SUMS OF DUFRESNE RANDOM VARIABLES

BY

JEAN-FRANÇOIS CHAMAYOU (TOULOUSE)

Abstract. We study particular cases of sums of independent Dufresne random variables (which are essentially products of beta and gamma variates) such that the distribution of the sum is again a Dufresne random variable.

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1. INTRODUCTION

The gamma distribution with scale parameter 1 and shape parameter $c > 0$ is the probability on $(0, \infty)$ defined by

$$
\gamma_c(dx) = \frac{1}{\Gamma(c)} x^{c-1} e^{-x} 1_{(0,\infty)}(x) dx.
$$

The beta distribution with parameters $a, b > 0$ is the probability on $(0, 1)$ defined by

$$
\beta_{a,b}(dx) = \frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1} 1_{(0,1)}(x) dx,
$$

where $B(a, b) = \Gamma(a) \Gamma(b)/\Gamma(a + b)$.

Let us denote by $\odot$ and $\ast$ the multiplicative and additive convolutions, respectively, of probability distributions on $\mathbb{R}$; this means that if $X$ and $Y$ are independent variables with distributions $\alpha$ and $\beta$, then the distributions of $XY$ and $X + Y$ are denoted by $\alpha \odot \beta$ and $\alpha \ast \beta$. Recall that the Laplace transform of a probability distribution $\alpha$ on $\mathbb{R}$ is

$$
L(\alpha)(s) = \int_{\mathbb{R}} e^{sx} \alpha(dx).
$$
It satisfies \( L(\alpha \ast \beta)(s) = L(\alpha)(s) L(\beta)(s) \). Similarly, the Mellin transform of a probability distribution \( \alpha \) defined on \([0, \infty)\) is

\[
M(\alpha)(s) = \int_{[0, \infty)} x^s \alpha(dx)
\]

and satisfies \( M(\alpha \circ \beta)(s) = M(\alpha)(s) M(\beta)(s) \).

The Pochhammer symbol is defined for \( a, s > 0 \) by

\[
(a)_s = \frac{\Gamma(a + s)}{\Gamma(a)}.
\]

and satisfies \( (a \circ \beta)_s = (a)_s (\beta)_s \).

Similarly, the Mellin transform of a probability distribution \( \alpha \) defined on \([0, \infty)\) is

\[
M(\alpha)(s) = \int_{[0, \infty)} x^s \alpha(dx)
\]

and satisfies \( M(\alpha \circ \beta)(s) = M(\alpha)(s) M(\beta)(s) \).

The aim of this paper is to give some unexpected properties of additive and multiplicative convolutions of the \( D(\alpha, \beta) \) laws. We shall use the following classical notation for the hypergeometric functions defined for \(-1 < s < 1\):

\[
pF_q(a; b; s) = \sum_{n=0}^{\infty} \frac{(a)_n s^n}{(b)_n n!}.
\]

These hypergeometric functions arise naturally as Laplace transforms of the Dufresne laws: for \(-1 < s < 1\) we obtain

\[
L(D(\alpha; \beta))(s) = pF_q(a; b; s).
\]

The double gamma distribution \( \gamma_c \) with parameter \( c > 0 \) is the probability on \( \mathbb{R} \) with Laplace transform \( L(\xi_c)(s) = (1 - s^2)^{-c} \) defined for \( s \in (-1, 1) \). If \( \tilde{\gamma}_c \) is the image of \( \gamma_c \) by \( x \mapsto -x \), we clearly have

\[
\lambda_c = \gamma_c \ast \tilde{\gamma}_c,
\]

and this proves the existence of \( \lambda_c \). It is easily seen that \( \lambda_1(dx) = e^{-|x|} dx / 2 \), which has various names in the literature: double exponential, bilateral exponential, Laplace distribution of first kind. If \( c \) is an integer, the density of

\[
L(D(\alpha; \beta); c) = \sum_{n=0}^{\infty} \frac{(a)_n s^n}{(b)_n n!}.
\]

* When writing \( D(\alpha; \beta) \) instead of \( \mathcal{D}(\alpha; \beta) \) we depart from the tradition of hypergeometric functions, which furthermore displays, respectively, by \( p \) and \( q \) the lengths of the sequences \( a \) and \( b \). Although there is some redundancy in the traditional notation, we still keep it for convenience of the reader when dealing with the hypergeometric functions \( pF_q \) or the Kampé de Fériet functions \( F^D_{k,p,q} \) in Section 4.
\( \lambda_c \) can be obtained by expansion into partial fractions of \( s \mapsto (1-s^2)^{-c} \). It is easily checked that if \( n \) is a non-negative integer, we have

\[
\lambda_{n+1}(dx) = e^{-|x|} \sum_{k=0}^{n} \frac{(2n-k)!}{n! (n-k)! 2^{n-2k+1}|x|^k} dx.
\]

If \( c \) is not an integer, the density is obtained by using the Mac Donald functions (modified Bessel functions of second kind). The lambda law \( \lambda_c(a; b) = \lambda_c \circ D(a; b) \) or bilateral Dufresne laws are introduced in Chamayou (2004). However, in the present paper we shall mainly use the laws \( \lambda_c(a; b) = \lambda_c \circ \beta_{a,b-a} \) which are the multiplicative convolutions of double gammas with parameter \( c \) by betas with parameters \( (a, b-a) \).

The symmetrical \textit{beta} distribution of third kind with parameter \( a = b = v+1/2 > 0 \) is defined by

\[
\beta_{a,b}^{(3)}(dx) = \frac{1}{2^{a+b-1} B(a, b)} (1-x)^{a-1} (1+x)^{b-1} 1_{(-1,1)}(x) dx.
\]

The Fourier transform is given in terms of Bessel functions (see Prudnikov et al. (1992), Vol. 3, formula 7.13.1, p. 594): \( \phi(t/2, -s^2/4) \).

2. SUMS OF SEVERAL DUFRESNE RANDOM VARIABLES

We first present a property of additive infinite divisibility for some Dufresne laws.

**Proposition 2.1.** If \( t > 0 \), the Laplace transform of \( D(t/2, t/2+1/2; t+1) \) is as follows:

\[
\phi(t/2, t/2+1/2; t+1; s) = (2/(1+\sqrt{1-s}))^t.
\]

This is an infinitely divisible law associated with the \textit{L\'evy} process \( t \mapsto X(P(t)) \), where \( P \) and \( X \) are independent \textit{L\'evy} processes whose respective laws at time \( t \) are \( \gamma_{t,2} \) and the inverse Gaussian law

\[
v_{1/2,t}(dx) = \frac{t}{2\sqrt{\pi}} x^{-3/2} \exp \left( -\frac{(x-t)^2}{t} \right) 1_{(0,\infty)}(x) dx,
\]

respectively.

**Proof.** The Laplace transform of a Dufresne law is given by the Gauss hypergeometric function: \( _2F_1(a, b; c; s) \). Then the Laplace transform of \( D(t/2, t/2+1/2; t+1) \) is

\[
_2F_1(t/2, t/2+1/2; t+1; s) = (2/(1+\sqrt{1-s}))^t
\]

according to Abramovitz and Stegun (1970), p. 556, formula 15-1-13. Moreover, by Seshadri (1993), p. 40, formula 2-9, we know that if for \( \alpha > 0 \) and \( p > 0 \)
we define

\( v_{a,p}(dx) = \frac{p \sqrt{x}}{\sqrt{2\pi}} x^{-3/2} \exp\left( -\frac{(x - px)^2}{2ax} \right) 1_{(0,\infty)}(x)\ dx, \)

then for \( s < 1/(2a) \) we get

\( \int_0^\infty e^{sx} v_a(dx) = \exp(p(1 - \sqrt{1 - 2as})). \)

Moreover, for every \( \beta < 2 \) we have

\( \int_0^\infty e^{\beta p} \gamma_{t,2}(dp) = (2 - \beta)^{-t}. \)

Using (2.3) for \( \beta = 1 - \sqrt{1 - 2as} \) we see that for \( s < 1 \)

\( \left( 2/(1 + \sqrt{1 - s}) \right)^t = \int_0^\infty \int_0^\infty e^{sx} v_{1/2,p}(dx) \gamma_{t,2}(dp), \)

which completes the proof. \( \Box \)

Note. This law is considered by Imhof (1986). The corresponding density of probability \( \int_0^\infty v_{1/2,p}(dx) \gamma_{t,2}(dp) \) can be expressed by using the parabolic cylinder functions:

\( f(t, x) = \frac{t^{2(3/2)t}}{\sqrt{2\pi}} e^{-x^{3/2}} x^{t/2 - 1} D_{-(t+1)}(\sqrt{2x}) \ 1_{(0,\infty)}(x)\ dx. \)

For this point see Prudnikov et al. (1992), Tome V, p. 45, formula 2.1.9-1. Note that the law on integers \( \sum_{m=0}^\infty f(m, x) \delta_m \) is considered in Chamayou (1984).

We now apply the previous Laplace transform computation to the bilateral Dufresne laws:

**Corollary 2.1.** If \( t > 0 \), the Laplace transform of the law \( \lambda_{1+1/2}(t; 2t+1) \) is as follows:

\( _2F_1(t/2, t/2 + 1/2; t + 1; s^2) = (2/(1 + \sqrt{1 - s^2}))^t. \)

This is an infinitely divisible law associated with the Lévy process \( t \mapsto X(P(t)) \), where \( P \) and \( X \) are independent Lévy processes whose respective laws at time \( t \) are \( \gamma_{t,2} \) and the Bessel law

\( v_p(dx) = \frac{pe^p}{\pi} \frac{K_1(\sqrt{x^2 + p^2})}{\sqrt{x^2 + p^2}} 1_{(\infty, \infty)}(x)\ dx, \)

where \( K_1 \) is the order-one modified Bessel function of second kind.
Proof. The Laplace transform of the density of a bilateral Dufresne variable $\lambda_d(a; c)$ is given by the generalized hypergeometric function:

$$3F_2(d, a/2, (a+1)/2; c/2, (c+1)/2; s^2)$$

(see Chamayou (2004)). If we set $d = a+1/2 = c/2$, the function (2.6) takes the form $3F_1(a/2, (a+1)/2; a+1; s^2)$, which allows us to use Proposition 2.1 to establish the result, where

$$2F_1(a/2, a/2+1/2; a+1; s^2) = (2/(1+\sqrt{1-s^2}))^a.$$  

We get the previous Laplace transform by integration of the expression

$$\frac{1}{\Gamma(a)} \int_0^\infty \exp(-p(1+\sqrt{1-s^2})/2) p^{a-1} dp.$$  

Then we get

$$\int_{-\infty}^\infty e^{\alpha x} v_p(dx) = \exp(p(1-\sqrt{1-s^2})).$$

For that point, see Prudnikov et al. (1992), Tome 2, p. 357, formula 2-16-12-4, and Abramowitz and Stegun (1970) for the representation of $K_{1/2}$. Thus, if $P(a)$ is a random variable following a gamma law $\gamma_a(ds)$ and if $(X(p))_{p \geq 0}$ is a Bessel process independent of $P(a)$ such that $X(p)$ has the law $v_p$, then $(2/(1+\sqrt{1-s^2}))^a$ is the Laplace transform of the law $X(P(a)/2)$, i.e. follows the above Dufresne law. 

Note. The probabilistic interpretation of the law of $\lambda_{t+1/2}(t; 2t+1)$ from the law of $D(t/2, t/2+1/2; t+1)$ is standard: if $X$ and $Z$ are independent random variables with respective laws $N(0, 1)$ and $D(t/2, t/2+1/2; t+1)$, then the consideration of the Laplace transform of $Z\sqrt{2|X|}$ shows that it is a $\lambda_{t+1/2}(t; 2t+1)$ law. We also remark, using the asymptotical representation of $K_1$ (see Abramovitz and Stegun (1970)), that for $p \to \infty$ the law of $X/\sqrt{p}$ tends to an $N(0, 1)$ law.

3. SUMS OF TWO DISTINCT DUFRESNE VARIABLES WITH THE SAME LAWS

Proposition 3.1. Let $a, b, c$ be such that $0 < a, b < 1$,
\[
\max(a, b) < c < \min(1+a, 1+b),
\]
and let $D_1, D_2, D'_1, D'_2$ be independent random variables with respective laws $D(a, b; c)$, $D(1-a, 1-b; 2-c)$, $D(c-a, c-b; c)$, $D(a+1-c, b+1-c; 2-c)$. Then $D_1 + D_2 \sim D'_1 + D'_2$. 
Proof. The Laplace transform of the two sums is written as the product of two Gauss hypergeometrical functions, i.e.

\[ _2F_1(a, b; c; s) \times _2F_1(1-a, 1-b; 2-c; s) \]

and

\[ _2F_1(c-a, c-b; c; s) \times _2F_1(a+1-c, b+1-c; 2-c; s) \]

The result is obtained by application of the "Darling products" identity (see Darling (1932)).

**Corollary 3.1.** Let \( a, c \) be such that \( 0 < a < 1/2, a+1/2 < c < a+1, \) and let \( \Lambda_1, \Lambda_2, \Lambda'_1, \Lambda'_2 \) be four independent random variables with respective laws

\[ \lambda_{c+1/2}(2a; 2c), \lambda_{3/2-c}(1-2a; 3-2c), \lambda_{c+1/2}(2(c-a)-1; 2-c), \lambda_{3/2-c}(2(1+a-c); 3-2c) \]

Then \( \Lambda_1 + \Lambda_2 \sim \Lambda'_1 + \Lambda'_2. \)

**Proof.** The proof is identical to the previous one for the Laplace transforms of the lambda laws, i.e.

\[ _3F_2(d, a/2, (a+1)/2; c/2, (c+1)/2; s^2) \]

by choosing \( d = c + 1/2 \) and \( d = 3/2 - c \), respectively.

**Proposition 3.2.** Let \( a, b \) be such that \( a > 1/2, 0 < b < a - 1/2, \) and let \( \Gamma_1, \Lambda_1 \) be independent random variables with respective laws

\[ D(2a, 2b, a-1/2; a+b+1/2, 2a-1), D(b; -), D(b, a-b-1/2; a+b+1/2) \]

Then \( \Lambda_0 \sim \Gamma_1 + \Lambda_1. \)

**Proof.** The Laplace transform of \( \Lambda_0 \) and \( \Gamma_1 + \Lambda_1 \) are written by using the hypergeometrical functions

\[ _3F_2(2a, 2b, a-1/2; a+b+1/2, 2a-1; s) \]

and

\[ \frac{1}{(1-s)^6} ( _2F_1(b, a-b-1/2; a+b+1/2; s) ) \]

respectively. The result is obtained by application of formula 14, p. 498, in Prudnikov et al. (1992), Vol. 3.

**Corollary 3.2.** Let \( \Lambda_0, \Lambda_1, \Lambda_2, \Lambda'_0, \Lambda'_1, \Lambda'_2 \) be independent random variables with respective laws

\[ \lambda_4(3; 6), \lambda_1, \lambda_3(1; 6), \lambda_8(7; 13), \lambda_2, \lambda_6(3; 12) \]

Then \( \Lambda_0 \sim \Lambda_1 + \Lambda_2 \) and \( \Lambda'_0 \sim \Lambda'_1 + \Lambda'_2. \)
Proof. We use again formula 14, p. 498, in Prudnikov et al. (1992), Vol. 3, with argument $s^2$:

$$3F_2(2a, 2b, a - 1/2; a + b + 1/2, 2a - 1; s^2)$$

$$= \frac{1}{(1 - s^2)^b} \left(2F_1(b, a - b - 1/2; a + b + 1/2; s^2)\right)$$

as an identity for the Laplace transforms of the lambda laws of $A_0^{(a,b)}$ and $A_1^{(a,b)}$. By choosing $a = b + 1, b = 1$ and $a = b + 2, b = 2$, respectively, we obtain the assertion. □

**Proposition 3.3.** Let $a, b$ be such that $a > 0, 1/2 < b < a + 1/2$ and let $D_0, \Gamma_1, A_1$ be independent random variables with respective laws

$$D(2a + 1, b + 1/2; 2a - b + 3/2), D(1; -), D(2a, a + 1, b - 1/2; a, 2a - b + 3/2).$$

Then $D_0 \sim \Gamma_1 + A_1$.

**Proof.** The Laplace transform of $D_0$ and $\Gamma_1 + A_1$ are written by using the hypergeometrical functions

$$2F_1(2a + 1, b + 1/2; 2a - b + 3/2; s)$$

and

$$\frac{1}{1 - s} \left(3F_2(2a, a + 1, b - 1/2; a, 2a - b + 3/2; s)\right),$$

respectively. The result is obtained by application of formula 15, p. 498, in Prudnikov et al. (1992), Vol. 3. □

4. PRODUCTS OF ONE DUFRESNE VARIABLE BY SUMS OF TWO DUFRESNE VARIABLES WITH THE SAME LAW AS THE SUM OF TWO OTHER DUFRESNE VARIABLES

**Proposition 4.1.** Consider the following eight independent Dufresne variables with respective laws:

- $D_3 \sim D(1, 1, 1; 2, 2)$, i.e. multiplicative convolution of an exponential distribution by two uniform distributions;
- $A_2 \sim D(1, 1; 3/2);
- A_{3,0}, A_{3,1}, A_{3,2} \sim D(1, 1; 3/2, 2)$, i.e. multiplicative convolution of a uniform distribution by $D(1, 1; 3/2)$;
- $D_{1,1} \sim D(2; 5/2)$, i.e. $\beta_{2,1/2};$
- $D_{1,2} \sim D(2; 3)$, i.e. $\beta_{2,1};$
- $D_0 \sim D(1; -)$, i.e. an exponential distribution.
Then

\( D_{1,1} (D_0 + D_3) \sim A_2 + A_{3,0}, \)

\( D_{1,2} (A_2 + A_{3,0}) \sim A_{3,1} + A_{3,2}. \)

**Proof.** The Laplace transform of the product of a beta distribution by the sum of Dufresne variables can be written as the Euler integral defining the Kampé de Fériet series, i.e.

\[
\begin{align*}
F_{1;1;\beta}^{1;C}(\begin{array}{c}
-a; \\
+a+b; \\
s, s \\
\end{array})
&= \frac{1}{B(a, b)} \int_0^1 x^{a-1} (1-x)^{b-1} \left( c F_D((c); (d); s) c F_D((c'); (d'); s) \right) dx
\end{align*}
\]

(see Exton (1978), p. 39, formula 2.1.5.7). Krupnikov (2001) gives the following reduction formula:

\[
(4.3) \quad F_{1;0;3}^{1;1;3}(\begin{array}{c}
2; \\
5/2; \\
1, 1, 1; \\
\end{array}) = (2F_1(1, 1; 3/2; s))(3F_2(1, 1, 1; 3/2, 2; s)),
\]

whence we can derive the result, iterating the previous integral relation. For that purpose, see Exton (1978), p. 184, formula A 1-2-100, we write

\[
(4.4) \quad F_{2;0;2}^{1;1;3}(\begin{array}{c}
2, 2; \\
5/2, 3; \\
1, 1, 1; \\
\end{array}) = (3F_2(1, 1, 1; 3/2, 2; s))^2
\]

we obtain the second result. \( \blacksquare \)

**PROPOSITION 4.2.** Let \( a, b, c, d \) be positive real numbers such that \( c < a < b + c \) and consider the following five independent Dufresne random variables with respective laws:

- \( D_3 \sim D(c, d; a); \)
- \( D_2 \sim D(b; -), \) i.e. \( \gamma_b; \)
- \( D_1 \sim D(a; b + c), \) i.e. \( \beta_{a, b + c - a}; \)
- \( D'_2 \sim D(b + c + d - a, c; b + c); \)
- \( D'_1 \sim D(a - c; -), \) i.e. \( \gamma_{a - c}. \)
Then
\begin{equation}
D_1(D_2 + D_3) \sim D'_1 + D'_2.
\end{equation}

**Proof.** The Laplace transform of $D_1(D_2 + D_3)$ is given (see the integral representation of the previous proposition) by the following Kampé de Fériet series:
\begin{equation}
F_{1:1;2}^{1:1:2} \left[ \begin{array}{c}
a; b; c, d; \\
b + c; -; a; s, s \end{array} \right],
\end{equation}
which can be reduced to $\frac{1}{2} F_0(a - c; -; s) F_2(b + c + d - a, c; b + c; s)$. We thank E. D. Krupnikov for this remark (personal communication).

**PROPOSITION 4.3.** Let us consider the following five independent Dufresne random variables with respective laws:
- $D_1 \sim D(2c; 1 + 2c)$, i.e. $\beta_{2c,1}$;
- $D_2 \sim D(a; b)$;
- $D_3, D'_1, D'_2 \sim D(c, a; 1 + c, b)$, where $a = (a_1, ..., a_{n+1})$ and $b = (b_1, ..., b_n)$.

Then
\begin{equation}
D_1(D_2 + D_3) \sim D'_1 + D'_2.
\end{equation}

**Proof.** The Laplace transform of $D_1(D_2 + D_3)$ is given (see the integral representation of Proposition 4.1) by the following Kampé de Fériet series:
\begin{equation}
F_{1:1;2}^{1:2+r_1:1+r} \left[ \begin{array}{c}
2c; c, a; a; \\
1 + 2c; 1 + c, b; b; s, s \end{array} \right],
\end{equation}
which can be reduced to $(r + 2F_{r+1}(c, a; c + 1, b; s))^2$. We thank E. D. Krupnikov for this remark (personal communication).

**COROLLARY 4.1.** Let us consider the following five independent mono- and bilateral Dufresne random variables with respective laws:
- $D_1 \sim D(2c; 1 + 2c)$, i.e. $\beta_{2c,1}$;
- $A_2 \sim \lambda c (a; b)$;
- $A_3, A_1, A'_2 \sim \lambda c (a; b)$, where $a$ and $b$ are of length $r$.

Then
\begin{equation}
D_1(A_2 + A_3) \sim A'_1 + A'_2.
\end{equation}

**Proof.** The Laplace transform of $D_1(A_2 + A_3)$ is given by the following Kampé de Fériet series:
\begin{equation}
F_{1:2r+1;2r+1}^{1:2r+1;2r+1} \left[ \begin{array}{c}
2c; c, a/2; (a + 1)/2; 1 + c, a/2, (a + 1)/2; s, s \\
1 + 2c; b/2; (b + 1)/2; b/2, (b + 1)/2; s, s \end{array} \right],
\end{equation}
which, as before, can be reduced to $(2 \lambda_1 F_{r+1}(c, a/2; (a + 1)/2; b/2, (b + 1)/2; s))^2$. 

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5. RANDOM DIFFERENCE EQUATIONS

We present now a series of examples concerning the random induction:

\[ X_{n+1} \sim A_n(X_n + Y_n), \quad n \in \mathbb{N}, \]

where \( X_n, Y_n \) are independent Dufresne random variables and \( A_n \) are products of independent beta variables.

**Proposition 5.1.** Let \( 0 < a < 1 \), where \( a \neq 1/2 \). Let \( X_n, Y_n, A_n, B_n \) be independent random variables with respective laws:

\[ X_n \sim D(a, a+1/2; 2a), \quad Y_n \sim D(1-a, 3/2-a; 2(1-a)) \]

and

\[ A_n \sim D(a, a-1/2; 1, 2a) \text{ for } 1/2 < a < 1, \]

\[ B_n \sim D(a, 1/2-a; 1, 2(1-a)) \text{ for } 0 < a < 1/2. \]

Then for \( 1/2 < a < 1 \)

\[ X_{n+1} \sim A_n(X_n + Y_n) \tag{5.1} \]

and for \( 0 < a < 1/2 \)

\[ Y_{n+1} \sim B_n(X_n + Y_n). \tag{5.2} \]

**Proof.** The Laplace transform of \( A_n(X_n + Y_n) \) is given (see the integral representation of the previous proposition) by the following Kampé de Fériet series:

\[ \text{Kampé de Fériet series:} \]

\[ F_{2;1;1}^{2;2;2} \left[ a, a+1/2; a, a+1/2; 1-a, 3/2-a; \ 2a; \ 2(1-a); \ s, s \right], \tag{5.3} \]

which can be reduced to \( 2F_1(a, a+1/2; 2a; s) \) according to Srivastava and Karlsson (1985), p. 32, equation (51). The Laplace transform of \( B_n(X_n + Y_n) \) is reduced to \( 2F_1(1-a, 3/2-a; 2(1-a); s) \).

**Proposition 5.2.** Let \( a > 0, b > 0, \text{ and } \max(a, b) < c < a+b. \) Let \( X_n, Y_n, A_n \) be independent random variables with respective laws:

\[ D(a, b; c), \ D(c-a-b; -), \ D(a, b; c-a, c-b). \]

Then

\[ X_{n+1} \sim A_n(X_n + Y_n). \tag{5.4} \]

**Proof.** The Laplace transform of \( A_n(X_n + Y_n) \) is given (see the integral representation in Proposition 4.1) by the following Kampé de Fériet
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series:

\[
F_{2;1}^{2;2;1;0} \left[ \begin{array}{cccc}
  a, b; & a, b; & c-a-b; & s, s \\
  c-b, c-a; & c; & -;
\end{array} \right],
\]

which can be reduced to \( _2F_1(a, b; c; s) \) according to Srivastava and Karlsson (1985), p. 28, equation (34).

**Proposition 5.3.** Let \( a > 0 \) and \( b > 0 \) be such that \( 1/2 < a+b < 1 \). Let \( X_n, Y_n, A_n \) be independent random variables with respective laws:

\[
D(a, b; a+b+1/2), \ D(a, b; a+b-1/2), \ D(a, b, 2a+2b-1; 2a, 2b, a+b).
\]

Then

\[
X_{n+1} \sim A_n(X_n + Y_n).
\]

**Proof.** The Laplace transform of \( A_n(X_n + Y_n) \) is given (see the integral representation in Proposition 4.1) by

\[
F_{3;1}^{3;2;2} \left[ \begin{array}{cccc}
  a, b, 2(a+b)-1; & a, b; & a, b; & s, s \\
  2a, 2b, a+b; & a+b+1/2; & a+b-1/2;
\end{array} \right],
\]

which can be reduced to \( _2F_1(a, b; a+b+1/2; s) \) according to Srivastava and Karlsson (1985), p. 29, equation (37).

**Proposition 5.4.** Let \( 0 < a < 1/2 \) and \( 1 < b < 1+2a \). Let \( X_n, Y_n, A_n \) be independent Dufresne random variables of respective laws

\[
D(a, b-1; a+b-1/2), \ D(a, b; a+b-1/2), \ D(a, b-1, 2(a+b-1); a+b-1, 2a, 2b).
\]

Then

\[
X_{n+1} \sim A_n(X_n + Y_n)
\]

for \( 1/2 > a > 0 \) and \( 1 < b < 1+2a \).

**Proof.** The Laplace transform of \( A_n(X_n + Y_n) \) is given (see the integral representation in Proposition 4.1) by the following Kampé de Féret series:

\[
F_{3;1}^{3;2;2} \left[ \begin{array}{cccc}
  a, b-1, 2(a+b-1); & a, b-1; & a, b; & s, s \\
  a+b-1, 2a, 2b-1; & a+b-1/2; & a+b-1/2;
\end{array} \right],
\]

which can be reduced to \( _2F_1(a, b-1; a+b-1/2; s) \) according to Srivastava and Karlsson (1985), p. 29, equation (39).

**Proposition 5.5.** Let \( a, b > 1/2 \). Let \( X_n, Y_n, A_n \) be independent Dufresne random variables with respective laws:

\[
D(a+1/2, b+1/2; a+b+1/2), \ D(a-1/2, b-1/2; a+b-1/2), \ D(a, b, a+b; 2a-1/2, 2b-1/2, 2a+2b-1).
\]
Then

\[ X_{n+1} \sim A_n(X_n + Y_n). \]  

**Proof.** The Laplace transform of \( A_n(X_n + Y_n) \) is given (see the integral representation in Proposition 4.1) by the following Kampé de Fériet series:

\[ F^{3:3;2}_{3:3;1} \left[ \begin{array}{c}
 a, b, a+b; \\
 2a-1/2, 2b-1/2, 2(a+b)-1;
\end{array} \right. \left. \begin{array}{c}
 a+1/2, b+1/2; \\
 a-1/2, b-1/2; \\
 s, s;
\end{array} \right] \]

which can be reduced to \( _2F_1 (a+1/2, b+1/2; a+b+1/2; s) \) according to Srivastava and Karlsson (1985), p. 30, equation (40). \( \square \)

**Proposition 5.6.** Let \( 0 < 2b < a < b+1 \). Let \( X_n, Y_n, A_n \) be independent Dufresne random variables with respective laws:

\[ D(a, 1+a/2, b; 1+a-b, a/2), D(1; -), \]

i.e. exponential random variables, and

\[ D(a, b, 1+a/2; a+b+1, b+a/2, 2+3a/2), \]

i.e. products of three independent beta variables. Then

\[ X_{n+1} \sim A_n(X_n + Y_n). \]

**Proof.** The Laplace transform of \( A_n(X_n + Y_n) \) is given (see the integral representation in Proposition 4.1) by the following Kampé de Fériet series:

\[ F^{3:3;1}_{3:3;0} \left[ \begin{array}{c}
 a, b, 1+a/2; \\
 1+a, 1+b, a/2;
\end{array} \right. \left. \begin{array}{c}
 a, 1+a/2, b; \\
 1; \\
 s, s;
\end{array} \right] \]

which can be reduced to \( _3F_2 (a, 1+a/2, b; 1+a-b, a/2; s) \) according to Lavoie and Grondin (1994), p. 395, equation (7). \( \square \)

**Proposition 5.7.** Let \( 0 < 2b < a < b+1, b+c < a, c > 0 \). Let \( X_n, Y_n, A_n \) be independent Dufresne random variables with respective laws:

\[ D(a, 1+a/2, b, c, a-b-c; 1+a-b, 1+a-c, 1+b+c, a/2), D(1; -), \]

i.e. exponential random variables, and

\[ D(a, b, 1+a/2, c, a-b-c; a+b+1, b+a/2, 2+3a/2, 2c+1, 1+2(a-b-c)), \]

i.e. products of five independent beta variables. Then

\[ X_{n+1} \sim A_n(X_n + Y_n). \]

**Proof.** The Laplace transform of \( A_n(X_n + Y_n) \) is given (see the integral representation in Proposition 4.1) by the following Kampé de Fériet series:

\[ F^{5:4;3}_{5:4;1} \left[ \begin{array}{c}
 a, b, 1+a/2, c, a-b-c; \\
 1+a, 1+b, a/2, 1+c, 1+a-b-c;
\end{array} \right. \left. \begin{array}{c}
 a, 1+a/2, b, c, a-b-c; \\
 1; \\
 s, s;
\end{array} \right] \]
which can be reduced to
\[ _2F_1(a, 1 + a/2, b, c, a - b - c; 1 + a - b, 1 + a - c, 1 + b + c, a/2; s) \]

**Proposition 5.8.** Let \( 0 < b < c < a < b + 1 \). Let \( X_n, Y_n, A_n \) be independent Dufresne random variables with respective laws:

\[
X_n \sim D(a, b, 1 + ab/(a + b - c); 1 + c, ab/(a + b - c)), \quad Y_n \sim D(1; -)
\]

(they are exponential variates) and

\[
A_n \sim D\left( a, b, 1 + ab/(a + b - c); a + b + 1, b(1 + a/(a + b - c)), 2 + a(1 + b/(a + b - c)) \right)
\]

(a product of three independent beta variables). Then

\[ X_{n+1} \sim A_n(X_n + Y_n). \]

**Proof.** The Laplace transform of \( A_n(X_n + Y_n) \) is given (see integral representation in Proposition 4.1) by the following Kampé de Fériet series:

\[ F^{3:3:1}_{3:2:0} \left[ \begin{array}{c} a, b, 1 + ab/(a + b - c): \quad a, b, 1 + ab/(a + b - c): \quad 1; \quad s, s \\ 1 + a, 1 + b, ab/(a + b - c): \quad 1 + c, ab/(a + b - c): \quad s \end{array} \right], \]

which can be reduced to \( _3F_2(a, b, 1 + ab/(a + b - c); 1 + c, ab/(a + b - c); s) \) according to Lavoie and Grondin (1994), p. 397, equation (13).

In the next corollary we offer an example among other possible choices of interrelations between the parameters \( a, b, c \) when they fulfill the reduction formula (5.19) below.

**Corollary 5.1.** Let \( b > 0, a + b > c, a = b + 1/2, c = b + 1/2 + \sqrt{2b + 1} \) and let \( X_n, Y_n, A_n \) be independent mono- and bilateral Dufresne random variables such that

\[
X_n \sim \lambda_{1+ab/(a+b-c)}(2b; 1+2c), \quad Y_n \sim \lambda_1(1; -)
\]

(they are double exponential random variables) and

\[
A_n \sim D\left( a, b, 1 + ab/(a + b - c); a + b + 1, b(1 + a/(a + b - c)), 2 + a(1 + b/(a + b - c)) \right)
\]

(a product of three independent beta variables). Then

\[ X_{n+1} \sim A_n(X_n + Y_n). \]

**Proof.** The Laplace transform of \( A_n(X_n + Y_n) \) is given by the following Kampé de Fériet series:

\[ F^{3:3:1}_{3:2:0} \left[ \begin{array}{c} a, b, 1 + ab/(a + b - c): \quad a, b, 1 + ab/(a + b - c): \quad 1; \quad s^2, s^2 \\ 1 + a, 1 + b, ab/(a + b - c): \quad 1 + c, ab/(a + b - c): \quad - \end{array} \right]. \]
which can be reduced to \( _3F_2(a, b, 1 + ab/(a + b - c); 1 + c, ab/(a + b - c); s^2) \) according to the previous reference. 

6. RANDOM ARITHMETICAL ENCOUNTERS OF THIRD KIND

In this section we consider four independent random variables \( B_1, B_2, G_1, G_2 \) with beta distribution of third kind and gamma distribution, and we give the distributions of the random variables \( Z = (G_1, G_2)(B_1 + B_2) \) for some values of the parameters.

**Proposition 6.1.** Let \( a, b, g_1, g_2 \) be positive numbers, \( B_1, B_2, G_1, G_2 \) be positive independent random variables such that \( G_j \sim \gamma_{g_j}, j = 1, 2, \) and put \( Z = (G_1, G_2)(B_1 + B_2) \) and \( F_Z(s) = E(e^{isZ}). \) Then:

If \( B_1 \sim \beta^{(3)}_{a+1/2, a+1/2} \) and \( B_2 \sim \beta^{(3)}_{b+1/2, b+1/2} \), then

\[
F_Z(s) = _4F_3(g_1, g_2, 1/2(a + b), 1/2(a + b) + 1; a + 1/2, b + 1/2, a + b; -s^2).
\]

In particular:
(1) for \( g_1 = 1/2, b = 0 \) we have \( Z \sim \lambda_{g_2}(a; 2a); \)
(2) for \( g_1 = 3/2, b = 1 \) we have \( Z \sim \lambda_{g_2}(a + 1; 2a + 1). \)

**Proof.** The Fourier transform of \( B_1 + B_2 \) is

\[
\mathcal{F}(B_1 + B_2) = \frac{1}{\Gamma(g_1)\Gamma(g_2)} \times
\int_0^\infty \int_0^\infty e^{-x-y} x^{g_1-1} y^{g_2-1} _2F_3(1/2(a + b), 1/2(a + b) + 1; a + 1/2, b + 1/2, a + b; -xys^2) dxdy.
\]

From Gradsteyn and Ryzhyk (1980), formula 7.52.9, we get (6.44). Replacing the parameters by the indicated values and using the possible reduction of \( _4F_3 \) to the canonical form \( _3F_2 \) of the lambda law, we complete the proof of parts (1) and (2). 

**Proposition 6.2.** Let \( a, b \) be positive numbers and let \( G_1, G_2; G_1', G_2' \) be four gamma independent random variables with respective parameters \( a, b; a, b. \) If \( G_0 \sim \gamma_{a+b} \) is independent of the lambda variable \( \Lambda_0 \sim \lambda_1(b; b+1/2), \) then

\[
\frac{1}{2}(G_1, G_2 - G_1'G_2') \sim G_0 \Lambda_0.
\]

**Proof.** The Laplace transforms of \( G_1, G_2 \) and \( G_1', G_2' \) are \( _2F_0(a, b; -; s) \); the Laplace transform of the half of their difference is

\[
(6.2) \ _2F_0(a, b; -; s/2) \ _2F_0(a, b; -; -s/2) = _4F_1\left(a, b, \frac{a+b}{2}, \frac{a+b+1}{2}; a+b; s^2\right);
\]
this equality comes from Erdélyi et al. (1954), p. 186, formula 4. Therefore for the Laplace transform

\[
L_{\alpha_0 \alpha_0} (s) = \frac{1}{\Gamma (a+b)} \int_0^\infty e^{-x} x^{a+b-1} 2F_1 (a, b; a+b; (xs)^2) dx dy.
\]

From Gradsteyn and Ryzhik (1980), formula 7.525.1, we get the right-hand side of the previous relation, which completes the proof. ■

**Proposition 6.3.** Let \( d \) be a positive parameter and assume that:

- \( B_1 \sim D(1; 2 + d/2) \), i.e. \( \beta_{1.1+d/2} \);
- \( G_1 \sim D(2; -), \) i.e. \( \gamma_2 \);
- \( G_2 \sim D(d; -), \) i.e. \( \gamma_d \);
- \( G_0 \sim D(1; -), \) i.e. an exponential variable;
- \( D_0 \sim D(1 + d, d/2; 2 + d/2) \).

Then

\[ B_1 (G_2 - G_1) \sim D_0 - G_0. \]

**Proof.** The Laplace transform of the product of a beta variable and the difference of two gamma variables \( B_1 (G_2 - G_1) \) is

\[
F_{1:1:1:0}^{1:1:1:1} \left[ \begin{array}{ccc} a & c & d; a+b & - & -; & -s, s \\ \end{array} \right] = \frac{1}{B(a, b)} \int_0^1 x^{a-1} (1-x)^{b-1} dx
\]

(see Exton (1978), p. 39, formula 2.1.5.7). Therefore using the transformation of the Kampé de Fériet functions:

\[
(6.3) \quad F_{1:1:0:0}^{1:1:1:1} \left[ \begin{array}{ccc} a & c & d; a+b & - & -; & -s, s \\ \end{array} \right] = \frac{1}{(1-s)^{a-b}} F_{1:1:0:1}^{1:2:1:0} \left[ \begin{array}{ccc} a+b-c & a, d & b; a+b-c; a+b-c; -s, s \\ \end{array} \right]
\]

(see Exton (1978), p. 39, formula 2.1.5.7) we reduce it for \( a = 1 + d/2, c = 2, b = 1 \) to the form

\[
F_{1:1:0:1}^{1:2:1:1} \left[ \begin{array}{ccc} a+b-c & 1+d/2, d; 1; a+b; d/2; -; -s, s \\ \end{array} \right] = \Gamma (1+d, a+b-c; a+b; s)
\]

according to Lavoie and Grondin (1994). Thus the proof is completed. Note that \( d = 2 \) is the only symmetrical case. ■
7. SUMMARY TABLES

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<tr>
<th>parameters conditions</th>
<th>law $D_0$</th>
<th>law $D'_1$</th>
<th>law $D'_2$</th>
</tr>
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<tbody>
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<td>$2a, 2b, a-1/2$</td>
<td>$b$</td>
<td>$b, a-b-1/2$</td>
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<th>law $A'_1$</th>
<th>law $A'_2$</th>
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<td>4(3; 6) 8(7; 13)</td>
<td>1(--; --)</td>
<td>3(1; 6)</td>
</tr>
</tbody>
</table>

Then for all these cases:

(7.1) \[ D_0 \sim D'_1 + D'_2. \]

<table>
<thead>
<tr>
<th>parameters conditions</th>
<th>law $D_1$</th>
<th>law $D_2$</th>
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<th>law $D'_2$</th>
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<td>$1-a, 1-b$</td>
<td>$c-a, c-b$</td>
<td>$a+1-c, b+1-c$</td>
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<td>$c$</td>
<td>$2-c$</td>
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<th>law $A_2$</th>
<th>law $A'_1$</th>
<th>law $A'_2$</th>
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<td>$1-2a; 3-2c$</td>
<td>$2(c-a)-1; 2-c$</td>
<td>$2(1+a-c); 3-2c$</td>
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Then for all these cases:

(7.2) \[ D_1 + D_2 \sim D'_1 + D'_2. \]

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<th>law $D_1$</th>
<th>law $D_2$</th>
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<td>$c, d; a$</td>
<td>$a-c; -$</td>
<td>$b+c+d-a, c; b+c$</td>
<td></td>
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<td>$2c; 1+2c$</td>
<td>$a; b$</td>
<td>$c, a; 1+c, b$</td>
<td>$c, a, 1+c, b$</td>
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<table>
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<th>law $A_2$</th>
<th>law $A'_1$</th>
<th>law $A'_2$</th>
</tr>
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<tbody>
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<td>$\lambda_{c+1}(a; b)$</td>
<td>$\lambda_c(a; b)$</td>
<td>$\lambda_c(a; b)$</td>
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</tbody>
</table>

Note. $a$ and $b$ are of length $r+1$ and $r$, respectively.
Then for all these cases:

\[(7.3)\]
\[D_1 (D_2 + D_3) \sim D_1' + D_2'.\]

<table>
<thead>
<tr>
<th>law of (A_0)</th>
<th>law of (G_1)</th>
<th>law of (G_2)</th>
<th>law of (G_1)</th>
<th>law of (A_2)</th>
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</thead>
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<tr>
<td>(\lambda_1(a; 2a))</td>
<td>(g; -)</td>
<td>(1/2; -)</td>
<td>(\beta_{1/2,1/2}^{(3)})</td>
<td>(\beta_{a+1/2,a+1/2}^{(3)})</td>
</tr>
<tr>
<td>(\lambda_1(a+1; 2a+1))</td>
<td>(g; -)</td>
<td>(3/2; -)</td>
<td>(\beta_{3/2,3/2}^{(3)})</td>
<td>(\beta_{a+1/2,a+1/2}^{(3)})</td>
</tr>
</tbody>
</table>

Then for all these cases:

\[(7.4)\]
\[A_0 \sim G_1 G_2 (A_1 + A_2).\]

<table>
<thead>
<tr>
<th>law of (G_1)</th>
<th>law of (G_1)</th>
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<th>law of (G_2)</th>
<th>law of (G_0)</th>
<th>law of (A_0)</th>
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<tr>
<td>(a)</td>
<td>(a)</td>
<td>(b)</td>
<td>(b)</td>
<td>(a+b)</td>
<td>(\lambda_1(b; b+1/2))</td>
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</table>

Then for all these cases:

\[(7.5)\]
\[\frac{1}{2} (G_1 G_2 - G_1' G_2') \sim G_0^2 A_0.\]

<table>
<thead>
<tr>
<th>law of (B_1)</th>
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<th>law of (G_2)</th>
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<td>(1+d, d/2; 2+d/2)</td>
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Then for all these cases:

\[(7.6)\]
\[B_1 (G_2 - G_1) \sim D_0 - G_0.\]

<table>
<thead>
<tr>
<th>parameters</th>
<th>conditions</th>
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<th>law of (Y_n)</th>
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<td>(a, 1/2-a)</td>
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<td>(a &gt; 0) and (b &gt; 0)</td>
<td>(a, b)</td>
<td>(c-a-b)</td>
<td>(b)</td>
<td>(c-a, c-b)</td>
</tr>
<tr>
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<td>(-)</td>
<td>(-)</td>
<td>(-)</td>
</tr>
<tr>
<td>(a &gt; 0) and (b &gt; 0)</td>
<td>(a, b)</td>
<td>(a, b)</td>
<td>(a, b, 2a+2b-1)</td>
<td>(2a, 2b, a+b)</td>
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<tr>
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<td>(a+b+1/2)</td>
<td>(a+b-1/2)</td>
<td>(-)</td>
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<td>(a-1/2, b-1/2)</td>
<td>(a, b, a+b)</td>
<td>(2a-1/2, 2b-1/2, 2a+2b-1)</td>
</tr>
</tbody>
</table>
Then for all these cases:

\[
X_{n+1} \sim A_n(X_n + Y_n).
\]

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**REFERENCES**


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Laboratoire de Statistique et Probabilités
Université Paul Sabatier
31062 Toulouse, France
E-mail: chamayou@cict.fr

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