CRITERIONS OF THE SIMILARITY FOR RANDOM WALKS AND BIRTH-AND-DEATH PROCESSES*

BY

ANNA POSKROBKO (BIALYSTOK)

Abstract. This paper is devoted to study the similarity of birth-and-death processes with a discrete and continuous time. We discuss some relations between the measures of orthogonality of the associated polynomials and the first return probabilities of two $\alpha$-similar random walks and two $\nu$-similar birth-and-death processes. We give the necessary and sufficient conditions for $\alpha$-similarity of two random walks both in terms of the corresponding spectral measures. We consider analogous conditions for $\nu$-similarity of two birth-and-death processes.

2000 Mathematics Subject Classification: Primary 60J80; Secondary 42C05.

Key words and phrases: Associated polynomials, measures of orthogonality, similar random walks, similar birth-and-death processes, first return probabilities.

1. INTRODUCTION

This work was intended as an attempt to study the similarity of birth-and-death processes with a discrete and continuous time. In Section 1 we are interested in a random walk with similar transition probabilities. We introduce a brief summary of such a process and its well-known properties. We recall the definition of $\alpha$-similarity. Moreover, we give necessary and sufficient conditions for measures of orthogonality of the associated polynomials of the corresponding random walks $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathcal{Y}$ is $\alpha$-similar to $\mathcal{X}$. For such random walks we establish the relations between their first return probabilities. Section 2 contains a discussion of a birth-and-death process with a continuous time. We introduce the notion of $\nu$-similarity and we obtain the analogous theorems but for the birth-and-death processes $\mathcal{Y}$ and $\mathcal{Z}$, where $\mathcal{Z}$ is $\nu$-similar to $\mathcal{Y}$.

This work was inspired by the results of the papers by Schiefermayr (2003), Dette (2000) and Lenin et al. (2000).

* The research was supported by Bialystok Technical University under Grant no W/IMF/1/04.
Let \( X = \{ X(n), n = 0, 1, \ldots \} \) denote a random walk on the nonnegative integers \( \{0, 1, 2, \ldots \} \) and let
\[
P_{ij}(n) = \Pr \{ X(m+n) = j | X(m) = i \}, \quad i, j \geq 0,
\]
be the \( n \)-step transition probabilities. We will use the notation \( p_j = P_{j,j+1}(1), \)
\( q_j = P_{j+1,j}(1), r_j = P_{jj}(1), j \geq 0, \) and \( P_{ij}(1) = 0 \) for \( |i-j| > 1, i, j \geq 0. \) We assume that \( p_j > 0, q_j > 0, r_j \geq 0, j \geq 0, \) and \( p_j + q_j + r_j \leq 1, j \geq 1. \) The inequality \( p_j + q_j + r_j < 1, j \geq 0, \) corresponds to a permanent absorbing state \( j^* \) which can only be reached from state \( j \) with probability \( 1 - (p_j + q_j + r_j). \)

Karlin and McGregor (1959) have shown that the \( n \)-step transition probability can be represented in the form
\[
P_{ij}(n) = \pi_j \int_{-1}^{1} x^n Q_i(x) Q_j(x) d\psi(x), \quad i, j \geq 0, \ n \geq 0,
\]
where
\[
\pi_0 = 1, \quad \pi_j = \frac{p_0 p_1 \cdots p_{j-1}}{q_1 q_2 \cdots q_j}, j \geq 1.
\]
\( \psi \) is a unique Borel measure with total mass 1 and infinite support in \([-1, 1]\], called the random walk measure of \( X \), and \( Q_j(x) \) is a random walk polynomial of degree \( j \) defined recursively as follows:
\[
Q_{-1}(x) = 0, \quad Q_0(x) = 1,
\]
\[
xQ_j(x) = q_j Q_{j-1}(x) + r_j Q_j(x) + p_j Q_{j+1}(x), \quad j \geq 0.
\]
The polynomials \( Q_j \) are orthogonal with respect to the random walk measure, i.e.
\[
\pi_j \int_{-1}^{1} Q_i(x) Q_j(x) d\psi(x) = \delta_{ij},
\]
where \( \delta_{ij} \) denotes Kronecker's symbol.

Given the random walk, polynomials \( Q_j \) define the corresponding sequence of first associated polynomials \( Q_j^{(1)} \) by replacing \( p_j, q_j \) and \( r_j \) by \( p_{j+1}, q_{j+1} \) and \( r_{j+1} \), respectively, in the recurrence relation (1). Therefore the first associated polynomials satisfy the recurrence relation
\[
Q_{-1}^{(1)}(x) = 0, \quad Q_0^{(1)}(x) = 1/p_0,
\]
\[
xQ_j^{(1)}(x) = q_{j+1} Q_{j-1}^{(1)}(x) + r_{j+1} Q_j^{(1)}(x) + p_{j+1} Q_{j+1}^{(1)}(x), \quad j \geq 0.
\]
It follows from the arguments of Karlin and McGregor (1959) that there exists a random walk measure \( \psi^{(1)} \) on the interval \([-1, 1]\) such that the first
associated polynomials are orthogonal with respect to this one, i.e.

$$\pi_{j+1} p_0 q_j \frac{1}{\delta_{ij}} \int_{-1}^{1} Q_i^{(1)}(x) Q_j^{(1)}(x) d\psi^{(1)}(x) = \delta_{ij}.$$ 

In the proofs we will use the monic associated polynomials

$$R_j^{(1)} = p_0 p_1 p_2 \ldots p_j Q_j^{(1)}(x), \quad j \geq 0,$$

which satisfy the recurrence relation

$$R_j^{(1)}(x) = 0, \quad R_0^{(1)}(x) = 1,$$

$$R_{j+1}^{(1)}(x) = (x - r_{j+1}) R_j^{(1)}(x) - p_j q_{j+1} R_{j-1}^{(1)}(x), \quad j \geq 0.$$  

**Definition 1.** For \( \alpha > 0 \), we call a random walk \( \mathcal{X} \) \( \alpha \)-similar to \( \mathcal{X} \) if there exist constants \( C_{ij} > 0 \), \( i, j \geq 0 \), such that

$$\tilde{P}_{ij}(n) = \alpha^{-n} C_{ij} P_{ij}(n), \quad i, j \geq 0, \quad n \geq 1.$$ 

In the following we will consider the random walk \( \tilde{\mathcal{X}} \), \( \alpha \)-similar to \( \mathcal{X} \) \((\alpha > 0)\), with parameters \( \tilde{p}_j, \tilde{q}_j, \tilde{r}_j, j \geq 0 \), its first associated polynomials \( \tilde{Q}_j^{(1)} \) orthogonal with respect to the measure \( \tilde{\psi}^{(1)} \) and the \( n \)-step transition probability \( \tilde{P}_{ij}(n) \). We will use the same letter to denote the measure and its distribution function.

**Theorem 1.** The random walk \( \mathcal{X} \) is \( \alpha \)-similar to \( \mathcal{X} \) if and only if the distribution functions of the random walk measures satisfy

$$\tilde{\psi}^{(1)}(x) = \psi^{(1)}(\alpha x), \quad x \in \mathbb{R},$$

and \( \alpha \geq \sup(\text{supp}(\psi^{(1)})) \). In the case where \( \mathcal{X} \) is \( \alpha \)-similar to \( \mathcal{X} \), we have the equalities for the first return probabilities to the origin:

$$\tilde{P}_{i0}(n) = \alpha^{-n} \frac{1}{\sqrt{\pi_i/\pi_{i-1}}} P_{i0}(n), \quad i \geq 1, \quad n \geq 1,$$

$$\tilde{P}_{00}(n) = \alpha^{-n} P_{00}(n), \quad n \geq 2.$$ 

**Proof.** Schiefermayr (2003) showed that the necessary and sufficient condition of \( \alpha \)-similarity of \( \mathcal{X} \) is the connection between parameters

$$\tilde{r}_j = \alpha^{-1} r_j, \quad \tilde{p}_j \tilde{q}_{j+1} = \alpha^{-2} p_j q_{j+1}, \quad j \geq 0.$$ 

**Necessity.** From the above remark and (2) we conclude that \( \tilde{R}_j^{(1)}(x) = \alpha^{-j} R_j^{(1)}(\alpha x) \), which gives the equality

$$\tilde{Q}_j^{(1)}(x) = \sqrt{\pi_j/\pi_{j-1}} Q_j^{(1)}(\alpha x).$$

We proceed to show that \( \tilde{\psi}^{(1)}(x) = \psi^{(1)}(\alpha x) \). We have

$$\delta_{ij} = \pi_{j+1} p_0 q_1 \int_{-1}^{1} Q_i^{(1)}(x) Q_j^{(1)}(x) d\psi^{(1)}(x) =$$
Since $\text{supp}(\psi^{(1)}) \subseteq [-1, 1]$, the parameter $\alpha$ has to satisfy $\alpha \ge \sup(\text{supp}(\psi^{(1)}))$.

**Sufficiency.** Let $\psi_j^{(1)}(x) = \psi_j^{(1)}(ax)$, $\alpha \ge \sup(\text{supp}(\psi^{(1)}))$ and $R_j^{(1)}$ be the corresponding system of monic orthogonal polynomials of $\mathcal{X}$ satisfying the recurrence relation (2). Define

$$R_j^{(1)}(x) = \alpha^{-j} R_j^{(1)}(ax).$$

Hence, for $i \neq j$,

$$0 = \int_{-1}^1 R_i^{(1)}(x) R_j^{(1)}(x) d\psi^{(1)}(x) = \int_{-1}^1 R_i^{(1)}(ax) R_j^{(1)}(ax) d\psi^{(1)}(ax)$$

$$= \alpha^{i+j} \int_{-1}^1 \tilde{R}_i^{(1)}(x) \tilde{R}_j^{(1)}(x) d\tilde{\psi}^{(1)}(x).$$

Thus $\tilde{R}_j^{(1)}$ is the corresponding system of monic orthogonal polynomials of $\tilde{\mathcal{X}}$. Using (3) we obtain the equivalent recurrence relation of $\tilde{R}_j^{(1)}(x)$, i.e.

$$R_j^{(1)}(x) = (ax - \alpha \tilde{r}_j) R_j^{(1)}(x) - \alpha^2 \tilde{p}_j \tilde{q}_{j+1} R_{j+1}^{(1)}(x).$$

Consequently, it is obvious that the parameters of $\mathcal{X}$ and $\tilde{\mathcal{X}}$ satisfy the conditions $\tilde{r}_j = \alpha^{-1} r_j$ and $\tilde{p}_j \tilde{q}_{j+1} = \alpha^{-2} p_j q_{j+1}$. This completes the proof of $\alpha$-similarity.

Using the results of Dette’s (2000) work we can show the connections of the first return probabilities to the origin of $\mathcal{X}$ and $\tilde{\mathcal{X}}$. We have

$$\tilde{P}_{10}(n) = \tilde{p}_0 \tilde{q}_1 \int_{-1}^1 x^{n-1} \tilde{Q}_{i-1}^{(1)}(x) d\tilde{\psi}^{(1)}(x)$$

$$= \alpha^{-2} p_0 q_1 \sqrt{\frac{\pi_i-1}{\pi_{i-1}}} \int_{-1}^1 x^{n-1} Q_{i-1}^{(1)}(ax) d\psi^{(1)}(ax)$$

$$= \alpha^{-n-1} p_0 q_1 \sqrt{\frac{\pi_i-1}{\pi_{i-1}}} \int_{-1}^1 (ax)^{n-1} Q_{i-1}^{(1)}(ax) d\psi^{(1)}(ax)$$

$$= \alpha^{-n-1} \sqrt{\frac{\pi_i-1}{\pi_{i-1}}} P_{10}(n).$$
and
\[
\tilde{P}_{00}(n) = \tilde{p}_0 \tilde{q}_1 \int_{-1}^{1} x^n - 2 \, d\tilde{\psi}^{(1)}(x) = \alpha^{-2} p_0 q_1 \int_{-1}^{1} x^n - 2 \, d\psi^{(1)}(xk) = \alpha^{-n} p_0 q_1 \int_{-1}^{1} (xk)^n - 2 \, d\psi^{(1)}(xk) = \alpha^{-n} P_{00}(n).
\]

This is our claim. ■

The criterion of \(\alpha\)-similarity does not depend on the initial transition probability for a sufficiently small population.

The \(k\)th associated orthogonal polynomials fulfil the recurrence relation
\[
Q^{(k)}_1(x) = 0, \quad Q^{(k)}_0(x) = 1/p_{k-1},
\]
\[
xQ^{(k)}_j(x) = q_{j+k} Q^{(k)}_{j-1}(x) + r_{j+k} Q^{(k)}_j(x) + p_{j+k} Q^{(k)}_{j+1}(x), \quad j \geq 0, \quad k \geq 0,
\]
and the corresponding measure of orthogonality \(\psi^{(k)}\) plays a similar role in the consideration of connection between the first return probabilities \(P_{ij}(n)\) and \(\tilde{P}_{ij}(n), i > j,\) of \(X\) and \(\tilde{X},\) respectively.

**Corollary 1.** The random walk \(\tilde{X}\) is \(\alpha\)-similar to \(X\) if and only if the measures satisfy
\[
\tilde{\psi}^{(k)}(x) = \psi^{(k)}(xk), \quad x \in \mathbb{R}, \quad k \geq 0,
\]
and \(\alpha \geq \sup (\text{supp}(\psi^{(k)})).\) In this case the relation between the first return probabilities to the state \(k\) of the systems \(X\) and \(\tilde{X}\) is the following:
\[
\tilde{P}_{ik}(n) = \alpha^{-n-1} \sqrt{\frac{\pi_{i-k-1}}{\pi_{i-k-1}}} P_{ik}(n), \quad i > k, \quad i \geq 0.
\]

**Proof.** Consider the random walk \(X^k\) with one-step probabilities
\[
u_j^k = u_{j+k}, \quad r_j^k = r_{j+k}, \quad q_j^k = q_{j+k}
\]
and the first associated orthogonal polynomials
\[
\varphi^{(1)}_1(x) = 0, \quad \varphi^{(1)}_0(x) = 1/p^k_0;
\]
\[
x \varphi^{(1)}_j(x) = q_{j+1}^k \varphi^{(1)}_{j-1}(x) + r_{j+1}^k \varphi^{(1)}_j(x) + p_{j+1}^k \varphi^{(1)}_{j+1}(x), \quad j \geq 0, \quad k \geq 0.
\]

We can build the monic associated polynomials for the above ones, and proceed analogously to the proof of Theorem 1 to give the conclusion for the systems \(X^k\) and \(\alpha\)-similar \(\tilde{X}^k\) and the measures \(\tilde{\psi}^{(1)}\) and \(\tilde{\psi}^{(1)}\). The assertion of Corollary 1 follows from the recursive relation for the \((k+1)\)st associated orthogonal
polynomials. Using again results of Dette's (2000) work we can obtain the relation between the first return probabilities to the state $k$ of $\mathcal{X}$ and $\tilde{\mathcal{X}}$:

$$
\tilde{P}_{ik}(n) = \tilde{p}_k \tilde{q}_{k+1} \int_{-1}^{1} x^{n-1} \tilde{Q}_{l-k-1}^{(k+1)}(x) d\tilde{\psi}^{(k+1)} (x)
= \alpha^{-2} p_k q_{k+1} \sqrt{\frac{\pi_{i-k-1}}{\pi_{i-k-1}}} \int_{-1}^{1} x^{n-1} Q_{l-k-1}^{(k+1)}(xx) d\psi^{(k+1)} (xx)
= \alpha^{-n-1} p_k q_{k+1} \sqrt{\frac{\pi_{i-k-1}}{\pi_{i-k-1}}} \int_{-1}^{1} (xx)^{n-1} Q_{l-k-1}^{(k+1)}(xx) d\psi^{(k+1)} (xx)
= \alpha^{-n-1} \sqrt{\frac{\pi_{i-k-1}}{\pi_{i-k-1}}} P_{ik}(n).
$$

This completes our proof. ■

By proving Theorem 1 and Corollary 1 we have also shown that $\psi^{(k)}(xx)$ is a measure of orthogonality if and only if $\alpha \geq \sup (\text{supp} (\psi^{(k)}))$, $k \geq 1$.

**EXAMPLE 1.** Let us consider a random walk $\mathcal{X}$ with constant parameters $p_j = p$, $q_j = q$, $r_j = 0$, $j \geq 0$, and $p + q = 1$. In this case the first associated polynomials are of the form

$$
Q_j^{(1)}(x) = \left( \frac{\sqrt{q}}{p} \right)^j U_j \left( \frac{x}{2 \sqrt{pq}} \right), \quad j \geq 0,
$$

where $U_j(x)$ denotes the Chebyshev polynomials of the second kind. In such a situation $\text{sup} (\text{supp} (\psi^{(1)})) = 2 \sqrt{pq}$. Since $\alpha \geq 2 \sqrt{pq}$, let $b \geq 1$ such that $\alpha = 2b \sqrt{pq}$.

Schiefermayr (2003) showed that for $\mathcal{X}$ as in this example there exists a unique $\alpha$-similar random walk $\tilde{\mathcal{X}}$ with parameters $\tilde{p}_j$, $\tilde{q}_j$, $\tilde{r}_j$ given by

$$
\tilde{p}_j = \alpha^{-1} \frac{Q_{j+1}^{(1)}(x)}{Q_j^{(1)}(x)} p_j, \quad \tilde{q}_{j+1} = \alpha^{-1} \frac{Q_j^{(1)}(x)}{Q_{j+1}^{(1)}(x)} q_{j+1}, \quad \tilde{r}_j = \alpha^{-1} r_j, \quad j \geq 0,
$$

where $\tilde{q}_0 = 0$. In our example $\tilde{\pi}_j = (Q_j^{(1)}(x))^2 \cdot \pi_j$ and

$$
P_{10}(n) = p_0 q_1 \int_{-1}^{1} x^{n-1} Q_{l-1}^{(1)}(x) d\psi^{(1)} (x)
= \frac{2}{\pi} p q \int_{-1}^{1} x^{n-1} \left( \sqrt{\frac{p}{q}} \right)^{i-1} U_{i-1} \left( \frac{x}{2 \sqrt{pq}} \right) \sqrt{1 - \frac{x^2}{4pq}} dx
$$
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\[ P_{10}(n) = \frac{1}{U_{i-1}(b)} \frac{i}{n(2b)^{n+1}} \left( \begin{array}{c} n \\ (n-i)/2 \end{array} \right) \]

if \( i + n \) is even. For odd \( i + n \), \( P_{10} = 0 \). See Dette (2000) for more details.

Using the results of Theorem 1 we can calculate the first return probability to the origin for \( \mathcal{Y} \):

\[ \tilde{P}_{10}(n) = \frac{\sin(\text{arc cos } b)}{\sin(i \text{ arc cos } b)} \frac{i}{n(2b)^{n+1}} \left( \begin{array}{c} n \\ (n-i)/2 \end{array} \right). \]

3. BIRTH-AND-DEATH PROCESSES WITH SIMILAR TRANSITION PROBABILITIES

We will deduce analogous criterions of the similarity for the birth-and-death processes, relations between their measures and first return probabilities.

Let \( \mathcal{Y} = \{Y(t), t \geq 0\} \) denote a birth-and-death process, i.e. a stationary Markov process whose transition probability function

\[ P_{ij}(t) = \Pr \{ \{ Y(t) = j \} | \{ Y(0) = i \} \} \]

satisfies the conditions

\[ P_{j,j+1}(t) = \lambda_j t + o(t), \]

\[ P_{j,j}(t) = 1 - (\lambda_j + \mu_j) t + o(t), \]

\[ P_{j,j-1}(t) = \mu_j t + o(t) \]

as \( t \to 0 \). Constants \( \lambda_j \) (birth rates) and \( \mu_j \) (death rates) may be thought of as the rates of absorption from state \( j \) into states \( j + 1 \) and \( j - 1 \), respectively (\( \lambda_j > 0, \mu_j > 0, j = 0, 1, \ldots, \mu_0 \geq 0 \)). Karlin and McGregor (1957) have shown that the transition probabilities \( P_{ij} \) can be represented as

\[ P_{ij}(t) = \kappa_j \int_0^\infty e^{-xt} G_1(x) G_j(x) dQ(x), \]

\[ \kappa_0 = 1, \quad \kappa_j = \frac{\lambda_0 \lambda_1 \ldots \lambda_{j-1}}{\mu_1 \mu_2 \ldots \mu_j}, \quad j > 0. \]
\{G_j(x)\} is a sequence of birth-and-death polynomials defined recursively:

\begin{equation}
G_{-1}(x) = 0, \quad G_0(x) = 1, \\
-xG_j(x) = \mu_j G_{j-1}(x) - (\lambda_j + \mu_j) G_j(x) + \lambda_j G_{j+1}(x), \quad j \geq 1,
\end{equation}

and orthogonal with respect to the spectral measure \( \varrho \), i.e.

\[ \kappa_j \int_0^\infty G_i(x) G_j(x) d\varrho(x) = \delta_{ij}. \]

It is shown in the paper of Karlin and McGregor (1957) that there is at least one such measure with total mass 1 on \([0, \infty)\).

In the proofs we will use the monic polynomials

\[ W_j(x) = (-1)^j \lambda_0 \lambda_1 \ldots \lambda_{j-1} G_j(x), \quad j \geq 1, \]

which satisfy the recurrence relation

\begin{equation}
W_{-1}(x) = 0, \quad W_0(x) = 1, \\
W_{j+1}(x) = (x - \lambda_j - \mu_j) W_j(x) - \lambda_{j-1} \mu_j W_{j-1}(x), \quad j \geq 0.
\end{equation}

**Definition 2.** The birth-and-death process \( \mathcal{B} \) is said to be \( v \)-similar to the birth-and-death process \( \mathcal{Y} \) for some real number \( v \) if there are constants \( c_{ij}, i, j \geq 0 \), such that

\[ \bar{P}_{ij}(t) = c_{ij} e^{vt} P_{ij}(t), \quad i, j \geq 0, \quad t \geq 0. \]

See Lenin et al. (2000) for more details.

\( \mathcal{B} \) is the process with parameters \( \bar{x}_j, \bar{\mu}_j, j \geq 0 \), and polynomials \( G_j \) orthogonal with respect to the measure \( \bar{\varrho} \).

**Theorem 2.** The birth-and-death process \( \mathcal{B} \) is \( v \)-similar to \( \mathcal{Y} \) if and only if the distribution functions of the spectral measures satisfy

\[ \hat{\varrho}(x) = \varrho(x-v), \quad x \in \mathbb{R}, \]

and \( v \leq \inf(\text{supp}(\varrho)) \).

**Proof. Necessity.** We claim that

\begin{equation}
\bar{W}_j(x) = W_j(x-v).
\end{equation}

This is implied by the fact that for the birth-and-death processes \( \mathcal{Y} \) and \( \mathcal{B} \), where \( \mathcal{B} \) is \( v \)-similar to \( \mathcal{Y} \), their rates are related as follows:

\begin{equation}
\bar{x}_j + \bar{\mu}_j = \lambda_j + \mu_j - v, \quad \bar{x}_j \bar{\mu}_{j+1} = \lambda_j \mu_{j+1}, \quad j \geq 0.
\end{equation}

We conclude from (6) that \( \bar{G}_j(x) = \sqrt{\kappa_j/\bar{k}_j} G_j(x-v) \).
Next we claim that $\tilde{q}(x) = q(x-v)$ since

$$\delta_{ij} = \kappa_j \int_0^\infty G_i(x) G_j(x) dq(x) = \kappa_j \int_0^\infty G_i(x-v) G_j(x-v) dq(x-v)$$

$$= \kappa_j \int_0^\infty \tilde{G}_i(x) \tilde{G}_j(x) d\tilde{q}(x) = \kappa_j \int_0^\infty \tilde{G}_i(x) \tilde{G}_j(x) d\tilde{q}(x).$$

Since supp$(\tilde{q}) \subset [0, \infty)$, the parameter $v$ has to satisfy $v \leq \inf$(supp$(q)$).

**Sufficiency.** Let $q$ and $\tilde{q}$ be the spectral measures of $\mathcal{U}$ and $\mathcal{\tilde{U}}$, respectively. Let $\tilde{q}(x) = q(x-v)$, $v \leq \inf$(supp$(q)$), and $W_j$ be the corresponding system of monic orthogonal polynomials of $\mathcal{U}$ satisfying the recurrence relation (5).

Let $\tilde{W}_j$ be defined by (6). Hence, for $i \neq j$,

$$0 = \int_0^\infty W_i(x) W_j(x) dq(x) = \int_0^\infty W_i(x-v) W_j(x-v) dq(x-v)$$

$$= \int_0^\infty \tilde{W}_i(x) \tilde{W}_j(x) d\tilde{q}(x).$$

It follows that $\tilde{W}_j$ is the system of orthogonal polynomials of $\mathcal{\tilde{U}}$. From the equation (6) we obtain the recurrence relation of $\tilde{W}_j$, i.e.

$$W_{j+1}(x) = (x-\tilde{\lambda}_j + \tilde{\mu}_j - v)) W_j(x) - \tilde{\lambda}_{j-1} \tilde{\mu}_j W_{j-1}(x).$$

Comparing (5) and (8) we obtain the equalities for rates of $\mathcal{U}$ and $\mathcal{\tilde{U}}$ as in (7). Such connections of rates prove the v-similarity of $\mathcal{\tilde{U}}$, as shown by Lenin et al. (2000).

We can formulate the analogous theorem for measures of the orthogonality $q^{(1)}$ and $\tilde{q}^{(1)}$ of the first associated polynomials $G_j^{(1)}$ and $\tilde{G}_j^{(1)}$, where

$$G^{(1)}_0(x) = 0, \quad G^{(1)}_1(x) = -1/\lambda_0,$$

$$-x G^{(1)}_j(x) = \mu_{j+1} G^{(1)}_{j+1}(x) - (\lambda_{j+1} + \mu_{j+1}) G^{(1)}_j(x) + \lambda_{j+1} G^{(1)}_{j+1}(x), \quad j \geq 0.$$

The monic form of these polynomials is

$$W^{(1)}_j(x) = (-1)^{j+1} \lambda_1 \ldots \lambda_j G^{(1)}_j(x), \quad j \geq 0,$$

and satisfies the recurrence relation

$$W^{(1)}_0(x) = 0, \quad W^{(1)}_1(x) = 1,$$

$$W^{(1)}_{j+1}(x) = (x-\lambda_{j+1} - \mu_{j+1}) W^{(1)}_j(x) - \lambda_j \mu_{j+1} W^{(1)}_{j-1}(x), \quad j \geq 0.$$

**Theorem 3.** The necessary and sufficient condition of the v-similarity of $\mathcal{\tilde{U}}$ when considering $\mathcal{U}$ is the equality of measures

$$\tilde{q}^{(1)}(x) = q^{(1)}(x-v), \quad x \in \mathbb{R},$$
and \( v \leq \inf(\text{supp}(q^{(1)})) \). If \( \mathcal{W} \) is \( v \)-similar to \( \mathcal{Y} \), we have the following relation for the first return probabilities to the origin:

\[
P_{10}(t) = e^{-vt} \sqrt{\frac{\kappa_{i-1}}{\kappa_{i-1}}} P_{10}, \quad i \geq 1, \quad t \geq 0.
\]

**Proof.** The proof of the equivalence is analogous to the one of Theorem 2, but with using the polynomials \( G^{(1)}_j(x) \) and \( G^{(1)}_j(x) \) and their monic forms \( W^{(1)}_j(x) \) and \( W^{(1)}_j(x) \). Next we use the well-known formula for the probability of the first return to the origin (see van Doorn (2003)):

\[
\bar{P}_{10}(t) = \tilde{\lambda}_0 \tilde{v}_1 \int_0^\infty e^{-x \tilde{t}} \tilde{Q}^{(1)}_{i-1}(x) \, d\tilde{q}^{(1)}(x)
\]

\[
= \tilde{\lambda}_0 \tilde{v}_1 \sqrt{\frac{\kappa_{i-1}}{\kappa_{i-1}}} \int_0^\infty e^{-x \tilde{t}} Q^{(1)}_{i-1}(x-v) \, dq^{(1)}(x-v)
\]

\[
= e^{-vt} \sqrt{\frac{\kappa_{i-1}}{\kappa_{i-1}}} \tilde{\lambda}_0 \tilde{v}_1 \int_0^\infty e^{-(x-v)t} Q^{(1)}_{i-1}(x-v) \, dq^{(1)}(x-v) = e^{-vt} \sqrt{\frac{\kappa_{i-1}}{\kappa_{i-1}}} P_{10}(t).
\]

This is the desired conclusion. \( \blacksquare \)

This result can be generalized. Let us consider the \( k \)th associated polynomials

\[
G^{(k)}_j(x) = 0, \quad G^{(k)}_0(x) = -1/\lambda_{k-1},
\]

\[
-xG^{(k)}_j(x) = \mu_{j+k} G^{(k)}_{j-1}(x) - (\lambda_{j+k} + \mu_{j+k}) G^{(k)}_j(x) + \lambda_{j+k} G^{(k)}_{j+1}(x), \quad j \geq 0, \quad k \geq 0,
\]

orthogonal to the measure \( q^{(k)}(x) \). Our extension deals with the \( v \)-similarity and relation between such measures.

**Corollary 2.** The birth-and-death process \( \mathcal{W} \) is \( v \)-similar to \( \mathcal{Y} \) if and only if the measures satisfy

\[
\tilde{q}^{(k)} = q^{(k)}(x-v), \quad x \in \mathbb{R}, \quad k \geq 0,
\]

and \( v \leq \inf(\text{supp}(q^{(k)})) \).

**Proof.** We can proceed analogously to the proof of Corollary 1 from the previous section. We can build the birth-and-death process \( \mathcal{Y}^k \) with parameters

\[
\mu_j^k = \mu_{j+k}, \quad \lambda_j^k = \lambda_{j+k}
\]

and with the corresponding monic associated polynomials orthogonal to the measure \( q^{(1)}(x) \). The assertion is obtained by Theorem 3. \( \blacksquare \)

**Remark.** The proofs of Corollary 2 and Theorem 3 yield an additional information. It follows that \( q^{(k)}(x-v) \) is also a measure of the orthogonality of the \( k \)th associated polynomials if and only if \( v \leq \inf(\text{supp}(q^{(k)})) \).
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Instytut Matematyki i Fizyki
Politechnika Białostocka
Wiejska 45A
15-351 Białystok, Poland
E-mail: aposkrobko@wp.pl

Received on 25.4.2005;
revised version on 15.7.2005