the decomposition (3.15) is unique, we have

\[ M^{x,f_k}_{s,t} - M^{x,f_l}_{s,t} = M^{x,f_k}_{s,t} - f_l, \quad N^{x,f_k}_{s,t} - N^{x,f_l}_{s,t} = N^{x,f_k}_{s,t} - f_l, \quad t \in [s, T], \]

for \( k, l \in \mathbb{N} \). Hence, by (3.2) and (2.2),

\[ E_{s,x} (M^{x,f_k}_{s,t} - M^{x,f_l}_{s,t}) + E_{s,x} (N^{x,f_k}_{s,t} - N^{x,f_l}_{s,t}) \leq C_3 \|\psi(f_k - f_l)\|^2_{\frac{\delta}{2}} \]

for some \( C_3 \) depending only on \( \lambda, A, d, p \) and \( T \). Moreover, from the first part of the proof it follows that we can assume that for each \( k \in \mathbb{N} \) all trajectories of \( M^{x,f_k}_{s,t} \) and \( N^{x,f_k}_{s,t} \) are right-continuous. Therefore there exist an \( (\mathcal{F}^r_t), P_{s,x} \)-square-integrable martingale \( M^{x,\psi}_{s,t} \) and an \( (\mathcal{F}^r_t), P_{s,x} \)-square-
-integrable martingale $N_{s,}^\theta$ such that

$$E_{s,} \langle M_{s,}^{X,\varphi} - M_{s,}^{X,\varphi} \rangle_T \to 0, \quad E_{s,} \langle N_{s,}^{X,\varphi} - N_{s,}^{X,\varphi} \rangle_T \to 0.$$  

Moreover, if we define $V_{s,}^{X,\varphi}$ by (3.3), then

$$\lim_{k \to \infty} E_{s,} \sup_{s \leq t \leq T} |V_{s,}^{X,\varphi} - V_{s,}^{X,\varphi}|$$  

$$\leq \lim_{k \to \infty} \int_s^T \{(aVp(s, x, u, \cdot), V(f_k - \varphi))_2 \, du = 0,$$

the last equality being a consequence of the assertion (i) of Theorem 2.1 in the case $d > 1$, and the assertions (i) and (iii) for $d = 1$ because, by the Schwarz inequality,

$$\left( \int_s^T \int_{s \leq t \leq T} \left| p'(s, x, t, y)(f_k - \varphi)'(y) \right| dy \right)^2$$  

$$\leq \int_s^T \left( (t-s)^{1/4} \frac{(p')^2}{p} (s, x, t, y) \, dt ight)^2 \int_s^T \left( (t-s)^{-1/4} |(f_k - \varphi)'(y)|^2 p(s, x, t, y) \right) dy.$$  

From (3.15)–(3.19) and the continuous mapping theorem we obtain (1.2) and (3.1)–(3.3), so what is left is to show (3.10). Since $A_{s,}^{X,\varphi}$ has the representation (3.1) and $V_{s,}^{X,\varphi}$ is a process of finite variation on $[s, T]$, (3.10) will be proved once we prove that $\langle M_{s,}^{X,\varphi} - N_{s,}^{X,\varphi} \rangle_T = 0$. It is easily seen, however, that the last assertion follows from (3.18) and the fact that $\langle M_{s,}^{X,\varphi} - N_{s,}^{X,\varphi} \rangle_T = \langle A_{s,}^{X,\varphi} \rangle_T = 0$. 

We will see in Section 4 that $X^\varphi$ admits a decomposition of the form (1.2), (3.1) with $M^{X,\varphi}, N^{X,\varphi}, V^{X,\varphi}$ satisfying the assertion (i) even if we drop the assumption (1.3). We do not know, however, whether in this case $A^{X,\varphi}$ is a zero quadratic variation process, and, consequently, we neither know that $X^\varphi$ is a Dirichlet process nor that the triple is unique.

**4. ADDITIVE FUNCTIONALS**

In this section we give conditions on $a$ and $\varphi$ under which for each $T > 0$ the additive functional (AF) $X^\varphi = \{X_{s,t}^\varphi, 0 \leq s \leq t \leq T\}$ admits a unique decomposition into a martingale additive functional (MAF) of finite energy and a continuous additive functional (CAF) of zero energy. We begin with basic definitions.

Put $\mathcal{P} = \{P_{s,t}: \mu \text{ is a probability measure on } \mathcal{B}\}$, where $\mathcal{B}$ is a Borel $\sigma$-field of $\mathbb{R}^d$, $P_{s,t}(\cdot) = \int_{\mathbb{R}^d} P_{s,x}(\cdot) \mu(dx)$ and define $\mathcal{F}$ as the completion of $\mathcal{F}_s^T$ with respect to the family $\mathcal{P}$, and then $\mathcal{G}_s^T$ as the completion of $\mathcal{F}_s^T$ in $\mathcal{F}$ with respect to $\mathcal{P}$ (see [7], Section 1.3, for more details).

We say that the family of random variables $A = \{A_{s,t}, 0 \leq s \leq t \leq T\}$ is an AF of $(X, P_{s,t})$ on $[0, T]$ if $A_{s,t}$ is $\mathcal{G}_s^T$-measurable for every $0 \leq s \leq t \leq T$ and $P_{s,t}(A_{s,t} = A_{s,u} + A_{u,t}, s \leq u \leq t \leq T) = 1$.
for every \((s, x) \in [0, T] \times \mathbb{R}^d\). If, in addition,
\[
P_{s,x}(\{\omega \in \Omega: [s, T] \ni t \mapsto A_{s,t}(\omega) \text{ is continuous}\}) = 1
\]
for every \((s, x) \in [0, T] \times \mathbb{R}^d\), then \(A\) is called a CAF.

For an AF \(A\) of \((X, P_{s,x})\) we define \(e(A)\) and \(\bar{e}(A)\) as in Section 1. If \(e(A) < \infty\) \((e(A) = 0)\), we call \(A\) an AF of finite energy \((zero\ energy)\).

Changing the variable \(t \mapsto \alpha t\) and using Fubini's theorem yields
\[
\alpha^2 e_\alpha(A) = \alpha \int_0^\infty I_{[0,\alpha T]}(\tau) e^{-\tau} \bar{e}_{\alpha/\tau}(A) d\tau.
\]

Therefore, if \((for\ instance)\) \(\sup_{0 < t \leq T} (1/t) \bar{e}_t(A) < \infty\), then applying the Lebesgue dominated convergence theorem shows that if \(\bar{e}(A)\) exists, then \(e(A)\) exists and \(e(A) = \bar{e}(A)\). In particular, if
\[
A_{s,t} = \int_s^t f(u, X_u) du, \quad 0 \leq s < t \leq T,
\]
for some \(f \in L^2(0, T)\), then using Schwarz's inequality and Fubini's theorem we obtain
\[
\bar{e}_t(A) \leq \int_0^{T-t} t \left( \int_s^{s+t} \|f(u, \cdot)\|_2^2 du \right) ds \leq t^2 \|f\|_{L^2(0,T)}^2.
\]

Hence \(e(A) = \bar{e}(A) = 0\).

We say that \(M = \{M_{s,t}, 0 \leq s < t \leq T\}\) is a continuous MAF of \((X, P_{s,x})\) if it is a CAF such that \(E_{s,x} M_{s,t}^2 < \infty\), \(E_{s,x} M_{s,t} = 0\) for every \(0 \leq s < t \leq T\), \(x \in \mathbb{R}^d\).

Let us remark that if \(M\) is an MAF of \((X, P_{s,x})\), then \(M_{s,t}\) is a \((\mathcal{G}_t^s, P_{s,x})\)-martingale on \([s, T]\), because, by the Markov property,
\[
E_{s,x} (M_{s,t} | \mathcal{G}_u^s) = E_{s,x} (M_{s,u} + M_{u,t} | \mathcal{G}_u^s) = M_{s,u} + E_{u,X_u} M_{u,t} = M_{s,u}
\]
for all \(s < u \leq t \leq T\) and \(x \in \mathbb{R}^d\). By the Markov property we also have
\[
E_{s,u} M_{s,u+u+v}^2 = E_{s,u} M_{s,u}^2 + E_{s,u+v} M_{s,u+v}^2 + E_{s-u} M_{s-u}^2 + E_{s+u,v} M_{s+u,v}^2
\]
for \(s, u, v \geq 0\) such that \(s+u+v \leq T\), and hence, by an elementary computation, \(\bar{e}_{u+v}(M) \leq \bar{e}_u(M) + \bar{e}_v(M)\) for all \(u, v > 0\) such that \(u+v \leq T\). Consequently, by the well-known properties of subadditive functions, \(\bar{e}(M)\) is well defined and
\[
\bar{e}(M) = \sup_{0 < t \leq T} t^{-1} \bar{e}_t(M).
\]

Notice also that in the case where \(\langle M_{s,t} \rangle_t = A_{s,t}\) and \(A\) is given by (4.1) with some non-negative integrable \(f\), we have
\[
\bar{e}(M) = \lim_{t \downarrow 0} \frac{1}{t} \int_0^{T-t} \int_s^{s+t} \|f(u, \cdot)\|_2 du \, ds = \|f\|_{L^1(0,T)}.
\]
A decomposition of any CAF of \((X, P_{s,x})\) into a continuous MAF of finite energy and a CAF of zero energy is unique in the sense that we have

**Proposition 4.1.** Suppose \(1M_1\) and \(2M_2\) are continuous MAFs of finite energy and \(1A_1\) and \(2A_2\) are CAFs of zero energy such that 
\[
(1M_{s,t} + 1A_{s,t}) = (2M_{s,t} + 2A_{s,t}),
\]
for all \(t \in [s, T],\) \(P_{s,m}-a.s.,\) and for a.e. \(s \in [0, T].\) Then
\[
P_{s,x}(1M_{s,t} = 2M_{s,t}, 1A_{s,t} = 2A_{s,t}, t \in [s, T]) = 1
\]
for every \((s, x) \in [0, T) \times \mathbb{R}^d.\)

**Proof.** Fix \((s, x) \in [0, T) \times \mathbb{R}^d\) and \(t \in (0, T - s).\) Since \(M = 1M - 2M\) is a continuous MAF of zero energy, it follows from (4.2) that \(\tilde{e}_t(M) = 0.\) Therefore there is \(\{\delta_n\} \subset (s, T - t)\) such that \(\delta_n \downarrow s\) and \(E_{\delta_n, n}M_{\delta_n, \delta_n + t}^2 = 0, n \in \mathbb{N}.\) For each \(n \in \mathbb{N}\) we have

\[
E_{s,x}M_{\delta_n, \delta_n + t}^2 \leq 2E_{s,x}M_{\delta_n}^2 + 2E_{s,x}M_{\delta_n, \delta_n + t}^2 = 2E_{s,x}M_{\delta_n}^2,
\]
because

\[
E_{s,x}M_{\delta_n, \delta_n + t}^2 = E_{s,x}(E_{s,x}(M_{\delta_n, \delta_n + t}^2 | \mathcal{F}_{\delta_n})) = E_{s,x}(E_{\delta_n, \delta_n}M_{\delta_n, \delta_n + t}^2)
\]

\[
\leq \int_{\mathbb{R}^d} E_{\delta_n, y}(M_{\delta_n}^2) p(s, x, \delta_n, y) dy = 0.
\]

Since \(M\) is continuous, letting \(n \uparrow \infty\) in (4.4) we get \(E_{s,x}M_{\delta_n, \delta_n + t}^2 = 0,\) and hence the desired conclusion.

**Theorem 4.2.** Assume (1.1) holds and let \(\varphi \in W^1_p \cap W^1_2\) with \(p = 2\) if \(d = 1\) and \(p > d\) if \(d > 1.\) Then there exists a unique continuous MAF \(M^{[\varphi]} = \{M^{[\varphi]}_{s,t}, 0 \leq s \leq t \leq T\}\) of finite energy and a unique CAF \(A^{[\varphi]} = \{A^{[\varphi]}_{s,t}, 0 \leq s \leq t \leq T\}\) of zero energy such that (1.4) is satisfied for every \((s, x) \in [0, T) \times \mathbb{R}^d.\) Moreover,

\[
\langle M^{[\varphi]}_{s,t} \rangle_t = \int_{s}^{t} (a \nabla \varphi, \nabla \varphi)(u, X_u) du, \quad t \in [s, T], \quad P_{s,x}-a.s.,
\]

for every \((s, x) \in [0, T) \times \mathbb{R}^d.\)

**Proof.** First we show that there is no loss of generality in assuming that \(\varphi \in C_0^2.\) To see this, choose a sequence \(\{f_k\} \subset C_0^2\) such that \(f_k \to \varphi\) uniformly on compact sets in \(\mathbb{R}^d,\) in \(W^1_p\) and \(W^1_2.\) If the theorem were true for functions of the class \(C_0^2,\) for each \(k \in \mathbb{N}\) we would have

\[
X_{s,t}^{[f_k]} = M_{s,t}^{[f_k]} + A_{s,t}^{[f_k]}, \quad t \in [s, T], \quad P_{s,x}-a.s.,
\]
for every \((s, x) \in [0, T) \times \mathbb{R}^d,\) where \(M^{[f_k]}\) is a continuous MAF of finite energy, \(A^{[f_k]}\) is a CAF of zero energy and (1.4), (4.5) are satisfied with \(\varphi\) replaced by \(f_k.\) By Proposition 4.1 the decomposition (4.6) is unique. Consequently,
$M^{[fk]} - M^{[f]} = M^{[fk-f]}$ for $k, l \in N$. On the other hand, by (2.2), there is $C_3$ depending only on $\lambda, A, d, p, T$ such that

$$E_{s,x} \langle M^{[fk-f]} \rangle_T \leq C_3 \| V (f_k - f_l) \|_2.$$ 

Therefore, by Doob's inequality, there is a subsequence $\{k_n\} \subset N$ such that

$$P_{s,x} \left( \sup_{s \leq t \leq T} |M^{[fk_{n+1}]} - M^{[fk_n]}| > 2^{-n} \right) \leq 2^{-n}$$

for every $(s, x) \in [0, T] \times \mathbb{R}^d$. Hence, if we set

$$\Omega_s = \{ \omega \in \Omega : \{M^{[fk_n]}\}_{n \in N} \text{ converges uniformly in } t \text{ on } [s, T] \},$$

then $P_{s,x}(\Omega_s) = 1$ for every $x \in \mathbb{R}^d$, by the Borel–Cantelli lemma. For given $s \in [0, T)$ let

(4.7) $M^{[x]}_{s,t}(\omega) = \lim_{n \to \infty} M^{[fk_n]}_{s,t}(\omega), \quad \omega \in \Omega_s, \quad M^{[x]}_{s,t}(\omega) = 0, \quad \omega \notin \Omega_s, \quad t \in [s, T],$

and

(4.8) $A^{[x]}_{s,t}(\omega) = X^{[x]}_{s,t}(\omega) - M^{[x]}_{s,t}(\omega), \quad \omega \in \Omega, \quad t \in [s, T].$

From (4.7), (4.8) it follows that $M^{[x]}$ and $A^{[x]}$ are continuous AFs satisfying (1.4) and (4.22). Moreover, by (4.5) and (4.3), $M^{[x]}$ is an MAF with

(4.9) $e(M^{[x]}) = (aV \varphi, V \varphi)_{2;0,T} < \infty.$

Since $e(A^{[x]}) = 0$,

(4.10) $e(A^{[x]}) = e(X^x - M^{[x]}) = 3e(X^{f_k-\varphi}) + 3e(M^{[fk]} - M^{[f]}) + 3e(A^{[fk]})$

$$= 3e(X^{f_k-\varphi}) + 3e(M^{[fk]} - M^{[f]}).$$

Put $g_k = f_k - \varphi, \quad k \in N$. We have

$$\alpha^2 e_x(X^{g_k}) = 2\alpha^2 \int_0^T ds \int_0^{T-s} e^{-st}(g_k - P^{s,s+t}g_k)_2 dt \leq 2\alpha (g_k, g_k - \alpha R_2 g_k)_{2;0,T},$$

and so, by (2.5),

(4.11) $\lim_{k \to \infty} e(X^{f_k-\varphi}) = 0.$

For $k, l \in N$ we also have

$$e_t(M^{[fk]} - M^{[f_l]}) = \int_0^{T-t} ds \int_s^{s+t} (a(u, \cdot)V (f_k - f_l), V (f_k - f_l))_2 du$$

$$\leq A (T-t) t \| V (f_k - f_l) \|_2^2.$$
Therefore

\[(4.12) \quad \lim_{k \to \infty} \epsilon(M^{f_{k\lambda}} - M^{f_{k\phi}}) \leq \Lambda T \lim_{k \to \infty} \liminf \|f_k - f_i\|_2^2 = 0\]

by Fatou's lemma. Combining (4.10)-(4.12) gives \(e(A^{(\phi)}) = 0\).

By what has already been proved, it suffices to prove the theorem for \(\phi \in C_0^2\). For this purpose, we define \(\varphi_k, \psi_k, M^{\varphi_k}, M^{\psi_k}, M^{x, \delta}\) as in the proof of Theorem 3.1. In view of (2.4), (3.6) and Doob's inequality we can choose a subsequence \(\{k_n\} \subset N\) such that for each \(\delta \in (0, T - s)\)

\[P_{s,r}(\sup_{s < t < T} |M^{x, \phi_{k_n+1}} - M^{x, \phi_{k_n}}| > 2^{-n}) < 2^{-n} \delta^{-d/2}\]

for all \(n \in N\). Let us set

\[\Gamma^1_s = \{\omega \in \Omega: \{M^{x, \phi_{k_n}}\}_{n \in N} \text{ converges uniformly in } t \text{ on } [s, T]\}\]

By the Borel–Cantelli lemma, \(P_{s,r}(\Gamma^1_s) = 1\) for every \(x \in \mathbb{R}^d\). Next, for given \(t \in [0, T - s)\) let

\[M^{[\delta]}(\omega) = \lim_{n \to \infty} M^{x, \phi_{k_n}}(\omega), \quad t \in [s, T], \quad \text{for } \omega \in \Gamma^1_s\]

and

\[M^{[\delta]}(\omega) = 0 \quad \text{for } \omega \notin \Gamma^1_s.\]

Since \(P_{s,r}(M^{x, \delta} = M^{[\delta]}, t \in [s, T]) = 1\), it follows from (3.7)-(3.9) that there is a subsequence \(\{\delta_m\} \subset (0, T - s)\) such that \(\delta_m \downarrow 0\) and \(P_{s,r}(\Gamma^2_s) = 1\) for every \(x \in \mathbb{R}^d\), where

\[\Gamma^2_s = \{\omega \in \Omega: \{M^{[\delta_m]}\}_{m \in N} \text{ converges uniformly in } t \text{ on } [s, T]\}\]

Define now

\[M^{[\delta]}(\omega) = \lim_{m \to \infty} M^{[\delta_m]}(\omega), \quad t \in [s, T], \quad \text{for } \omega \in \Gamma^1_s \cap \Gamma^2_s,\]

\[M^{[\delta]}(\omega) = 0 \quad \text{for } \omega \notin \Gamma^1_s \cap \Gamma^2_s\]

and \(A^{[\phi]}\) by (4.8). Then obviously (1.4) and (4.5) are satisfied and \(M^{[\phi]}\) is continuous. Moreover, since \(M^{x, \phi_k}\) is \(\mathcal{F}^t\)-measurable for every \(k \in N\), \(M^{[\delta]}\) is \(\mathcal{G}^t\)-measurable for every \(\delta \in (0, T - s)\), and hence \(M^{[\phi]}\) is also \(\mathcal{G}^t\)-measurable. Finally, for fixed \(0 \leq s < u < t \leq T, \text{ and } n \in N\) and every \(m \in N\) such that \(\delta_m \leq \min\{u - s, t - u\}\) we have

\[(4.13) \quad M^{x, \phi_{k_n}} = M^{x, \phi_{k_n}} + M^{\psi_{k_n}} + M^{\psi_{k_n}} + M^{\phi_{k_n}} + \delta_m.\]
By (2.2) and (2.4),
\[ \lim_{m \to \infty} \limsup_{n \to \infty} E_{s,n} \langle M^{\varphi_n} \rangle u + \delta_m \leq \lim_{m \to \infty} \limsup_{n \to \infty} \Lambda K_2 (u - s)^{-d/2} \| V \varphi_n \|_{p;u,u+\delta_m}^2 \leq \lim_{m \to \infty} \Lambda K_2 (u - s)^{-d/2} \delta_m \| \varphi \|_2^2 = 0. \]

Therefore letting \( n \uparrow \infty \) and then \( m \uparrow \infty \) in (4.13) we get \( M^{[\varphi]} = M^{[\varphi]}_E + M^{[\varphi]}_P \) \( P \)-a.s. From this and (4.5) and (4.9) we see that \( M^{[\varphi]} \) is an MAF of finite energy. By (4.8), \( A^{[\varphi]} \) is a CAF, so we only need to show that \( e (A^{[\varphi]}) = 0. \)

To see this, we first prove that
\[ \limsup_{k \to \infty} \sup_{a \to -\infty} \alpha (\psi_k, \psi_k - \alpha R_a \psi_k)_{2;0,T} = 0. \]

By (2.3) we have
\[ \alpha (R_a \psi_k, \psi_k)_{2;0,T} - \langle D_t R_a \psi_k, \psi_k \rangle_{0,T} + 2^{-1} (aV R_a \psi_k, V \psi_k)_{2;0,T} = (\psi_k, \psi_k)_{2;0,T}. \]

Since \( \psi_k, R_a \psi_k \in \mathcal{W}^\gamma (0, T) \), we have
\[ \langle D_t R_a \psi_k, \psi_k \rangle_{0,T} = -\langle D_t \psi_k, R_a \psi_k \rangle_{0,T} - (R_a \psi_k (0, \cdot), \psi_k (0, \cdot))_2 \]

(see, e.g., [20]) and, consequently,
\[ \alpha (\psi_k, \psi_k - \alpha R_a \psi_k)_{2;0,T} = 2^{-1} (aV (\alpha R_a \psi_k), V \psi_k)_{2;0,T} + \langle D_t \psi_k, \alpha R_a \psi_k \rangle_{0,T} + (\alpha R_a \psi_k (0, \cdot), \psi_k (0, \cdot))_2. \]

By (2.4),
\[ \limsup_{k \to \infty} \sup_{a \to -\infty} \| (aV (\alpha R_a \psi_k), V \psi_k)_{2;0,T} \|
\]
\[ \leq A \limsup_{k \to \infty} \| (V (\alpha R_a \psi_k - \psi_k), V \psi_k)_{2;0,T} \| + (V \psi_k, V \psi_k)_{2;0,T} = 0. \]

Since \( 2 < D_t \psi_k, \psi_k \rangle_{0,T} = - (\psi_k (0, \cdot), \psi_k (0, \cdot))_2 \), using (2.4) we also get
\[ \lim_{a \to -\infty} \langle D_t \psi_k, \alpha R_a \psi_k \rangle_{0,T} = \langle D_t \psi_k, \psi_k \rangle_{0,T} \leq 0, \]

because \( |\langle D_t \psi_k, \alpha R_a \psi_k - \psi_k \rangle_{0,T}| \leq \| D_t \psi_k \|_{w_2^{-1};0,T} \| \alpha R_a \psi_k - \psi_k \|_{w_2^{-1};(0,T)} \). Observe now that
\[ |\langle \alpha R_a \psi_k (0, \cdot), \psi_k (0, \cdot) \rangle_2 | = \left[ \int_0^T \alpha e^{-at} (p^{0,t} \psi_k (t, \cdot), \psi_k (0, \cdot))_2 dt \right] \]
\[ \leq \int_0^T \alpha e^{-at} \| \psi_k (t, \cdot) \|_2 \cdot \| \psi_k (0, \cdot) \|_2 dt = \int_0^T e^{-t} \| \psi_k (t/\alpha, \cdot) \|_2 \cdot \| \psi_k (0, \cdot) \|_2 dt. \]
Since \( \psi_k \in \mathcal{W}(0, T) \) and \( \mathcal{W} \subset C([0, T]; L_2) \), \( \|\psi_k(t/\alpha, \cdot)\|_2 \to \|\psi_k(0, \cdot)\|_2 \) as \( \alpha \to \infty \). At the same time, for any \( s \in [0, T) \),

\[
\|\psi_k(s, \cdot)\|_2^2 = \int_0^T \int_{\mathbb{R}^d} ke^{-kt} E_{s, x} \varphi(X_{s+t}) \, dt \, dx \leq \int_0^T ke^{-kt} E_{s, m} \|\varphi(X_{s+t})\|^2 \leq \|\varphi\|_2^2,
\]

so applying Fatou's lemma gives

\[
\limsup_{\alpha \to \infty} \int_0^T e^{-t} \|\psi_k(0, \cdot)\|_2^2 \, dt = \|\psi_k(0, \cdot)\|_2^2.
\]

Moreover,

\[
\int_0^T \int_{\mathbb{R}^d} ke^{-kt} E_{0, s} \varphi(X_s) - \varphi(x) \, \varphi(x) \, dx \leq \int_0^T ke^{-kt} E_{0, m} \varphi(X_s) - \varphi(X_0)^2 \, dt
\]

\[
= 2 \int_0^T ke^{-kt} (\varphi - P^{0, t/\alpha} \varphi, \varphi) \, dt = 2 \int_0^T e^{-t} (\varphi - P^{0, t/\alpha} \varphi, \varphi) \, dt.
\]

Clearly, \( (\varphi - P^{0, t/\alpha} \varphi, \varphi)_2 \leq 2 \|\varphi\|_2^2 \) and, by (2.1), \( (\varphi - P^{0, t/\alpha} \varphi, \varphi)_2 \to 0 \) as \( k \to \infty \), so applying once again Fatou's lemma we obtain

\[
\limsup_{k \to \infty} \|\psi_k(0, \cdot)\|_2^2 \leq \limsup_{k \to \infty} (4 \int_0^T e^{-t} (\varphi - P^{0, t/\alpha} \varphi, \varphi) \, dt + 2e^{-2kt} \|\varphi\|_2^2) = 0.
\]

Combining (4.15)-(4.19) gives (4.14). We have

\[
\int_0^{T-s} e^{-at} E_{s, m} (X_t)^2 \, dt
\]

\[
\leq \int_0^{T-s} e^{-a(t+s)} (\psi_k(s, t, \cdot), \psi_k(s, t, \cdot))_2 \, dt + \frac{T-s}{2} e^{-aT} (\psi_k(s, \cdot), \psi_k(s, \cdot))_2 \, dt
\]

\[
-2 \int_0^{T-s} e^{-at} (\psi_k(s, \cdot), P^{s, t} \psi_k(s, t, \cdot))_2 \, dt
\]

\[
= \int_0^{T-s} e^{-at} (\psi_k(s, t, \cdot), \psi_k(s, t, \cdot))_2 \, dt - \int_0^{T-s} e^{-at} (\psi_k(s, \cdot), \psi_k(s, \cdot))_2 \, dt
\]

\[
+ \frac{2}{\alpha} (1-e^{-aT}) (\psi_k(s, \cdot), \psi_k(s, \cdot))_2 - 2 (\psi_k, R_{\alpha} \psi_k(s, \cdot))_2,
\]

and hence, by (4.14),

\[
\limsup_{k \to \infty} \limsup_{\alpha \to \infty} \alpha^2 e_{\alpha}(X^{\psi_k}) \leq 2 \limsup_{k \to \infty} \limsup_{\alpha \to \infty} (\psi_k, \psi_k - R_{\alpha} \psi_k)_{2; 0,T} = 0.
\]
Furthermore, for any $t \in (T-s)$ and $\delta \in (0, t)$ we have
\[
E_{s,m}(M_{s,s+t}^{\varphi_k} - M_{s,s+\delta}^{\varphi_k})^2 \leq 3E_{s,m}(M_{s,s+t}^{\varphi_k} - M_{s,s+\delta}^{\varphi_k} - M_{s,s+t}^{[\varphi]} + M_{s,s+\delta}^{[\varphi]})^2
+ 3E_{s,m}(M_{s,s+\delta}^{\varphi_k} - M_{s,s+\delta}^{[\varphi]})^2.
\]
By Fatou's lemma,
\[
\lim_{t \to \infty} \int_0^{T-t} E_{s,m}(M_{s,s+t}^{\varphi_k} - M_{s,s+\delta}^{\varphi_k})^2 \, ds \leq \lim_{t \to \infty} \int_0^{T-t} E_{s,m}(M_{s,s+t}^{\varphi_k} - M_{s,s+\delta}^{[\varphi]})^2 \, ds.
\]
On the other hand, by (3.5) and Fubini's theorem,
\[
\int_0^{T-t} E_{s,m}(M_{s,s+t}^{\varphi_k})^2 \, ds = \int_0^{T-t} \int_0^{s+t} (aV(\varphi_k - \varphi_1)(u, \cdot), V(\varphi_k - \varphi_1)(u, \cdot))_2 \, du
\leq t \int_0^T (aV(\varphi_k - \varphi_1)(u, \cdot), V(\varphi_k - \varphi_1)(u, \cdot))_2 \, du \leq t\Lambda \|V(\varphi_k - \varphi_1)\|_{L^2;0,T}^2
\]
and
\[
\int_0^{T-t} E_{s,m}(M_{s,s+\delta}^{\varphi_k})^2 \, ds \leq \delta\Lambda (\|V\psi_k\|_{L^2;0,T}^2 + T\|V\|_2^2).
\]
From the above and (2.4) we get
\[
\bar{e}(M^{\varphi_k} - M^{[\varphi]}) \leq 3\Lambda \lim_{t \to \infty} \|V(\varphi_k - \varphi_1)\|_{L^2;0,T}^2 = 3\Lambda \|V\psi_k\|_{L^2;0,T}^2,
\]
and hence
\[
\lim_{k \to \infty} \bar{e}(M^{\varphi_k} - M^{[\varphi]}) = 0,
\]
again by (2.4). Finally, since $\psi_k \in L^2(0, T)$, we have $e(A^{[\varphi]}) = 0$. Thus (4.10) holds with $f_k$ replaced by $\varphi_k$ and, consequently, (4.20) and (4.21) give $e(A^{[\varphi]}) = 0$. □

The zero energy part of the decomposition (1.4) admits a unique representation of the form (3.1). More precisely, we have

**Proposition 4.3.** Under the assumptions of Theorem 4.2, for every $(s, x) \in [0, T) \times \mathbb{R}^d$ there exists a unique triple $(M_{s}^{x,\varphi}, N_{s}^{x,\varphi}, V_{s}^{x,\varphi})$ satisfying the assertion (i) of Theorem 3.1 and such that
\[
M_{s}^{[\varphi]} = M_{s}^{x,\varphi}, \quad A_{s}^{[\varphi]} = \frac{1}{2}(-M_{s}^{x,\varphi} + \tilde{N}_{s}^{x,\varphi} - 2V_{s}^{x,\varphi}), \quad t \in [s, T], \quad P_{s,x} -a.s.
\]
Moreover, the triple can be chosen so that $\langle M_{s}^{x,\varphi} \rangle, \langle N_{s}^{x,\varphi} \rangle$ are given by (3.2) and $V_{s}^{x,\varphi}$ by (3.3).
Proof. Uniqueness of the triple is a consequence of uniqueness of the decomposition of a continuous process into a continuous martingale and a continuous process of finite variation.

Existence of the triple can be proved by the same method as in the proof of Theorem 3.1. Indeed, for fixed \((s, x) \in [0, T) \times \mathbb{R}^d\), \(\varphi \in C_0^0\) let \((M_{s, x}^{\varphi}, N_{s, x}^{\varphi}, V_{s, x}^{\varphi})\) be the triple constructed in the first part of the proof of Theorem 3.1 (to construct the triple we did not use (1.3)). Comparing its construction with the proof of the decomposition (1.4) we see that (4.22) is satisfied. Take now a sequence \(\{f_k\}\) considered at the beginning of the proof of Theorem 4.2. Since \(\{f_k\} \subset C_0^0\), it follows from the above observation that for each \(k \in \mathbb{N}\) there is a triple \((M_{s, x}^{f_k}, N_{s, x}^{f_k}, V_{s, x}^{f_k})\) satisfying the assertion (i) of Theorem 3.1 and such that (3.2), (3.3) and (4.22) hold with \(f_k\) in place of \(\varphi\). The last property together with Proposition 4.1 implies (3.16), so letting \(k \to \infty\) proves existence of the representation (4.22) for general \(\varphi\). ■

Let us set \(\mathcal{Y} = [0, \infty) \times \mathbb{R}^d\) and suppose that \(a: \mathcal{Y} \to \mathbb{R}^d \otimes \mathbb{R}^d\) is symmetric and satisfies (1.1) for all \((t, x) \in \mathcal{Y}\). Then one can construct a time-homogeneous family \(\{(Y, \hat{\mathcal{P}}_y); y \in \mathcal{Y}\}\) by putting \(Y((s, \omega)) = (s+t, X_{s+t}(\omega))\) and \(\hat{\mathcal{P}}_y = (s, \omega) (\Gamma) = P_{s, \omega}((\omega: (s, \omega) \in \Gamma))\) (for a construction of such a family by using Dirichlet forms theory see [11] and [17]). Let \(Y^\varphi = \varphi(Y_t) - \varphi(Y_0),\ t \geq 0\). Since the decomposition (4.22) is unique, without ambiguity we can set

\[
\tilde{M}_t^{[\varphi]}(\omega) = (s, \varphi), \quad \tilde{A}_t^{[\varphi]}(\omega) = (Y_t^\varphi - \tilde{M}_t^{[\varphi]})(\omega) \quad \text{for} \ s, t \geq 0
\]

such that \(s+t \leq T\), where \(M^{[\varphi], T}\) is the MAF of Theorem 4.2 on \([0, T]\). For \(B : = Y^\varphi, \tilde{M}^{[\varphi]}, \tilde{A}^{[\varphi]}\) we then have \(\hat{\mathcal{P}}_y (B_{s+t} = B_s + B_t, \omega) = 1\) for any \(s, t \geq 0,\ y \in \mathcal{Y}\), where \(\hat{\theta}_t((s, \omega)) = (s+t, \omega)\). Moreover, if we denote by \(\hat{E}_{s, m}\) the expectation sign with respect to \(\hat{\mathcal{P}}_{(s, m)}(\cdot) = \int_{\mathbb{R}^d} \hat{\mathcal{P}}_{(s, \omega)}(\cdot) m(\omega) d\omega\), then

\[
e^\varphi_b(X^\varphi) = \int_0^T \int_0^{T-t} e^{-at} \hat{E}_{s, m}(Y^\varphi)^2 ds \, dt, \quad \tilde{e}_t(M^{[\varphi]}) = \int_0^{T-t} \hat{E}_{s, m}(\tilde{M}_t^{[\varphi]})^2 ds.
\]

Therefore (1.4) may be viewed as a strict version (i.e. for every starting point \(y \in \mathcal{Y}\)) of the decomposition of the functional \(Yf\) with \(f(t, x) = I_{[0, T]}(t) \varphi(x)\) obtained in [12] and [19]. Note, however, that the results obtained in [12] and [19] lead to a decomposition of \(Yf\) for general time-inhomogeneous diffusions \((Y, \mathcal{P}_y)\) and general time-dependent \(f\).

**THEOREM 4.4.** Under the assumptions of Theorem 4.2, if moreover (1.3) is satisfied, \(\tilde{e}(A^{[\varphi]}) = 0\).

**Proof.** We first prove that

\[
(4.23) \quad \tilde{e}(X^\varphi) = (a \nabla \varphi, \nabla \varphi)_{2;0,T}.
\]
To this end, we write

\[ \hat{e}_t(X^0) = 2 \int_0^{T-t} (\varphi, \varphi - P_{s+t}^s \varphi)_2 ds = \int_0^{T-t} ds \int_s^{s+t} (a(u, \cdot) V P_{s+t}^s \varphi, V \varphi)_2 du \]

\[ = \int_0^{s+t} ds \int_0^u f(s, u) ds + \int_0^{T-t} du \int_0^{u-t} f(s, u) ds + \int_0^{T-t} ds \int_0^{s+t} (a(u, \cdot) V \varphi, V \varphi)_2 du \]

\[ = I_1(t) + I_2(t) + I_3(t), \]

where \( f(s, u) = (a(u, \cdot) V (P_{s+t}^s \varphi - \varphi), V \varphi)_2 \). It is easily seen that \( \lim_{t \to 0} t^{-1} I_1(t) = 0 \). By (2.9), \( \{P_{s+t}^s \varphi - \varphi \} \) is bounded in \( W_{1/2} \) and \( P_{s+t}^s \varphi - \varphi \to 0 \) in \( L_2 \) for any \( \{s_n\} \subset (u-t, u) \) such that \( s_n \uparrow u \). Therefore the function \( (u-t, u] \mapsto f(s, u) \) is continuous, and hence \( t^{-1} \int_{u-t}^u f(s, u) ds \to f(u, u) = 0 \)

as \( t \downarrow 0 \). From this and the dominated convergence theorem we conclude that \( \lim_{t \to 0} t^{-1} I_2(t) = 0 \). Finally, since \( I_3(t) = \hat{e}_t(M^{[\varphi]}) \), by (4.9) we have \( \lim_{t \to 0} t^{-1} I_3(t) = (a \varphi, \varphi)_2; 0, r \), and (4.23) is proved. Following the proof of Theorem 4.1 in [15] we now show that

\[ (A^{[\varphi]})^2 = 2 \int_0^{s+t} (A^{[\varphi]} - A^{[\varphi]}_s, dA^{[\varphi]}_s) \]

\[ = 2 \int_0^{s+t} (\varphi(X_{s}, t) - \varphi(X_{u}), dA^{[\varphi]}_s - 2 \int_0^{s+t} (M^{[\varphi]}_s, dM^{[\varphi]}_s) \]

\[ = \varphi(X_{s+t}) (-M^{[\varphi]}_{s+t} + \tilde{N}_{s+t}) - \int_0^{s+t} \varphi(X_{u}) d(-M^{[\varphi]}_{s+t} + \tilde{N}_{s+t}) \]

\[ = \int_0^{s+t} \tilde{N}_{s+t} \tilde{N}_{s+t} - \int_0^{s+t} \tilde{N}_{s+t} \tilde{N}_{s+t} \]

where

\[ \int_0^{s+t} \varphi(X_{u}) d\tilde{N}_{s+t} = - \int_T^{T-t} \varphi(X_{u}) dN_{s+t} - \langle X^0_{s+t}, \tilde{N}_{s+t} \rangle \]

and

\[ \langle X^0_{s}, \tilde{N}_{s} \rangle_{s+t} = \langle M^{[\varphi]}_{s}, \tilde{N}_{s} \rangle_{s+t} = \langle M^{[\varphi]}_{s} \rangle_{s+t} \]

Clearly,

\[ E_{s,m}(\varphi(X_{s+t}) M^{[\varphi]}_{s} + M^{[\varphi]}_{s,s+t}) = E_{s,m}(X^0_{s,s+t} M^{[\varphi]}_{s,s+t}) \]
and, since $\varphi(X_{s+t})$ is $\sigma(X_u, u \in [s+t, T]) = \mathcal{F}^\pi_{T-t}$-measurable, we obtain

$$E_{s,x} (\varphi(X_{s+t}) N^\varphi_{x,s+t}) = E_{s,x} \{ \varphi(X_{s+t}) (E_{s,x}(N^\pi_{T-t} - N^\pi_{t}) | \mathcal{F}^\pi_{T-t}) \} = 0.$$ 

We have also

$$E_{s,m} \int_s^{s+t} X^\varphi_{u, s+t} dV^\varphi_{s,u} = 0,$$

because, by the Markov property,

$$E_{s,x} \int_s^{s+t} p^{-1} (aV p, V \varphi)(s, x, u, X_u) X^\varphi_{u, s+t} du$$

$$= \int_s^{s+t} E_{s,x} \{ p^{-1} (aV p, V \varphi)(s, x, u, X_u) E_{u,X_u} X^\varphi_{u, s+t} \} du$$

$$= \int_s^{s+t} \sum_{j=1}^{d} a_j (u, y) D_j p(s, x, u, y) D_j \varphi(y) (P^\varphi_{u, s+t} \varphi - \varphi)(y) dy dy.$$

By the above calculations, we get

$$E_{s,m} (A^{[\varphi]}_{s+t})^2 = E_{s,m} (-X^\varphi_{s,s+t} M^{[\varphi]}_{s,s+t} + (M^{[\varphi]}_{s,s+t})^2)$$

$$= E_{s,m} (-A^{[\varphi]}_{s,s+t} M^{[\varphi]}_{s,s+t} + (M^{[\varphi]}_{s,s+t})^2 + (M^{[\varphi]}_{s,s+t})^2)$$

$$= -E_{s,m} (A^{[\varphi]}_{s,s+t} M^{[\varphi]}_{s,s+t}),$$

and hence

$$\bar{e}_i(X^\varphi) = \bar{e}_i(M^{[\varphi]}) + \bar{e}_i(A^{[\varphi]}) + 2 \int_0^{T-t} E_{s,m} (A^{[\varphi]}_{s,s+t} M^{[\varphi]}_{s,s+t}) ds = \bar{e}_i(M^{[\varphi]}) - \bar{e}_i(A^{[\varphi]})$$

for every $t \in [0, T]$. Obviously, this gives (4.24) and proves the theorem for $\varphi \in C^2_0$ when combined with (4.23). To prove the general case, we approximate $\varphi$ in $W^1_2$ by a sequence $\{f_k\} \subset C^2_0$ and use the estimate

$$\bar{e}(A^{[\varphi]}) \leq 3e(X^{f_k - \varphi}) + 3e(M^{[f_k - \varphi]}) + 3e(A^{[f_k]}) = 6 (aV (f_k - \varphi), V (f_k - \varphi))_{2,0,T},$$

which follows from uniqueness of the decomposition (1.4), the fact that $\bar{e}(A^{[f_k]}) = 0$ and (4.9), (4.23).

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