AN ALMOST SURE LIMIT THEOREM FOR THE MAXIMA AND SUMS OF STATIONARY GAUSSIAN SEQUENCES

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Abstract. Let \( X_1, X_2, \ldots \) be some standardized stationary Gaussian process and let us put:

\[
M_k = \max(X_1, \ldots, X_k), \quad S_k = \sum_{i=1}^{k} X_i, \quad \sigma_k = \sqrt{\text{Var}(S_k)}.
\]

Our purpose is to prove an almost sure central limit theorem for the sequence \((M_k, S_k/\sigma_k)\) under suitable normalization of \(M_k\). The investigations presented in this paper extend the recent research of Csaki and Gonchigdanzan [1] and Dudziński [2].

2000 Mathematics Subject Classification: Primary 60F15; Secondary 60F05.

Key words and phrases: Extreme values, partial sums, almost sure central limit theorem, dependent stationary Gaussian sequences.

1. INTRODUCTION

Recently, in a number of papers the joint asymptotic distribution of the maxima \( M_k = \max(X_1, \ldots, X_k) \) and partial sums \( S_k = \sum_{i=1}^{k} X_i \) of weakly dependent random variables have been studied. Let \( r(k) = \text{Cov}(X_1, X_{1+k}) \), \( \sigma_k = \sqrt{\text{Var}(S_k)} \), and let \( \Phi \) denote the standard normal distribution function. Ho and Hsing were concerned in [3] with the case when \((X_i)\) is some standardized stationary Gaussian process. They proved that under certain additional assumptions

\[
\lim_{k \to \infty} P(a_k (M_k - b_k) \leq x, S_k/\sigma_k \leq y) = \exp(-e^{-x}) \Phi(y)
\]

for all \( x, y \in (-\infty, \infty) \), where

\[
a_k = (2 \log k)^{1/2}, \quad b_k = (2 \log k)^{1/2} - \frac{\log \log k + \log 4\pi}{2 (2 \log k)^{1/2}}.
\]

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In our considerations, we will also concentrate on the case when \((X_i)\) is some stationary standard normal process.

It turns out that the more general property may be proved, namely: if \((u_k)\) is a numerical sequence, satisfying the condition
\[
\lim_{k \to \infty} k(1 - \Phi(u_k)) = \tau \quad \text{for some } \tau, \ 0 \leq \tau < \infty,
\]
then under some extra assumptions on \(r(k)\) we have
\[
\lim_{k \to \infty} P(M_k \leq u_k, S_k/\sigma_k \leq y) = e^{-\tau} \Phi(y) \quad \text{for all } y \in (-\infty, \infty).
\]

We will use this fact to prove the main result of our paper, i.e. the so-called almost sure central limit theorem for the sequence \((M_k, S_k/\sigma_k)\). Namely, we will show that if (1) holds and some conditions on \(r(k)\) are satisfied, then
\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I(M_k \leq u_k, S_k/\sigma_k \leq y) = e^{-\tau} \Phi(y) \quad \text{a.s.}
\]
for all \(y \in (-\infty, \infty)\), where \(I\) denotes the indicator function.

Our research is an extension of recent works by Csaki and Gonchigdanzan [1] and Dudziński [2]. In both papers the almost sure central limit theorems for the maxima of certain stationary standard normal sequences have been proved.

2. NOTATION AND ASSUMPTIONS

Throughout the paper \(X_1, X_2, \ldots\) is a standardized stationary Gaussian process. Let us introduce (or recall from the previous section) the following notation:
\[
r(k) = \text{Cov}(X_1, X_{1+k}), \quad M_k = \max(X_1, \ldots, X_k), \quad M_{k,l} = \max(X_{k+1}, \ldots, X_l),
\]
\[
S_k = \sum_{i=1}^{k} X_i, \quad \sigma_k = \sqrt{\text{Var}(S_k)},
\]
\(\Phi\) denotes the standard normal distribution function, and \(I\) means the indicator function. Furthermore, \(f \ll g\) and \(f \sim g\) will stand for \(f = o(g)\) and \(f/g \to 1\), respectively.

In order to shorten the presentation of our results, we label the assumptions of our lemmas and theorems as follows:

(a1) \[
\sup_{s \gg n} \sum_{t=s-n}^{s-1} |r(t)| \ll \frac{(\log n)^{1/2}}{(\log \log n)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0;
\]

(a2) \[
\sum_{t=1}^{n} (n-t)r(t) \geq 0 \quad \text{for all } n \in \{1, 2, \ldots\};
\]
Maxima and sums of stationary Gaussian sequences

The main result is an almost sure central limit theorem for the sequence of maxima and partial sums of certain standardized stationary Gaussian processes.

**THEOREM 1.** Let $X_1, X_2, \ldots$ be a standardized stationary Gaussian process. Suppose moreover that conditions (a1)--(a3) are fulfilled. Then:

(i) If the numerical sequence $(u_k)$ satisfies (a4), we have

$$
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I(M_k \leq u_k, S_k/\sigma_k \leq y) = e^{-\tau} \Phi(y) \text{ a.s.}
$$

for all $y \in (-\infty, \infty)$ and some $\tau \in [0, \infty)$.

(ii) If

$$
a_k = (2 \log k)^{1/2}, \quad b_k = (2 \log k)^{1/2} - \frac{\log \log k + \log 4\pi}{2(2 \log k)^{1/2}},
$$

we have

$$
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I(a_k(M_k - b_k) \leq x, S_k/\sigma_k \leq y) = \exp(-e^{-x}) \Phi(y) \text{ a.s.}
$$

for all $x, y \in (-\infty, \infty)$.

**4. AUXILIARY RESULTS**

In this section we state and prove three lemmas, which will be useful in the proof of Theorem 1.

**LEMMA 1.** Let $X_1, X_2, \ldots$ be a standardized stationary Gaussian process satisfying assumptions (a1)--(a3). Suppose moreover that condition (a4) holds for the numerical sequence $(u_k)$. Then for all $y \in (-\infty, \infty)$, $k < l$ and some $\varepsilon > 0$

$$
E \left| I \left( M_1 \leq u_t, \frac{S_t}{\sigma_t} \leq y \right) - I \left( M_{k,l} \leq u_t, \frac{S_t}{\sigma_t} \leq y \right) \right| \leq \frac{1}{(\log \log l)^{1+\varepsilon} l}.
$$

**Proof.** We will start with the following observations. Let $1 \leq i \leq l$. Then

$$
\text{Cov} \left( X_i, \frac{S_i}{\sigma_i} \right) = \frac{1}{\sigma_i} \sum_{t=0}^{i-1} r(t) + \sum_{t=1}^{l-i} r(t) | < \frac{2}{\sigma_i} \sum_{t=0}^{l-1} r(t).
$$
Since in addition, by (a2),
\[ \sigma_i = \sqrt{1 + 2 \sum_{t=1}^{l} (l-t) r(t)} \geq l^{1/2}, \]
we have
\[ \left| \text{Cov} \left( X_i, \frac{S_i}{\sigma_i} \right) \right| \leq \frac{2}{l^{1/2}} \sum_{t=0}^{l-1} |r(t)| \quad \text{for all } 1 \leq i \leq l. \]
This together with (a1) implies that
\[ (2) \sup_{1 \leq i \leq l} \left| \text{Cov} \left( X_i, \frac{S_i}{\sigma_i} \right) \right| \leq \frac{(\log l)^{1/2}}{l^{1/2} (\log \log l)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0. \]
Since
\[ \lim_{l \to \infty} \frac{(\log l)^{1/2}}{l^{1/2} (\log \log l)^{1+\varepsilon}} = 0, \]
by (2) there exist numbers \( \lambda \) and \( l_0 \) such that
\[ (3) \sup_{1 \leq i \leq l} \left| \text{Cov} \left( X_i, \frac{S_i}{\sigma_i} \right) \right| < \lambda < 1 \quad \text{for all } l > l_0. \]
Let us recall now the following property, proved in Subsection 4.3 of Leadbetter et al. [4]. It states that if \( \delta(k) \to 0 \), then \( |r(k)| < 1 \) for all \( k \geq 1 \). Consequently, as (a3) is satisfied, we can write the relation
\[ (4) \sup_{l \geq 1} |r(l)| = \delta < 1. \]
Properties (2)–(4) will be intensively used in the following step of our proof.
Let \( y \) be an arbitrary real number and \( k < l \). We have
\[ E[I(M_l \leq u_i, S_l/\sigma_l \leq y) - I(M_{k,l} \leq u_i, S_l/\sigma_l \leq y)] = P(M_{k,l} \leq u_i, S_l/\sigma_l \leq y) - P(M_l \leq u_i, S_l/\sigma_l \leq y). \]
Let in addition \( Y_l \) be a random variable which has the same distribution as \( S_l/\sigma_l \) but is independent of \((X_1, \ldots, X_l)\). We can write that
\[ (5) E[I(M_l \leq u_i, S_l/\sigma_l \leq y) - I(M_{k,l} \leq u_i, S_l/\sigma_l \leq y)] \leq |P(M_l \leq u_i, S_l/\sigma_l \leq y) - P(M_l \leq u_i) P(Y_l \leq y)| + |P(M_{k,l} \leq u_i, S_l/\sigma_l \leq y) - P(M_{k,l} \leq u_i) P(Y_l \leq y)| + (P(M_{k,l} \leq u_i) - P(M_l \leq u_i)) =: A_1 + A_2 + A_3. \]
We now estimate all the components \( A_1, A_2, A_3 \) in (5).
As $Y_i$ is independent of $(X_1, \ldots, X_l)$, we have

$$A_1 = \left| P(X_1 \leq u_i, \ldots, X_l \leq u_i, S_l/\sigma_l \leq y) - P(X_1 \leq u_i, \ldots, X_l \leq u_i, Y_i \leq y) \right|.$$ 

Since $(X_1, \ldots, X_l, S_l/\sigma_l)$ as well as $(X_1, \ldots, X_l, Y_i)$ are standard normal vectors and conditions (3), (4) are satisfied, applying Theorem 4.2.1 in [4] (the so-called Normal Comparison Lemma) we obtain

$$A_1 \leq \sum_{i=1}^{l} \left| \text{Cov} \left( X_i, \frac{S_l}{\sigma_l} \right) \exp \left( -\frac{u_i^2 + y^2}{2(1 + \text{Cov}(X_i, S_l/\sigma_l))} \right) \right| < \sum_{i=1}^{l} \left| \text{Cov} \left( X_i, \frac{S_l}{\sigma_l} \right) \exp \left( -\frac{u_i^2}{2(1 + \lambda)} \right) \right|,$$

where $\lambda$ is such as in (3). From (6) and (2) we get

$$A_1 \leq l^{1/2} \frac{(\log l)^{1/2}}{l^{1/2}} \exp \left( -\frac{u_i^2}{2(1 + \lambda)} \right) = \frac{l^{1/2} (\log l)^{1/2}}{(\log \log l)^{1/2}} \exp \left( -\frac{u_i^2}{2(1 + \lambda)} \right) = \frac{l^{1/2} (\log l)^{1/2}}{(\log \log l)^{1/2}} \exp \left( -\frac{u_i^2}{2(1 + \lambda)} \right).$$

As the sequence $(u_k)$ satisfies assumption (a4), by relations (4.3.4 (i)) and (4.3.4 (ii)) in [4] we obtain

$$\exp \left( -\frac{u_i^2}{2(1 + \lambda)} \right) \sim K \frac{(\log l)^{1/2(1 + \lambda)}}{l^{1/2(1 + \lambda)}}.$$

Using (7) and (8), we have

$$A_1 \leq \frac{l^{1/2} (\log l)^{1/2}}{(\log \log l)^{1/2}} \frac{(\log l)^{1/2(1 + \lambda)}}{l^{1/2(1 + \lambda)}} = \frac{(\log l)^{1/2(1 + \lambda)}}{(\log \log l)^{1/2(1 + \lambda)}}.$$ 

Since $0 < \lambda < 1$, we have $1/(1 + \lambda) - \frac{1}{2} > 0$. Hence

$$(\log l)^{1/2(1 + \lambda)} \leq l^{1/(1 + \lambda) - 1/2}.$$ 

This together with (9) implies that

$$A_1 \leq \frac{1}{(\log \log l)^{1 + \epsilon}} \quad \text{for some } \epsilon > 0.$$

We now give the bound for the component $A_2$ in (5). Since $Y_i$ is independent of $(X_{k+1}, \ldots, X_l)$, we obtain

$$A_2 = \left| P(X_{k+1} \leq u_i, \ldots, X_l \leq u_i, S_l/\sigma_l \leq y) - P(X_{k+1} \leq u_i, \ldots, X_l \leq u_i, Y_i \leq y) \right|.$$ 

Applying Theorem 4.2.1 in [4] again and arguing as in the estimation of $A_1$, we have

$$A_2 \leq \frac{1}{(\log \log l)^{1 + \epsilon}} \quad \text{for some } \epsilon > 0.$$
Thus, it remains to estimate the last term $A_3$ in (5). It is easy to check that (see also the first lines in the proof of Lemma 2.4 from the paper of Csaki and Gonchigdanzan [1])

\begin{equation}
A_3 \leq |P(M_l \leq u_l) - \Phi^l(u_l)| + |P(M_{k,l} \leq u_l) - \Phi^{l-k}(u_l)| + (\Phi^{l-k}(u_l) - \Phi^l(u_l)) =: B_1 + B_2 + B_3.
\end{equation}

Since the covariance function $r(k)$ satisfies (4), by Theorem 4.2.1 in [4] we obtain

\begin{equation}
B_1 \leq \sum_{1 \leq i < j \leq l} |r(j-i)| \exp\left(-\frac{u_i^2}{1 + |r(j-i)|}\right) \leq l \sum_{i=1}^{l-1} |r(i)| \exp\left(-\frac{u_i^2}{1 + \delta}\right) \leq l \exp\left(-\frac{u_i^2}{1 + \delta}\right) \sum_{i=0}^{l-1} |r(i)|,
\end{equation}

where $\delta$ is such as in (4). It follows from (13), (8) and (a1) that

\begin{equation}
B_1 \leq l \frac{(\log l)^{1/(1+\delta)}}{l^{2/(1+\delta)}} \frac{(\log l)^{1/2}}{(\log \log l)^{1+\delta}} = \frac{(\log l)^{1/(1+\delta)+1/2}}{l^{2/(1+\delta)-1}} \frac{(\log \log l)^{1+\delta}}{l}. \quad \text{(14)}
\end{equation}

Since, by property (4), $0 \leq \delta < 1$, we obtain $2/(1+\delta) - 1 > 0$. Consequently, we have $(\log l)^{1/(1+\delta)+1/2} \leq l^{2/(1+\delta)-1}$ and

\begin{equation}
B_1 \leq \frac{1}{(\log \log l)^{1+\delta}} \quad \text{for some } \varepsilon > 0.
\end{equation}

Using similar methods to those in the estimation of $B_1$, we can check that

\begin{equation}
B_2 \leq \frac{1}{(\log \log l)^{1+\delta}} \quad \text{for some } \varepsilon > 0.
\end{equation}

In addition, from the estimation of $D_3$ in the proof of Lemma 2.4 in [1] we obtain the following bound for $B_3$ in (12):

\begin{equation}
B_3 \leq k/l. \quad \text{(16)}
\end{equation}

By (12) and (14)–(16) we have

\begin{equation}
A_3 \leq \frac{1}{(\log \log l)^{1+\delta} + \frac{k}{l}} \quad \text{for some } \varepsilon > 0.
\end{equation}

Relations (5), (10), (11) and (17) establish the assertion of Lemma 1. \hfill \square
The following lemma will be also needed in the proof of our main result.

**Lemma 2.** Let \( X_1, X_2, \ldots \) be a standardized stationary Gaussian process satisfying assumptions (a1)–(a3). Suppose moreover that condition (a4) holds for the numerical sequence \((u_k)\). Then there exist positive numbers \( \gamma \) and \( \varepsilon \) such that if

\[
 k < \frac{\gamma l (\log \log l)^{2+2\varepsilon}}{\log l} \quad \text{and} \quad k < l,
\]

then

\[
|\text{Cov}(I(M_k \leq u_k, S_k/\sigma_k \leq y), I(M_{k+l} \leq u_l, S_l/\sigma_l \leq y))| \leq \frac{k^{1/2} (\log l)^{1/2}}{l^{1/2} (\log \log l)^{1+\varepsilon}}
\]

for all \( y \in (-\infty, \infty) \).

**Proof.** Similarly to the proof of Lemma 1, we will begin with some observations.

Let \( i \geq k + 1 \). By assumptions (a1) and (a2) we obtain

\[
|\text{Cov}(X_i, \frac{S_k}{\sigma_k})| \leq \frac{1}{\sigma_k \sigma_i} \sum_{t=1}^{i-1} |r(t)| = \frac{\sum_{t=1}^{i-1} |r(t)|}{\sqrt{k + 2 \sum_{t=1}^{k} (k-t) r(t)}} \leq \frac{(\log k)^{1/2}}{k^{1/2} (\log \log k)^{1+\varepsilon}}.
\]

Since in addition

\[
\lim_{k \to \infty} \frac{(\log k)^{1/2}}{k^{1/2} (\log \log k)^{1+\varepsilon}} = 0,
\]

there exist numbers \( \mu \) and \( k_0 \) such that

\[
(19) \quad \sup_{i \geq k+1} |\text{Cov}(X_i, S_k/\sigma_k)| < \mu < 1 \quad \text{for all } k > k_0.
\]

We now estimate \(|\text{Cov}(S_k/\sigma_k, S_l/\sigma_l)|\), where \( k < l \). Using (a2), we have

\[
|\text{Cov}\left(\frac{S_k}{\sigma_k}, \frac{S_l}{\sigma_l}\right)| = \left| \frac{1}{\sigma_k \sigma_l} \left( \sigma_k^2 + \text{Cov}(X_1 + \ldots + X_k, X_{k+1} + \ldots + X_l) \right) \right|
\]

\[
= \left| \frac{1}{\sigma_k \sigma_l} \left( \sum_{t=k}^{l-1} r(t) + \sum_{t=k-1}^{l-2} r(t) + \ldots + \sum_{t=1}^{l-k} r(t) \right) \right|
\]

\[
< \frac{\sigma_k^2 + k \sum_{t=0}^{l-1} |r(t)|}{\sigma_k \sigma_l} \leq \frac{k + 2 \sum_{t=1}^{k} (k-t) r(t) + k \sum_{t=0}^{l-1} |r(t)|}{k^{1/2} l^{1/2}}
\]

\[
\leq \frac{k^{1/2}}{l^{1/2}} + \frac{2k}{k^{1/2} l^{1/2}} \sum_{t=1}^{k} |r(t)| + \frac{k^{1/2} l^{1/2}}{l^{1/2}} \sum_{t=0}^{l-1} |r(t)| < \frac{k^{1/2}}{l^{1/2}} + 3 \frac{k^{1/2} l^{1/2}}{l^{1/2}} \sum_{t=0}^{l-1} |r(t)|.
\]
This and assumption (a1) imply that

\[
(20) \quad \left| \text{Cov} \left( \frac{S_k}{\sigma_k}, \frac{S_l}{\sigma_l} \right) \right| \leq k^{1/2} \frac{(\log l)^{1/2}}{l^{1/2} (\log \log l)^{1/2} + \epsilon} \quad \text{for some } \epsilon > 0.
\]

By (20), there exist numbers \( C \) and \( l_1 \) such that

\[
\left| \text{Cov} \left( \frac{S_k}{\sigma_k}, \frac{S_l}{\sigma_l} \right) \right| \leq C \frac{k^{1/2} (\log l)^{1/2}}{l^{1/2} (\log \log l)^{1/2} + \epsilon} \quad \text{for all } l > k > l_1.
\]

Let \( q \) be a fixed real number satisfying the condition \( 0 < q < 1 \). Let in addition \( \gamma = (q/C)^2 \), where the constant \( C \) is defined in the inequality above. Then

\[
(21) \quad \left| \text{Cov} \left( \frac{S_k}{\sigma_k}, \frac{S_l}{\sigma_l} \right) \right| < q < 1 \quad \text{if } k < \frac{\gamma l (\log \log l)^{2+2\epsilon}}{\log l} \quad \text{and } l_1 < k < l.
\]

We will apply properties (19)–(21) in the following step of our proof.

Let \( y \) be an arbitrary real number and \( k < l \). We have

\[
|\text{Cov}(I(M_k \leq u_k, S_k/\sigma_k \leq y), I(M_{k,l} \leq u_l, S_l/\sigma_l \leq y))|
\]

\[
= |P(X_1 \leq u_k, \ldots, X_k \leq u_k, S_k/\sigma_k \leq y, X_{k+1} \leq u_l, \ldots, X_l \leq u_l, S_l/\sigma_l \leq y) - P(X_1 \leq u_k, \ldots, X_k \leq u_k, S_k/\sigma_k \leq y) | P(X_{k+1} \leq u_l, \ldots, X_l \leq u_l, S_l/\sigma_l \leq y)|.
\]

Let moreover \((\tilde{X}_{k+1}, \ldots, \tilde{X}_l, \tilde{Y}_l)\) be a random vector which has the same distribution as \((X_{k+1}, \ldots, X_l, S_l/\sigma_l)\) but is independent of \((X_1, \ldots, X_k, S_k/\sigma_k)\). Then

\[
|\text{Cov}(I(M_k \leq u_k, S_k/\sigma_k \leq y), I(M_{k,l} \leq u_l, S_l/\sigma_l \leq y))|
\]

\[
= |P(X_1 \leq u_k, \ldots, X_k \leq u_k, S_k/\sigma_k \leq y, X_{k+1} \leq u_l, \ldots, X_l \leq u_l, S_l/\sigma_l \leq y) - P(X_1 \leq u_k, \ldots, X_k \leq u_k, S_k/\sigma_k \leq y, \tilde{X}_{k+1} \leq u_l, \ldots, \tilde{X}_l \leq u_l, \tilde{Y}_l \leq y)|
\]

\[
= |P(X_1 \leq u_k, \ldots, X_k \leq u_k, X_{k+1} \leq u_l, \ldots, X_l \leq u_l, S_k/\sigma_k \leq y, S_l/\sigma_l \leq y) - P(X_1 \leq u_k, \ldots, X_k \leq u_k, \tilde{X}_{k+1} \leq u_l, \ldots, \tilde{X}_l \leq u_l, S_k/\sigma_k \leq y, \tilde{Y}_l \leq y)|.
\]

Since \((X_1, \ldots, X_k, X_{k+1}, \ldots, X_l, S_k/\sigma_k, S_l/\sigma_l)\) and \((X_1, \ldots, X_k, \tilde{X}_{k+1}, \ldots, \tilde{X}_l, S_k/\sigma_k, \tilde{Y}_l)\) are standard normal vectors and conditions (3), (4), (19) and (21) are satisfied, applying Theorem 4.2.1 in Leadbetter et al. [4] we can write

\[
(22) \quad |\text{Cov}(I(M_k \leq u_k, S_k/\sigma_k \leq y), I(M_{k,l} \leq u_l, S_l/\sigma_l \leq y))|
\]

\[
\leq \sum_{i=1}^k \left( \sum_{j=k+1}^l |r(j-i)| \exp \left( -\frac{u_k^2 + u_l^2}{2(1 + |r(j-i)|)} \right) + \sum_{i=1}^k \text{Cov} \left( X_i, \frac{S_i}{\sigma_i} \right) \exp \left( -\frac{u_k^2 + y^2}{2(1 + |\text{Cov}(X_i, S_i/\sigma_i)|)} \right) 
\]

\[
+ \sum_{j=k+1}^l \text{Cov} \left( X_i, \frac{S_k}{\sigma_k} \right) \exp \left( -\frac{u_l^2 + y^2}{2(1 + |\text{Cov}(X_i, S_k/\sigma_k)|)} \right) + \right)
\]
We now estimate all the components $D_1$, $D_2$, $D_3$, $D_4$ in (22).

Using the notation on $\delta$ in (4), we obtain the following bounds for $D_1$:

\begin{equation}
D_1 \leq k \sum_{t=1}^{l-1} |r(t)| \exp\left(-\frac{u^2_k + u^2_i}{2(1 + |r(t)|)}\right) < k \exp\left(-\frac{u^2_k + u^2_i}{2(1 + \delta)}\right) \sum_{t=0}^{l-1} |r(t)|.
\end{equation}

By (23), (8) and assumption (a1), for some $\varepsilon > 0$ we have

\begin{align*}
D_1 &\leq k \frac{(\log k)^{1/2(1+\delta)} (\log l)^{1/2(1+\delta)}}{k^{1/(1+\delta)} l^{1/(1+\delta)}} \frac{(\log l)^{1/2}}{(\log \log l)^{1+\varepsilon}} \\
&\leq k \frac{(\log l)^{1/(1+\delta)}}{k^{1/(1+\delta)} l^{1/(1+\delta)}} \frac{(\log l)^{1/2}}{(\log \log l)^{1+\varepsilon}} = \frac{k^{1-1/(1+\delta)} (\log l)^{1/(1+\delta)+1/2}}{l^{1/(1+\delta)} (\log \log l)^{1+\varepsilon}}.
\end{align*}

Since, by (4), $0 \leq \delta < 1$, we obtain $1 - 1/(1+\delta) < \frac{1}{2}$ and $1/(1+\delta) = \frac{1}{2} + \alpha$ for some $\alpha > 0$. Therefore

\begin{equation}
D_1 \leq \frac{k^{1/2} (\log l)^{1/(1+\delta)+1/2}}{l^{1/2} (\log \log l)^{1+\varepsilon}} \leq \frac{k^{1/2}}{l^{1/2} (\log \log l)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0.
\end{equation}

We now estimate the component $D_2$. Using its definition in (22) and the notation on $\lambda$ in (3), we have

\begin{equation}
D_2 < \exp\left(-\frac{u^2_k}{2(1 + \lambda)}\right) \sum_{i=1}^{k} \left|\text{Cov}\left(X_i, \frac{S_i}{\sigma_i}\right)\right|.
\end{equation}

It follows from (25), (8) and (2) that for some $\varepsilon > 0$

\begin{align*}
D_2 &\leq \frac{(\log k)^{1/2(1+\lambda)}}{k^{1/(1+\lambda)} l^{1/2}} \frac{(\log l)^{1/2}}{(\log \log l)^{1+\varepsilon}} = \frac{k^{1-1/(1+\lambda)} (\log k)^{1/2(1+\lambda)} (\log l)^{1/2}}{l^{1/2} (\log \log l)^{1+\varepsilon}}.
\end{align*}

Since $0 < \lambda < 1$, we have $1/(1+\lambda) - \frac{1}{2} > 0$. Hence $(\log k)^{1/2(1+\lambda)} \leq k^{1/(1+\lambda)-1/2}$ and

\begin{equation}
D_2 \leq \frac{k^{1/2} (\log l)^{1/2}}{l^{1/2} (\log \log l)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0.
\end{equation}

We now estimate the component $D_3$. From its definition in (22) and the notation on $\mu$ in (19) we obtain

\begin{equation}
D_3 \leq \exp\left(-\frac{u^2_k}{2(1 + \mu)}\right) \sum_{i=k+1}^{l} \left|\text{Cov}\left(X_i, \frac{S_i}{\sigma_i}\right)\right|.
\end{equation}
Let us observe that

\[ \sum_{i=k+1}^{l} \left| \text{Cov} \left( X_i, \frac{S_k}{\sigma_k} \right) \right| \leq \frac{1}{\sigma_k} \sum_{i=k+1}^{l} \left| r(i-1) + r(i-2) + \ldots + r(i-k) \right| \]

\[ \leq \frac{1}{\sigma_k} \left( \sum_{i=k+1}^{l} \left| r(i-1) \right| + \sum_{i=k+1}^{l} \left| r(i-2) \right| + \ldots + \sum_{i=k+1}^{l} \left| r(i-k) \right| \right) \]

\[ = \frac{1}{\sigma_k} \left( \sum_{i=k-1}^{l-k} \left| r(i-1) \right| + \sum_{i=k-2}^{l-k} \left| r(i-2) \right| + \ldots + \sum_{i=k-k}^{l-k} \left| r(i-k) \right| \right) \]

\[ = \frac{1}{\sigma_k} \left( \sum_{t=k}^{l-1} \left| r(t) \right| + \sum_{t=k-1}^{l-2} \left| r(t) \right| + \ldots + \sum_{t=1}^{l-k} \left| r(t) \right| \right) \leq \frac{k}{\sigma_k} \sum_{t=0}^{l-1} \left| r(t) \right| \]

\[ = \frac{k}{\sqrt{k+2} \sum_{t=1}^{k} (k-t) r(t)} \sum_{t=0}^{l-1} \left| r(t) \right| . \]

By assumptions (a1) and (a2) we have

\[ \sum_{i=k+1}^{l} \left| \text{Cov} \left( X_i, \frac{S_k}{\sigma_k} \right) \right| \leq \frac{k^{1/2} (\log l)^{1/2}}{(\log \log l)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0. \quad (28) \]

From (27), (8) and (28) we obtain

\[ D_3 \leq \frac{(\log l)^{1/2(1+\mu)}}{l^{1/(1+\mu)}} \frac{k^{1/2} (\log l)^{1/2}}{(\log \log l)^{1+\varepsilon}} = \frac{k^{1/2} (\log l)^{1/2(1+\mu)+1/2}}{l^{1/(1+\mu)} (\log \log l)^{1+\varepsilon}}. \]

Since \( 0 < \mu < 1 \), we have \( 1/(1+\mu) > \frac{1}{2} \). Hence \( 1/(1+\mu) = \frac{1}{2} + \beta \) for some \( \beta > 0 \). This yields that

\[ D_3 \leq \frac{k^{1/2} (\log l)^{1/2(1+\mu)+1/2}}{l^{1/2} \beta (\log \log l)^{1+\varepsilon}} \leq \frac{k^{1/2}}{l^{1/2} (\log \log l)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0. \quad (29) \]

Thus, it remains to estimate the last term \( D_4 \) in (22). Obviously, we have

\[ D_4 \leq |\text{Cov} (S_k/\sigma_k, S_l/\sigma_l)|. \]

This and (20) imply the following property:

\[ D_4 \leq \frac{k^{1/2} (\log l)^{1/2}}{l^{1/2} (\log \log l)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0. \quad (30) \]

From (22), (24), (26), (29), (30) we infer that if

\[ k < \frac{\gamma l (\log \log l)^{2+2\varepsilon}}{\log l} \quad \text{and} \quad k < l, \]
then

\[ |\text{Cov} \left( I(M_k \leq u_k, S_k/\sigma_k \leq y), I(M_{k,l} \leq u_l, S_l/\sigma_l \leq y) \right)| \leq \frac{k^{1/2} (\log l)^{1/2}}{l^{1/2} (\log \log l)^{1+\varepsilon}} \]

for all \( y \in (-\infty, \infty) \) and some \( \varepsilon > 0 \). This completes the proof of Lemma 2. ■

In the proof of our main result we will also apply the following lemma.

**Lemma 3.** Let \( X_1, X_2, \ldots \) be a standardized stationary Gaussian process satisfying assumptions (a1)–(a3). Suppose moreover that condition (a4) holds for the numerical sequence \((u_k)\). Then

\[ \lim_{k \to \infty} P(M_k \leq u_k, S_k/\sigma_k \leq y) = e^{-\tau} \Phi(y) \]

for all \( y \in (-\infty, \infty) \) and some \( \tau \in [0, \infty) \).

**Proof.** Let \( y \) be an arbitrary real number and let, for each natural \( k \), \( Y_k \) denote the random variable which has the same distribution as \( S_k/\sigma_k \) but is independent of \((X_1, \ldots, X_k)\). From the estimation of \( A_1 \) in the proof of Lemma 1 we have

\[ |P(M_k \leq u_k, S_k/\sigma_k \leq y) - P(M_k \leq u_k) P(Y_k \leq y)| \leq \frac{1}{(\log \log k)^{1+\varepsilon}} \]

for some \( \varepsilon > 0 \). This property and the fact that

\[ \lim_{k \to \infty} \frac{1}{(\log \log k)^{1+\varepsilon}} = 0 \]

imply the following relation:

(31) \[ \lim_{k \to \infty} P(M_k \leq u_k, S_k/\sigma_k \leq y) = \lim_{k \to \infty} P(M_k \leq u_k) P(Y_k \leq y). \]

As \( X_1, X_2, \ldots \) is a standard normal process, the covariance function \( r(k) \) and the sequence \((u_k)\) satisfy assumptions (a3) and (a4), respectively, by Theorem 4.3.3 in Leadbetter et al. [4] we have

(32) \[ \lim_{k \to \infty} P(M_k \leq u_k) = e^{-\tau} \quad \text{for some } \tau, \ 0 \leq \tau < \infty. \]

Since in addition \( Y_k \)'s have the standard normal distribution, from (31) and (32) we obtain

\[ \lim_{k \to \infty} P(M_k \leq u_k, S_k/\sigma_k \leq y) = e^{-\tau} \Phi(y) \]

for all \( y \in (-\infty, \infty) \) and some \( \tau \in [0, \infty) \). This completes the proof of Lemma 3. ■
5. PROOF OF THE MAIN RESULT

We now give the proof of Theorem 1. It makes an extensive use of the
results in Lemmas 1–3.

Proof of Theorem 1. The idea of this proof is similar to that of Theo-
rem 1.1 in Csaki and Gonchigdanzan [1].

From Lemma 3 we infer that if \((u_k)\) satisfies (a4) with some \(t \in [0, \infty)\), then

\[
\lim_{k \to \infty} P(M_k \leq u_k, S_k/\sigma_k \leq y) = e^{-t} \Phi(y) \quad \text{for all} \ y \in (-\infty, \infty).
\]

Hence, arguing as in the proof of Theorem 1.1 (i) in [1], in order to prove part
(i) of Theorem 1, it is enough to show that

\[
\text{Var}\left(\sum_{k=1}^{n} \frac{1}{k} I(M_k \leq u_k, S_k/\sigma_k \leq y)\right) \leq \frac{(\log n)^2}{(\log \log n)^{1+\varepsilon}}
\]

for all \(y \in (-\infty, \infty)\) and some \(\varepsilon > 0\).

Let \(\xi_k = I(M_k \leq u_k, S_k/\sigma_k \leq y) - P(M_k \leq u_k, S_k/\sigma_k \leq y)\). We have

\[
\text{Var}\left(\sum_{k=1}^{n} \frac{1}{k} I(M_k \leq u_k, \frac{S_k}{\sigma_k} \leq y)\right) = \text{E}\left(\sum_{k=1}^{n} \frac{1}{k} \xi_k^2\right)
\leq \sum_{k=1}^{n} \frac{1}{k^2} E\xi_k^2 + 2 \sum_{1 \leq k < l \leq n} \frac{1}{kl} |E(\xi_k \xi_l)| =: F_1 + F_2.
\]

Since \(\xi_k\)'s are bounded, we get

\[
F_1 \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.
\]

We now estimate the component \(F_2\) in (34). Using similar methods to
those in the estimation of \(|E(\eta_k, \eta_l)|\) in [1], it is easy to check that

\[
|E(\xi_k \xi_l)| \leq E|I(M_i \leq u_i, S_i/\sigma_i \leq y) - I(M_k \leq u_k, S_k/\sigma_k \leq y)| + |\text{Cov}(I(M_k \leq u_k, S_k/\sigma_k \leq y), I(M_k \leq u_k, S_k/\sigma_k \leq y))|.
\]

Lemmas 1 and 2 imply that for all natural \(k\) and \(l\) such that

\[
k < \frac{\gamma l (\log \log l)^{2+2\varepsilon}}{\log l} \quad \text{and} \quad k < l
\]

as well as for all \(y \in (-\infty, \infty)\) and some \(\varepsilon > 0\) we have

\[
E|I(M_i \leq u_i, \frac{S_i}{\sigma_i} \leq y) - I(M_k \leq u_k, \frac{S_k}{\sigma_k} \leq y)| \leq \frac{1}{(\log \log l)^{1+\varepsilon}} + \frac{k}{l},
\]

\[
|\text{Cov}(I(M_k \leq u_k, \frac{S_k}{\sigma_k} \leq y), I(M_k \leq u_k, \frac{S_k}{\sigma_k} \leq y))| \leq \frac{k^{1/2} (\log \log l)^{1/2}}{l^{1/2} (\log \log l)^{1+\varepsilon}}.
\]
Consequently, we infer that if $k < \gamma l (\log \log l)^{2+2\varepsilon} / (\log l)$ and $k < l$, then

$$|E(\xi_k^* \xi_l^*)| \ll \frac{1}{(\log \log l)^{1+\varepsilon}} + \frac{k^{1/2} (\log l)^{1/2}}{l^{1/2} (\log \log l)^{1+\varepsilon}}$$

for some $\varepsilon > 0$.

Hence

$$F_2 \ll \sum_{1 \leq k < l \leq n, \ k < \gamma l (\log \log l)^{2+2\varepsilon} / (\log l)} \frac{1}{kl (\log \log l)^{1+\varepsilon}} + \sum_{1 \leq k < l \leq n, \ k < \gamma l (\log \log l)^{2+2\varepsilon} / (\log l)} \frac{k^{1/2} (\log l)^{1/2}}{l^{1/2} (\log \log l)^{1+\varepsilon}} + \sum_{1 \leq k < l \leq n, \ k \geq \gamma l (\log \log l)^{2+2\varepsilon} / (\log l)} \frac{1}{kl}$$

$$= G_1 + G_2 + G_3.$$  

Let us note that

$$G_1 \ll \sum_{i=3}^{n} \frac{1}{i (\log \log l)^{1+\varepsilon}} \sum_{k=1}^{l-1} \frac{1}{k} \ll \sum_{i=3}^{n} \frac{\log l}{(\log \log l)^{1+\varepsilon}}.$$  

Since $f(t) = (\log \log l) / (\log \log t)^{1+\varepsilon}$ is an increasing function for sufficiently large $t$, we obtain

$$G_1 \ll \frac{\log n}{(\log \log n)^{1+\varepsilon}} \sum_{i=1}^{n} \frac{1}{i} \ll \frac{(\log n)^2}{(\log \log n)^{1+\varepsilon}}$$

for some $\varepsilon > 0$.

We have the following estimates for $G_2$:

$$G_2 \ll \sum_{k=2}^{n} \sum_{l=k+1}^{n} \frac{k^{1/2} (\log l)^{1/2}}{kl^{1/2} (\log \log l)^{1+\varepsilon}} \ll \frac{(\log n)^{1/2}}{(\log \log n)^{1+\varepsilon}} \sum_{k=1}^{n-1} \frac{1}{k^{1/2}} \sum_{l=k+1}^{\infty} \frac{1}{l^{3/2}}$$

$$\ll \frac{(\log n)^{1/2}}{(\log \log n)^{1+\varepsilon}} \sum_{k=1}^{n-1} \frac{1}{k} \ll \frac{(\log n)^{3/2}}{(\log \log n)^{1+\varepsilon}}$$

for some $\varepsilon > 0$.

To estimate $G_3$ in (36), let us note that, since $k \geq \gamma l (\log \log l)^{2+2\varepsilon} / (\log l)$, we have

$$\frac{1}{kl} \ll \frac{\log l}{\gamma l^2 (\log \log l)^{2+2\varepsilon}}.$$  

Therefore, we can write that

$$G_3 \ll \sum_{1 \leq k < l \leq n, \ \gamma l^2 (\log \log l)^{2+2\varepsilon}} \frac{\log l}{(\log \log n)^{2+2\varepsilon} \sum_{k=1}^{n-1} \frac{1}{k}} \ll \frac{(\log n)^2}{(\log \log n)^{2+2\varepsilon}}$$

for some $\varepsilon > 0$. 

Maxima and sums of stationary Gaussian sequences
From (36)–(39) we obtain

\[ F_2 \leq \frac{(\log n)^2}{(\log \log n)^{1+\varepsilon}} \text{ for some } \varepsilon > 0. \]

Relations (34), (35) and (40) imply that condition (33) holds for all \( y \in (-\infty, \infty) \) and some \( \varepsilon > 0 \). Consequently, the assertion (i) of Theorem 1 is fulfilled.

In order to prove Theorem 1 (ii), let us observe that, by Theorem 4.3.3 (ii) in Leadbetter et al. [4],

\[ \lim_{k \to \infty} P(M_k \leq x/a_k + b_k) = \exp(-e^{-x}). \]

This together with Theorem 4.3.3 (i) in [4] implies that

\[ \lim_{k \to \infty} k(1 - \Phi(x/a_k + b_k)) = e^{-x}. \]

Thus, it is easily seen that the assertion (ii) of Theorem 1 is a special case of the assertion (i) of that theorem with \( u_k = x/a_k + b_k, \tau = e^{-x} \).

Acknowledgement. I wish to thank Professor W. Dziubdziela for his comments and support.

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