SOME REMARKS ON $\alpha$S, $\beta$-SUBSTABLE RANDOM VECTORS

BY

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Abstract. An $\alpha$S random vector $X$ is $\beta$-substable, $\alpha < \beta \leq 2$, if $X = \Theta^{1/\beta}$ for some symmetric $\beta$-stable random vector $Y$, $\Theta \geq 0$ a random variable with the Laplace transform $\exp \{-t^{\beta}\}$, $Y$ and $\Theta$ are independent. We say that an $\alpha$S random vector is maximal if it is not $\beta$-substable for any $\beta > \alpha$.

In the paper we show that the canonical spectral measure for every $\alpha$S, $\beta$-substable random vector $X$, $\beta > \alpha$, is equivalent to the Lebesgue measure on $S_{n-1}$. We show also that every such vector admits the representation $X = Y + Z$, where $Y$ is an $\alpha$S sub-Gaussian random vector, $Z$ is a maximal $\alpha$S random vector, $Y$ and $Z$ are independent. The last representation is not unique.

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Let us remind first the well-known definitions of symmetric $\alpha$-stable random variables, random vectors and stochastic processes, $\alpha \in (0, 2]$. The random variable $X$ is symmetric $\alpha$-stable if there exists a positive constant $A$ such that

$$E \exp \{itX\} = \exp \{-A \|t\|^\alpha\}.$$ 

A random vector $X = (X_1, \ldots, X_n)$ is symmetric $\alpha$-stable if for every $\xi = (\xi_1, \ldots, \xi_n)$ the random variable $\langle \xi, X \rangle = \sum_{k=1}^{n} \xi_k X_k$ is symmetric $\alpha$-stable. This is equivalent to the following condition:

$$\forall \xi = (\xi_1, \ldots, \xi_n) \ \exists c(\xi) > 0 \ \langle \xi, X \rangle \overset{d}{=} c(\xi) X_1.$$ 

It is well known that if $X$ is an $\alpha$S random vector on $\mathbb{R}^n$, then there exists a finite measure $\nu$ on $\mathbb{R}^n$ such that

$$E \exp \{i \langle \xi, X \rangle\} = \exp \{-\int_{\mathbb{R}^n} |\langle \xi, x \rangle|^\alpha \nu(dx)\}.$$ 

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The measure $v$ is called the spectral measure for an $S\alpha S$ random vector $X$. If $v$ is concentrated on the unit sphere $S_{n-1} \subset \mathbb{R}^n$, then it is called the canonical spectral measure for $X$. The canonical spectral measure for a given $S\alpha S$ vector $X$ is uniquely determined.

An $S\alpha S$ random vector $X$ is $\beta$-substable, $\alpha < \beta \leq 2$, if there exists a symmetric $\beta$-stable random vector $Y$ such that

$$X \overset{d}{=} Y\Theta^{1/\beta},$$

where $\Theta \geq 0$ is an $\alpha/\beta$-stable random variable with the Laplace transform $\exp\{-t^{\alpha/\beta}\}$, $Y$ and $\Theta$ are independent.

**Definition 1.** An $S\alpha S$ random vector $X$ is maximal if for every $\beta \geq \alpha$ and every $S\beta S$ random vector $Y$, and every $\Theta$ independent of $Y$ the equality $X \overset{d}{=} Y\Theta$ implies that $\alpha = \beta$ and $\Theta = \text{const}.$

A stochastic process $\{X_t : t \in T\}$ is symmetric $\alpha$-stable if all its finite-dimensional distributions are symmetric $\alpha$-stable, i.e., if for every $n \in \mathbb{N}$ and every choice of $t_1, \ldots, t_n \in T$ the random vector $(X_{t_1}, \ldots, X_{t_n})$ is symmetric $\alpha$-stable.

For more information on stable random vectors, processes and distributions see [2]. Almost all $S\alpha S$ random vectors and stochastic processes studied in literature are maximal; and even more, almost all of them have pure atomic spectral measure. In [1] one can find some results on characterizing maximal $S\alpha S$ random vectors in the language of geometry of reproducing kernel spaces, however, except some trivial cases, these results are given only for infinite-dimensional $S\alpha S$ random vectors. The following, surprisingly simple theorem characterizes maximal symmetric $\alpha$-stable random vectors on $\mathbb{R}^n$:

**Theorem 1.** Assume that a random vector $X = (X_1, \ldots, X_n)$ is symmetric $\alpha$-stable and $\beta$-substable for some $\beta \in (\alpha, 2]$. Then the canonical spectral measure $v$ for the vector $X$ has a continuous density function $f(u)$ with respect to the Lebesgue measure on the unit sphere $S_{n-1} \subset \mathbb{R}^n$, and $f(u) > 0$ for every $u \in S_{n-1}$.

**Proof.** From the assumptions we infer that there exists a symmetric $\beta$-stable random vector $Y = (Y_1, \ldots, Y_n)$ such that $X \overset{d}{=} Y\Theta^{1/\beta}$, where $\Theta > 0$ independent of $Y$ is $\alpha/\beta$-stable with a Laplace transform $\exp\{-t^{\alpha/\beta}\}$. Assume that

$$E \exp\{it \langle \xi, Y \rangle\} = \exp\{-c(\xi)^\beta |t|^{\beta}\}.$$  

This means that for every $\xi$ we have

$$\langle \xi, Y \rangle \overset{d}{=} c(\xi) Y_0,$$

where $E \exp\{it Y_0\} = \exp\{-|t|^{\beta}\}$.

In particular,

$$E|\langle \xi, Y \rangle|^\beta = c(\xi)^\beta E|Y_0|^\beta.$$
Since \( \alpha < \beta \), we have \( c^{-1} = E|Y_0|^\alpha < \infty \) and \( c(\xi)^\alpha = cE\langle \xi, Y \rangle^\alpha \). Calculating now the characteristic function for the vector \( X \) we obtain

\[
E \exp \{i \langle \xi, X \rangle \} = E \exp \{i \langle \xi, Y^{1/\beta} \rangle \} = E \exp \{-c \langle \xi, Y \rangle \} = \exp \{-c \langle \xi, Y \rangle \} = \exp \{-\int_{\mathbb{R}^n} \langle \xi, x \rangle^\alpha c f_\beta(x) \, dx \},
\]

where \( f_\beta(x) \) denotes the density function of the \( S\beta S \) random vector \( Y \). This means that the function \( cf_\beta(x) \) is the density of a spectral measure for the random vector \( X \).

To get the canonical spectral measure \( v_0 \) for the \( S\alpha S \) random vector \( X \) from this spectral measure it is enough to make the spherical substitution \( x = ru \) and integrate out the radial part. Consequently, for every Borel set \( A \subset S_{n-1} \) we obtain

\[
v_0(A) = \int_{S_{n-1}} \cdots \int_{S_{n-1}} c f_\beta(ru)^{n-1+s} \, dr \, w(du),
\]

where \( w \) is the Lebesgue measure on \( S_{n-1} \). Since \( f_\beta \) is uniformly continuous on \( \mathbb{R}^n \) and \( f_\beta > 0 \) everywhere, \( g(\mathbf{u}) \) is a continuous function and \( g(\mathbf{u}) > 0 \) everywhere. The uniqueness of the canonical spectral measure implies that the function \( g(\mathbf{u}) \) is the density of the measure \( v_0 \), which completes the proof.

**Corollary 1.** Every random vector with a pure atomic spectral measure is maximal. In fact, for maximality of the \( S\alpha S \) random vector it is enough that its spectral measure \( \mu \) is zero on a set in \( S_{n-1} \) of positive Lebesgue measure.

**Corollary 2.** Let \( (E, \mathcal{B}, \mu) \) be a \( \sigma \)-finite measure space and let \( Y = \{ Y(B); B \in \mathcal{B}, \mu(B) < \infty \} \) be an independently scattered \( S\alpha S \) random measure on \( (E, \mathcal{B}) \) controlled by the measure \( \mu \). We say that a stochastic process \( X = \{X_t; t \in T\} \) is a set-indexed \( S\alpha S \)-process if there exists a map \( S \) from \( T \) to \( \mathcal{B} \) such that

\[
X_t = Y(S_t).
\]

Every set-indexed \( S\alpha S \)-process is maximal.

**Proof.** Notice that any finite-dimensional marginal distribution of a set-indexed \( S\alpha S \)-process has a pure point spectrum. For example, the 3-dimensional marginal characteristic function is
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\[ E \exp \{ i (z_1 X_{1t} + z_2 X_{2t} + z_3 X_{3t}) \} = E \exp \{ i (z_1 Y(S_1) + z_2 Y(S_2) + z_3 Y(S_3)) \} \]

\[ = \exp \{ |z_1|^{\eta} \mu(S_1 \cap S_2 \cap S_3^c) + |z_2|^{\eta} \mu(S_1 \cap S_2 \cap S_3^c) \]
\[ + |z_3|^{\eta} \mu(S_1 \cap S_2 \cap S_3) + |z_2 + z_3|^{\eta} \mu(S_1 \cap S_2 \cap S_3) \]
\[ + |z_3 + z_1|^{\eta} \mu(S_1 \cap S_2 \cap S_3) + |z_1 + z_2|^{\eta} \mu(S_1 \cap S_2 \cap S_3^c) \]
\[ + |z_1 + z_2 + z_3|^{\eta} \mu(S_1 \cap S_2 \cap S_3). \]

Some of important \(\mathcal{S}\alpha\mathcal{S}\)-processes are set-indexed processes: for example, multiparameter Lévy motion, multiparameter additive processes, generally linearly additive processes, a class of self-similar \(\mathcal{S}\alpha\mathcal{S}\)-processes (see, e.g., [3]–[6]). Moreover, all these processes have very interesting properties, called determinisms.

**Corollary 3.** If an \(\mathcal{S}\alpha\mathcal{S}\) random vector \(X\) is not maximal, i.e., if \(X\) is \(\beta\)-substable for some \(\beta > \alpha\), then there exist a symmetric Gaussian random vector \(Z\) and a maximal \(\mathcal{S}\alpha\mathcal{S}\) random vector \(Y\) such that

\[ X \overset{d}{=} Z \Theta^{1/2} + Y, \]

where \(\Theta \geq 0\) has the Laplace transform \(\exp \{-t^{\eta/2}\}\), \(Z\), \(Y\) and \(\Theta\) are independent.

**Proof.** Since every continuous function attains its extremes on very compact set, we have

\[ A = \inf \{g(u): u \in S_{n-1}\} > 0, \]

where \(g(u)\) is the density of the canonical spectral measure for \(X\) obtained in Theorem 1. Now it is easy to see that \(X \overset{d}{=} Z \Theta^{1/2} + Y\) for the Gaussian random vector \(Z\) with the characteristic function \(\exp \{-A^{1/\alpha} \sum_{k=1}^{n} z_k^2\}\), and the \(\mathcal{S}\beta\mathcal{S}\) random vector \(Y\) with the spectral measure given by the density function \(f(u) = g(u) - A\).

**Remark 1.** The representation obtained in Corollary 3 is not unique. In fact, for every \(\mathcal{S}\alpha\mathcal{S}\) \(\beta\)-substable random vector \(X\) and every symmetric Gaussian random vector \(Z\) taking values in the same space \(\mathbb{R}^n\) there exist a constant \(c > 0\) and a maximal \(\mathcal{S}\alpha\mathcal{S}\) random vector \(Y\) such that

\[ X \overset{d}{=} cZ \Theta^{1/2} + Y, \]

where \(\Theta\) as in Corollary 3, \(Y\), \(Z\) and \(\Theta\) are independent.

**Proof.** The representation \((*)\) for the characteristic function of an \(\mathcal{S}\alpha\mathcal{S}\) random vector holds for every \(\alpha \in (0, 2]\) including the Gaussian case. However, for \(\alpha = 2\) we do not have uniqueness for the spectral measure \(\nu\). In fact, \(\nu\) can always be taken here from the class of pure atomic measures on \(S_{n-1}\), but such a representation is not useful for our construction. We will use the measure \(\nu_A\) constructed as follows:
Let \( v = v_1 \) be the uniform distribution on the unit sphere \( S_{n-1} \subset \mathbb{R}^n \), and let \( U = (U_1, \ldots, U_n) \) be the random vector with the distribution \( v \). Then we have

\[
\exp \left\{ -\int_{S_{n-1}} \langle \xi, u \rangle^2 c_n v(du) \right\} = \exp \left\{ -\frac{1}{2} \langle \xi, \xi \rangle \right\},
\]

where \( c_n^{-1} = 2EU^2 \). Now let \( \Sigma \) be the covariance matrix for the random vector \( Z \) and let \( \Sigma = AA^T \). We denote by \( v_1 \) the distribution of the random vector \( AU \). Then

\[
\exp \left\{ -\int_{\mathbb{R}^n} \langle \xi, x \rangle^2 c_n v_1(dx) \right\} = \exp \left\{ -\int_{S_{n-1}} \langle A^T \xi, u \rangle^2 c_n v(du) \right\} = \exp \left\{ -\frac{1}{2} \langle A^T \xi, A^T \xi \rangle \right\} = \exp \left\{ -\frac{1}{2} \langle \xi, \Sigma \xi \rangle \right\},
\]

which is the characteristic function for the Gaussian vector \( Z \). It is easy to see now that for a suitable constant \( a > 0 \)

\[
\exp \left\{ -\int_{\mathbb{R}^n} \langle \xi, x \rangle^2 c_n v_1(dx) \right\} = \exp \left\{ -a(\langle \xi, \Sigma \xi \rangle)^{n/2} \right\},
\]

which is a characteristic function of the sub-Gaussian vector \( Z \Theta^{1/2} \). We define now the measure \( v_A \) as the projection (in the sense described in the proof of Theorem 1) of the measure \( v_1 \) to the sphere \( S_{n-1} \) and we obtain

\[
\int_{\mathbb{R}^n} \langle \xi, x \rangle^2 c_n v_1(dx) = \int_{S_{n-1}} \langle \xi, u \rangle^2 v_A(du).
\]

Since \( v_1 \) is absolutely continuous with respect to the Lebesgue measure, \( v_A \) has the same property and \( v_A(du) = f_A(u) \omega(du) \) for some continuous positive function \( f_A \). If \( g(u) \) is the density of the spectral measure for \( X \), then there exists \( c_0 > 0 \) such that

\[
c_0 = \sup \{ c > 0 : g(u) - cf_A(u) \geq 0 \}.
\]

Now it is enough to define the maximal \( \mathcal{S} \alpha \mathcal{S} \) random vector \( X \) by its canonical spectral measure absolutely continuous with respect to the Lebesgue measure with density \( h(u) = g(u) - c_0 f_A(u) \) and put \( c = c_0^{1/\alpha} \).

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