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ON THE EXIT TIME OF α-STABLE PROCESS

BY

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Abstract. In this paper we investigate the probability that α-stable Lévy process stays in convex body up to time t. This can be optimally estimated from below by the same probability but of the rotationally invariant process.

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INTRODUCTION

Let (X_t, P^x) be an α-stable process with values in $\mathscr{R}^d$. For $D \subset \mathscr{R}^d$, we define $\tau_D = \inf\{t \geqslant 0, X_t \notin D\}$. It is very important to know the behaviour of $P^x(\tau_D > t)$. For example, $\int_0^\infty P^x(\tau_D > t)\,dt$ estimates the Green function of D, and the behaviour of $\log P(\tau_D > t)$, for $t \to \infty$, estimates the eigenvalues of the generator (see [2]–[5] and [9]). So far, $P^x(\tau_D > t)$ has been described in the case when the distribution of X_t is rotationally invariant. This paper is devoted to the general case of α-stable processes. In fact, we prove that if D is symmetric and convex, then $P_X^0(\tau_D > t)$ is less than $P_{\hat{X}}^0(\tau_D > t)$, where $\hat{X}$ is a rotationally invariant α-stable process.

PRELIMINARIES

In this paper, (X_t, P^x) denotes *α-stable Lévy process* (i.e. a homogeneous process with independent increments) with values in $\mathscr{R}^d$, $0 < \alpha < 2$. Whenever we mention *α-stable process* we think about the process as described above.

The Fourier transform of X_t is given by the formula

$$E\exp\big(i(y, X_t)\big) = \exp\big(-t \int_{S^{d-1}} |\langle y, s\rangle|^\alpha \sigma(ds)\big),$$

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where σ is a certain symmetric, positive, finite measure concentrated on S^{d-1}, $\langle\cdot,\cdot\rangle$ denotes the standard scalar product, and $|\cdot| = (\cdot,\cdot)^{1/2}$ is a norm. Such a measure σ (called the *spectral measure*) determines the distribution of X_1, whence the distribution of the whole process [8]. It is well known that trajectories of (X_t) are right continuous and have left-hand limits a.s.

Now we show the main tool of our paper. First we introduce the following three families of random objects.

1. Let $(X_i)_{i=1}^{\infty}$ denote a sequence of i.i.d. real variables such that $P(X_i > t) = e^{-t}$. Put $\Gamma_n = X_1 + \ldots + X_n$.

2. $(Z_n)_{n=1}^{\infty}$ denotes a sequence of i.i.d. $\mathscr{R}^d$-valued symmetric vectors such that $E|Z_n|^{\alpha} < \infty$, that is $P(-Z_n \in \cdot) = P(Z_n \in \cdot)$.

3. $(U_n)_{n=1}^{\infty}$ denotes a sequence of i.i.d. real-valued variables with uniform distribution on $[0, 1]$.

Moreover, we assume that (Γ_n), (Z_n), (U_n) are independent families.

The following representation is crucial for our purposes.

PROPOSITION (the Series Representation, see [6], [7], [10]). *We have*:

(a) $\sum_{n=1}^{\infty} \Gamma_n^{-1/\alpha} \cdot Z_n \cdot \mathbb{1}_{[U_n, 1]}(t)$ *converges a.s. in* $D[0, 1]$ *both in the supremum and the Skorohod metrics.*

(b) $Y(t) = \sum_{n=1}^{\infty} \Gamma_n^{-1/\alpha} \cdot Z_n \cdot \mathbb{1}_{[U_n, 1]}(t)$, $0 \leqslant t \leqslant 1$, *is an* α-*stable process with independent and homogeneous increments.*

(c) *The Fourier transform of* $Y(t)$ *is equal to*

$$E \exp\big(i(y, Y(t))\big) = \exp\left(-C'_{\alpha} t E|(y, Z)|^{\alpha}\right),$$

where $C'_{\alpha} = \int_0^{\infty} x^{-\alpha} \sin x \, dx$ *and* $Z \stackrel{d}{=} Z_n$; *hence the spectral measure of* $Y(t)$ *is equal to*

$$\sigma(A) = C'_{\alpha} E \mathbb{1}_A\left(\frac{Z}{|Z|}\right)|Z|^{\alpha}.$$

COROLLARY. *Let* (X_t, P^x) *be an* α-*stable Lévy process with spectral measure* σ *and* $\sigma(S^{d-1}) = 1$. *Assume that* $(Z_n)_{n=1}^{\infty}$ *are i.i.d. and* $\mathscr{L}(Z_n) = \sigma$. *Let* $(g_n)_{n=1}^{\infty}$ *be a sequence of Gaussian variables, with distribution* $N(0, 1)$, *and assume that the families* (Γ_n), (Z_n), (U_n) *and* (g_n) *are independent. Then the series*

$$\left(\frac{1}{C'_{\alpha} E|g_1|^{\alpha}}\right)^{1/\alpha} \sum_{n=1}^{\infty} \Gamma_n^{-1/\alpha} \cdot Z_n \cdot g_n \cdot \mathbb{1}_{[U_n, 1]}(t)$$

is a representation of $X(t)$ (*in distribution on* $D[0, 1]$).

Since our proof is based on representation of the process via the mixture of Gaussian processes, we shall recall a definition and some nice features of Gaussian measures.

(*) X is a *Gaussian vector* if for every $y \in \mathscr{R}^d$ the real random variable (y, X) has distribution $N(m, \sigma^2)$, where $m = E(y, X)$ and $\sigma^2 = E(y, X)^2$.

(**) If X is a symmetric Gaussian random vector with values in $\mathscr{R}^d$, then there exist numbers $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_d \geqslant 0$ and an orthonormal system $\{v_1, v_2, \ldots, v_d\}$ such that

$$\mathscr{L}(X) = \mathscr{L}(\lambda_1 v_1 g_1 + \lambda_2 v_2 g_2 + \ldots + \lambda_d v_d g_d),$$

where g_i are i.i.d. with distribution $N(0, 1)$.

($\overset{**}{*}$) ANDERSON INEQUALITY [1]. *Let X be a symmetric Gaussian vector in $\mathscr{R}^d$, and V a symmetric convex set in $\mathscr{R}^d$. Then for every $a \in \mathscr{R}^d$*

$$P(X + a \in V) \leqslant P(X \in V).$$

The inequality above implies that if X is Gaussian and Y is any random vector independent of X, then

$$P(X + Y \in V) \leqslant P(X \in V).$$

From all α-stable Lévy processes on $\mathscr{R}^d$ we distinguish the special one, the so-called "rotation invariant" process denoted by $\hat{X}(t)$. Its characteristic functional depends on $|y|$: for every $y \in \mathscr{R}^d$,

$$E\exp\big(i(y, \hat{X}_t)\big) = \exp(-t|y|^\alpha).$$

THE MAIN RESULT

Now we can state and prove our theorem.

THEOREM. *Let (X_t, P^x) be an α-stable Lévy process with spectral measure σ and $\sigma(S^{d-1}) = 1$. Let $\hat{X}_t$ denote the rotationally invariant α-stable process. Take arbitrary $r \in N$ and let $V_1, V_2, \ldots, V_r$ be any convex symmetric sets in $\mathscr{R}^d$ and $0 \leqslant t_1 < t_2 < \ldots < t_r \leqslant 1$ be any sequence from $[0, 1]$. Then*

$$P^0\Big(\bigcap_{i=1}^{r} (X_{t_i} \in V_i)\Big) \geqslant P^0\Big(\bigcap_{i=1}^{r} (\hat{X}_{t_i} \in V_i)\Big).$$

Proof. First choose and fix any arbitrary orthonormal system in $\mathscr{R}^d$, say $\{e_1, e_2, \ldots, e_d\}$. Assume that $(g_{ik})_{\substack{i=1,\ldots,d \\ k=1,2,\ldots}}$ are independent and have identical distribution $N(0, 1)$. Put

$$M(t) = \left(\frac{1}{C'_\alpha E|g|^\alpha}\right)^{1/\alpha} \sum_{n=1}^{\infty} \Gamma_n^{-1/\alpha} \cdot (e_1 g_{1n} + \ldots + e_d g_{dn}) \cdot \mathbb{1}_{[U_n, 1]}(t)$$

(as usual, (g_{in}), (Γ_n), (U_n) are independent). $M(t)$ is an α-stable process. For $y \in \mathscr{R}^d$ we have

$$E\exp\big(i(y, M(t))\big) = \exp\left(-\frac{1}{E|g|^\alpha} t \cdot E|(y, e_1 g_1 + \ldots + e_n g_n)|^\alpha\right) = \exp(-t|y|^\alpha),$$

because $g_1, g_2, \ldots, g_n$ are independent $N(0, 1)$ variables. Consequently, $M(t)$ is a version of $\hat{X}(t)$. Let

$$X(t) = C_\alpha \sum_{n=1}^{\infty} \Gamma_n^{-1/\alpha} \cdot Z_n \cdot g_n \cdot \mathbb{1}_{[U_n, 1]}(t),$$

where $\mathscr{L}(Z_n) = \sigma$, g_n are independent $N(0, 1)$ and

$$C_\alpha = \left(\frac{1}{C'_\alpha E|g|^\alpha}\right)^{1/\alpha}.$$

Fix the points $0 = t_0 < t_1 < t_2 < \ldots < t_r \leqslant 1$. In the rest of the proof all probabilities and expectations are regarded as conditional: we fix (U_n, Γ_n, Z_n); then the distribution of

$$X(t) = C_\alpha \sum_{n=1}^{\infty} \Gamma_n^{-1/\alpha} \cdot Z_n \cdot g_n \cdot \mathbb{1}_{[U_n, 1]}(t)$$

is Gaussian.

Let us put $G_k = X_{t_k} - X_{t_{k-1}}$ and $Y_k = G_1 + \ldots + G_k$, $k = 1, \ldots, r$. If we fix (Γ_n), (U_n) and (Z_n), then $G_1, G_2, \ldots, G_r$ are independent Gaussian vectors with values in $\mathscr{R}^d$. It is easy to see that $(Y_1, Y_2, \ldots, Y_r)$ generates a Gaussian vector in $(\mathscr{R}^d)^r$. Observe that if $\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_r$ are other independent vectors such that $G_n \stackrel{d}{=} \tilde{G}_n$, then

$$\mathscr{L}\big((Y_1, Y_2, \ldots, Y_r)\big) = \mathscr{L}\big((\tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_r)\big), \quad \text{where } \tilde{Y}_k = \tilde{G}_1 + \ldots + \tilde{G}_k.$$

All we have to do now is to estimate the quantity

$$P\big((Y_1, Y_2, \ldots, Y_r) \in V_1 \times \ldots \times V_r\big).$$

Since, by virtue of (**),

$$G_k = X_{t_k} - X_{t_{k-1}} = C_\alpha \sum_{n=1}^{\infty} \Gamma_n^{-1/\alpha} \cdot Z_n \cdot g_n \cdot \mathbb{1}(t_{k-1} < U_n \leqslant t_k)$$

is a Gaussian vector, there exists an orthonormal system, say $\{v_{1k}, \ldots, v_{dk}\}$, and numbers $\lambda_{1k} \geqslant \lambda_{2k} \geqslant \ldots \geqslant \lambda_{dk} \geqslant 0$ such that

$$G_k \stackrel{d}{=} \lambda_{1k} v_{1k} g_{1k} + \lambda_{2k} v_{2k} g_{2k} + \ldots + \lambda_{dk} v_{dk} g_{dk}.$$

We can find λ_{1k} easily:

$$\lambda_{1k}^2 = \sup_{|x|=1} E(x, G_k)^2 = C_\alpha^2 \sup_{|x|=1} \sum_{n=1}^{\infty} \Gamma_n^{-2/\alpha} \cdot (x, Z_n)^2 \cdot \mathbb{1}(t_{k-1} < U_n \leqslant t_k)$$

$$\leqslant C_\alpha^2 \sum_{n=1}^{\infty} \Gamma_n^{-2/\alpha} \cdot \mathbb{1}(t_{k-1} < U_n \leqslant t_k).$$

By a similar argument,

$$(y, G_k^*) \stackrel{d}{=} g\,|y|\Big(\sum_{n=1}^{\infty} \Gamma_n^{-2/\alpha} \cdot \mathbb{1}(t_{k-1} < U_n \leqslant t_k)\Big)^{1/2}$$

(we use the fact that $\{v_{1k}, \ldots, v_{dk}\}$ is an orthonormal system). Taking the expectation of Γ_n, Z_n, U_n, we get the desired conclusion.

Remarks. 1. Taking $V_i = V$, V closed, and using standard approximation arguments, we get for $t \geqslant 0$ the estimate $P_X^0(\tau_V > t) \geqslant P_{\hat{X}}^0(\tau_V > t)$.

2. The spectral measure σ of $\hat{X}$ has the mass greater than 1 if $d > 1$. Indeed,

$$\sigma(S^{d-1}) = \frac{1}{E\,|g|^\alpha} E\,|e_1 g_1 + \ldots + e_d g_d|^\alpha = \frac{1}{E\,|g|^\alpha} E\,(g_1^2 + \ldots + g_d^2)^{\alpha/2}.$$

However, let us take any $v_1 \in \mathscr{R}^d$ such that $|v_1| = 1$ and consider

$$X(t) = C_\alpha \sum_{n=1}^{\infty} \Gamma_n^{-1} \cdot v_1 \cdot g_{1n} \cdot \mathbb{1}_{[U_n, 1]}(t),$$

and

$$\hat{X}(t) = C_\alpha \sum_{n=1}^{\infty} \Gamma_n^{-1/\alpha} \cdot \mathbb{1}_{[U_n, 1]}(t) \cdot (v_1 g_{1n} + v_2 g_{2n} + \ldots + v_d g_{dn}).$$

Put $V = \{x\colon |(v_1, x)| \leqslant 1\}$. Now,

$$\big(v_1, \hat{X}(t)\big) = C_\alpha \sum_{n=1}^{\infty} \Gamma_n^{-1/\alpha} \cdot \mathbb{1}_{[U_n, 1]}(t) \cdot g_{1n} \stackrel{d}{=} \big(v_1, X(t)\big);$$

hence

$$P_X^0(\tau_V > t) = P_{\hat{X}}^0(\tau_V > t).$$

But the spectral measure of $X(t)$ has a total mass equal to

$$\frac{1}{E\,|g|^\alpha} \cdot E\,|v_1 g|^\alpha = 1.$$

This proves that the inequality is optimal.

3. Assume that $X(t)$ has the spectral measure σ_X which is absolutely continuous with respect to the spectral measure $\sigma_{\hat{X}}$ of $\hat{X}$. Let $\sigma_X(ds) = f(s) \cdot \sigma_{\hat{X}}(ds)$ ($\sigma_{\hat{X}}$ is equal to uniform measure on S^{d-1} multiplied by $(E\,|g|^\alpha)^{-1} \cdot E\,(g_1^2 + \ldots + g_d^2)^{\alpha/2}$). Assume that $f(s) \geqslant C > 0$ for $s \in S^{d-1}$. Then, under the conditions of our theorem, we have

$$P^0\Big(\bigcap_{i=1}^{r} (X_{t_i} \in V_i)\Big) \leqslant P^0\Big(\bigcap_{i=1}^{r} (C^{1/\alpha} \hat{X}_{t_i} \in V_i)\Big).$$

For the proof, observe that $X_t \stackrel{d}{=} \bar{X}_t + C^{1/\alpha} \hat{X}_t$, where $\bar{X}_t$ and $\hat{X}_t$ are independent α-stable processes and $\bar{X}_t$ has a spectral measure $\sigma = \sigma_X - C\sigma_{\hat{X}}$. Using the Anderson inequality gives the desired result.

Let us put

$$G_k^* = g_{1k}\lambda_{1k}^* v_{1k} + g_{2k}\lambda_{1k}^* v_{2k} + \ldots + g_{dk}\lambda_{1k}^* v_{dk},$$

where

$$\lambda_{1k}^* = \sqrt{\textstyle\sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha}\cdot \mathbb{1}(t_{k-1} < U_n \leqslant t_k)}.$$

For a moment, let us denote by $(g'_{in})_{\substack{i=1,\ldots,d\\ n=1,2,\ldots}}$ a sequence of i.i.d. $N(0, 1)$ variables, independent of (g_{in}). Observe that

$$\begin{aligned} g_{1k}\lambda_{1k} v_{1k} + g_{2k}\lambda_{2k} v_{2k} + \ldots + g_{dk}\lambda_{dk} v_{dk} + g'_{1k}\cdot\sqrt{(\lambda_{1k}^*)^2 - \lambda_{1k}^2}\cdot v_{1k} \\ + g'_{2k}\cdot\sqrt{(\lambda_{1k}^*)^2 - \lambda_{2k}^2}\cdot v_{2k} + \ldots + g_{dk}\cdot\sqrt{(\lambda_{1k}^*)^2 - \lambda_{dk}^2}\cdot v_{dk} \\ \stackrel{d}{=} g_{1k}\lambda_{1k}^* v_{1k} + g_{2k}\lambda_{1k}^* v_{2k} + \ldots + g_{dk}\lambda_{1k}^* v_{dk}. \end{aligned}$$

Therefore, we can choose independent Gaussian vectors $\bar{G}_1, D_1, \bar{G}_2, D_2, \ldots, \bar{G}_r, D_r$ and independent Gaussian vectors $G_1^*, G_2^*, \ldots, G_r^*$ such that for $k = 1, \ldots, r$ we have

(a) $\bar{G}_k + D_k \stackrel{d}{=} G_k^*$,

(b) $\bar{G}_k \stackrel{d}{=} G_k$,

(c) $G_k^* \stackrel{d}{=} g_{1k}\lambda_{1k}^* v_{1k} + \ldots + g_{dk}\lambda_{1k}^* v_{dk}$.

Put $\bar{Y}_k = \bar{G}_1 + \ldots + \bar{G}_k$, $Z_k = D_1 + \ldots + D_k$, $Y_k^* = G_1^* + \ldots + G_k^*$. The Anderson inequality implies that

$$\begin{aligned} P\big((Y_1^*, \ldots, Y_r^*)\in V_1\times\ldots\times V_r\big) = P\big((\bar{Y}_1, \ldots, \bar{Y}_r) + (Z_1, \ldots, Z_r)\in V_1\times\ldots\times V_r\big) \\ \leqslant P\big((\bar{Y}_1, \ldots, \bar{Y}_r)\in V_1\times\ldots\times V_r\big) = P\big((Y_1, \ldots, Y_r)\in V_1\times\ldots\times V_r\big). \end{aligned}$$

Let us compute the distribution of (Y_k^*). Since

$$G_k^* = g_{1k}\lambda_{1k} v_{1k} + g_{2k}\lambda_{1k} v_{2k} + \ldots + g_{dk}\lambda_{1k} v_{dk},$$

it is easy to see that

$$\mathscr{L}(G_k^*) = \mathscr{L}\left(\frac{1}{C_\alpha}\big(\hat{X}(t_k) - \hat{X}(t_{k-1})\big)\right).$$

Indeed, let $y\in\mathscr{R}^d$; then

$$\begin{aligned} &\big(y, (\hat{X}(t_k) - \hat{X}(t_{k-1}))\big) \\ &= C_\alpha \sum_{n=1}^{\infty} \Gamma_n^{-1/\alpha}\cdot \mathbb{1}(t_{k-1} < U_n \leqslant t_k)\cdot\big((y, e_1) g_{1n} + \ldots + (y, e_d) g_{dn}\big) \\ &\stackrel{d}{=} C_\alpha \sum_{n=1}^{\infty} \Gamma_n^{-1/\alpha}\cdot \mathbb{1}(t_{k-1} < U_n \leqslant t_k)\cdot g_{1n}\sqrt{(y, e_1)^2 + \ldots + (y, e_d)^2} \\ &\stackrel{d}{=} g C_\alpha |y| \Big(\sum_{n=1}^{\infty} \Gamma_n^{-2/\alpha}\cdot \mathbb{1}(t_{k-1} < U_n \leqslant t_k)\Big)^{1/2}, \end{aligned}$$

where $\mathscr{L}(g) = N(0, 1)$.

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