SOME REMARKS ON THE MAXIMUM
OF A ONE-DIMENSIONAL DIFFUSION PROCESS

BY

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Abstract. For a certain class of one-dimensional diffusions \(X(t)\), we study the distribution of \(\max_{t \in [0,T]} X(t)\) and the distribution of the first instant at which \(X(t)\) attains the maximum by reducing \(X(t)\) to Brownian motion. Moreover, for \(T\) fixed or random, we study the asymptotics of threshold crossing probability, i.e. the rate of decay of \(P(\max_{s \in [0,T]} X(s) > z)\) as \(z\) goes to infinity. Some examples are also reported.

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1. INTRODUCTION

In this note, we consider a temporally homogeneous one-dimensional diffusion \(X\) and we study the distribution of the maximum process \(S_t = \max_{s \in [0,t]} X(s)\) or, equivalently, the distribution of the first-crossing time \(\tau_z\) of \(X(t)\) through a given threshold value \(z\). The knowledge of the distribution of \(S_t\) is very useful in a variety of applications ranging from biology and engineering to mathematical finance: for instance, when modelling neural activity, in queueing theory, in ruin problems, in modelling option prices.

Really, unlike the case of Brownian motion (BM), for general stochastic processes closed formulae for \(P(S_t \leq z)\) are not available; so in certain applications one is satisfied with the determination of the tail behaviour of \(P(S_T > z)\) for some fixed \(T > 0\). The evaluation of the tail probability \(P(S_T > z)\) for a fixed \(z\) is a key point in many statistical problems (for some examples in parametric and nonparametric statistics, in imaging processing and in genetical problems, see the references quoted in [10]).

For instance, a numerical method to compute the distribution of the maximum of a Gaussian random process was developed in [10]. Some theoretical results on Gaussian random fields were proved e.g. in [13], [14], [6]; in particular, if \(X(t)\) is a Gaussian process with stationary increments, under certain conditions on the
variance and covariance functions of the process, in [7] and [20] the following estimate was shown for \( z \to +\infty \):

\[
P(S_T > z) \sim \text{const} \cdot z^\beta \Psi \left( \frac{z}{\sigma_X(T)} \right),
\]

where \( \beta \) is a positive constant, \( \Psi(x) = P(W > x) \) is the tail distribution of a standard Gaussian random variable \( W \), and \( \sigma_X^2(t) \) is the variance function of \( X(t) \) (here \( f(x) \sim g(x) \) means that \( \lim_{x \to +\infty} f(x)/g(x) = 1 \)). In particular, as well known, if \( X(t) \) is (standard) BM, for any \( z \geq 0 \) the following equality holds:

\[
(1.1') \quad P(S_T > z) = 2\Psi \left( \frac{z}{\sqrt{T}} \right).
\]

A different type of asymptotics was obtained in [3] for a certain class of diffusions \( X(t) \) by studying the behaviour of \( P(S_T \leq z) \) for fixed \( z > 0 \), as \( T \to \infty \). By a variable change, \( X(t) \) transforms into a local martingale \( Y(t) \) (see Section 2); if its quadratic variation is bounded from above and below by two deterministic increasing functions \( \beta(t) \) and \( \alpha(t) \), then (see [3])

\[
(1.2) \quad \lim_{T \to \infty} \inf_{T \to \infty} P(S_T \leq z)\sqrt{\beta(T)} \geq c_-, \quad \lim_{T \to \infty} \sup_{T} P(S_T \leq z)\sqrt{\alpha(T)} \leq c_+,
\]

where \( c_- \) and \( c_+ \) are positive constants.

Another interesting aspect of the problem is to estimate \( P(S_T > z) \) in the case when \( T \) is a random variable independent of \( X \). The asymptotics of \( P(S_T > z) \), as \( z \to +\infty \), can be reduced, e.g., to those of an overflow probability in queueing theory (see [23]); another interpretation is that of a ruin probability for a certain risk process (see [23]).

Let \( X(t) \) be a Gaussian process with stationary increments, and assume that \( T \) is a nonnegative random variable, independent of \( X \), with regularly varying tail distribution at \( \infty \) with index \( \nu > 0 \), i.e. \( P(T > x) = L(x)x^{-\nu} \), \( L(\cdot) \) being a function slowly varying at \( \infty \). Moreover, suppose that the variance function \( \sigma_X^2(t) \) is continuous on \([0, +\infty)\) and it is regularly varying at infinity with index \( \alpha \in (0, 2] \). Then, under an additional condition on the behaviour of \( \sigma_X(t) \) at \( t = 0 \), the following holds (see [12]):

\[
(1.3) \quad P(S_T > z) \sim C \cdot P(T > \sigma_X^{-1}(z)) \quad \text{as} \quad z \to +\infty,
\]

where \( C = E \left( \max_{t \in [0,1]} B_\alpha(t) \right)^{\nu/\alpha} \) and \( B_\alpha(t) \) denotes a fractional Brownian motion (FBM) with Hurst parameter \( \alpha \in (0, 1] \) (i.e. a centered Gaussian process with stationary increments, continuous sample paths, \( B_\alpha(0) = 0 \) and \( \text{Var}(B_\alpha(t)) = t^{2\alpha} \)). In the special case when \( X(t) \) is FBM itself, formula (1.3) is nothing but the classical result of Breiman [8].
In this paper, motivated by the intention of extending the results above to Itô processes, we consider a class of temporally homogeneous one-dimensional diffusion processes \(X(t), t \geq 0\), characterized by drift \(b(x)\) and infinitesimal variance \(\sigma^2(x)\), where \(b(\cdot)\) and \(\sigma(\cdot)\) are regular enough functions. Therefore, our aim is to study the rate of decay of \(P(S_T > z)\) as \(z \to \infty\), and to show that results on the maximum process \(S_t\), analogous to (1.1) and (1.3), hold for \(X(t)\). Moreover, we show that the distribution of the first instant at which \(X(t)\) attains the maximum in the interval \([0, T]\) follows a compound arc-sine law. All this is done by reducing the process \(X(t)\) to BM, by using the arguments considered in [3] and [5], that is, by combining a deterministic transformation of the process \(X(t)\) with a random time-change.

2. NOTATION AND MAIN RESULTS

Let \(X(t) \in I = (r_1, r_2) (-\infty \leq r_1 \leq 0 < r_2 \leq +\infty)\) be the solution of the stochastic differential equation (SDE):

\[
(2.1) \quad dX(t) = b(X(t))\,dt + \sigma(X(t))\,dB_t, \quad X(0) = 0,
\]

where \(B_t\) is (standard) BM. Throughout the paper we will suppose that the usual conditions (see e.g. [15], [17]) for the existence and uniqueness of the solution of (2.1) are satisfied. Moreover, we require that \(X(t) \in I\) for all \(t \geq 0\) (for conditions implying this see e.g. [2], [16], [17]).

For \(x, y \in I = (r_1, r_2)\), let \(\tau_y(x) = \inf\{t > 0 : X(t) = y | X(0) = x\}\) be the first-hitting time of \(X\) to \(y\) when starting from \(x\). We recall that the diffusion \(X(t)\) is said to be regular if for any \(x, y \in I\) the condition \(P(\tau_y(x) < \infty) > 0\) holds, while it is said to be recurrent if for any \(x, y \in I\) the condition \(P(\tau_y(x) < \infty) = 1\) is satisfied (see e.g. [16], [3]). Let us consider now the infinitesimal generator \(L\) associated with the diffusion (2.1):

\[
(2.2) \quad Lh(x) = b(x)h'(x) + \frac{1}{2}h''(x)\sigma^2(x), \quad h \in C^2(I).
\]

The scale function \(u(x)\) is the solution of the problem:

\[
(2.3) \quad Lu(x) = 0, \quad x \in I; \quad u(0) = 0, \quad u'(0) = 1;
\]

\(u\) is strictly increasing and it is explicitly given by

\[
(2.4) \quad u(x) = \int_0^x \exp\left(-\int_0^t \frac{2b(z)}{\sigma^2(z)}\,dz\right)\,dt.
\]

If the boundaries \(r_1\) and \(r_2\) of \(I\) are unattainable (see e.g. [15], [16]), the recurrence of \(X(t)\) is equivalent to \(u(x) \to \infty\) as \(x \to r_i\). For instance, BM is recurrent, being
in this case \( u(x) = x \). The process \( Y(t) \equiv u(X(t)) \) turns out to be a local martingale, since by Itô’s formula we have \( dY(t) = u'(u^{-1}(Y(t))) \sigma(u^{-1}(Y(t))) dB_t \). The quadratic variation of the process \( Y(t) \) will be denoted by
\[
(Y)_t = \int_0^t [u'(X(s))\sigma(X(s))]^2 \, ds.
\]

Finally, we say that the diffusion process \( X(t) \in I \), which is the solution of the SDE (2.1), is conjugated to BM if there exists an increasing differentiable function \( v : I \to \mathbb{R} \) with \( v(0) = 0 \), such that the process \( Z(t) \equiv v(X(t)) \) is BM. Notice that if \( X(t) \) is conjugated to BM, then \( X \) is recurrent.

Now, we go to investigate the distribution of \( S_T = \max_{x \in [0,T]} X(s) \), where \( X(t) \) is the solution of (2.1). The maximum of the process \( X(t) \) is naturally related to \( \tau_z = \inf\{t > 0 : X(t) > z\} \); in order to make the first-crossing time problem meaningful, we shall assume that \( X(t) \) is recurrent.

**Theorem 2.1.** Let \( T > 0 \) be given, let us assume that the solution \( X(t) \) of (2.1) is recurrent, and that \( \langle Y \rangle_\infty = \infty \), where \( Y(t) = u(X(t)) \). Moreover, we suppose that there exist two deterministic, continuous increasing functions \( \alpha(t) \) and \( \beta(t) \), with \( \alpha(0) = \beta(0) = 0 \), such that, for every \( t < T \), \( \alpha(t) \leq \langle Y \rangle_t \leq \beta(t) \). Then, for any \( z > 0 \),
\[
(2.5) \quad 2\Psi\left(\frac{u(z)}{\sqrt{\alpha(T)}}\right) \leq P(S_T > z) \leq 2\Psi\left(\frac{u(z)}{\sqrt{\beta(T)}}\right),
\]
where
\[
\Phi(x) = 1 - \Psi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-t^2/2) \, dt,
\]
and \( u(\cdot) \) is given by (2.4).

**Proof.** For \( z > 0 \) we have
\[
P(S_T \leq z) = P\left(\max_{t \in [0,T]} u(X(t)) \leq u(z)\right) = P\left(\max_{t \in [0,T]} Y(t) \leq u(z)\right).
\]
Since \( \langle Y \rangle_\infty = \infty \), we can use a random time-change (see e.g. [21]), which implies that there exists a Wiener process \( \tilde{B}_t \) such that a.s. \( Y(t) = \tilde{B}_{\langle Y \rangle_t} \). Thus
\[
P(S_T \leq z) = P\left(\max_{t \in [0,T]} \tilde{B}_{\langle Y \rangle_t} \leq u(z)\right) = P\left(\max_{t \in [0,\langle Y \rangle_T]} \tilde{B}_t \leq u(z)\right).
\]
Since \( \alpha(t) \leq \langle Y \rangle_t \leq \beta(t) \), we obtain
\[
\max_{t \in [0,\alpha(T)]} \tilde{B}_t \leq \max_{t \in [0,\langle Y \rangle_T]} \tilde{B}_t \leq \max_{t \in [0,\beta(T)]} \tilde{B}_t.
\]
that is,
\[
P\left( \max_{t \in [0,T]} \tilde{B}_t \leq u(z) \right) \leq P\left( \max_{t \in [0,Y_T]} \tilde{B}_t \leq u(z) \right) \\
\leq P\left( \max_{t \in [0,T]} \tilde{B}_t \leq u(z) \right).
\]

Thus, (2.5) follows by (1.1').

If the quadratic variation \(\langle Y \rangle_t\) of \(Y(t)\) is deterministic, (2.5) becomes
\[
P(S_T > z) = 2\Psi\left( \frac{u(z)}{\sqrt{\alpha(T)}} \right)
\]
(note the affinity with (1.1)). Moreover, if \(X(t)\) is conjugated to BM by means of the function \(v\), then \(\langle Y \rangle_t = \alpha(t) = \beta(t) = t\), so we obtain
\[
P(S_T > z) = 2\Psi\left( \frac{v(z)}{\sqrt{t}} \right).
\]

Let \(T\) be a nonnegative random variable whose distribution \(F(t) = P(T \leq t)\) has regularly varying tails with index \(\nu \geq 0\), that is, \(P(T > t) = L(t)t^{-\nu}\), \(L(\cdot)\) being a function slowly varying at \(+\infty\) (i.e. \(\lim_{x,y \to +\infty} L(x)/L(y) = 1\)). We will write \(T \in RV(\nu)\), following the notation of [12]. A slightly weaker tail behaviour arises in the case when two positive constants \(a\) and \(b\) exist such that, as \(t \to +\infty\),
\[
a L(t)t^{-\nu} \leq 1 - F(t) \leq bL(t)t^{-\nu}.
\]
In this case we will write \(T \in V(\nu)\).

Before studying the case of a diffusion, we consider BM with drift \(\mu \in \mathbb{R}\). For \(T\) deterministic, the following holds (see [11]).

**Proposition 2.1.** Let \(T\) be given and fixed and let \(\mu < 0\); then, as \(z \to \infty\),
\[
P\left( \max_{t \in [0,T]} (B_t + \mu t) > z \right) \sim \Psi\left( \frac{z - \mu T}{\sqrt{T}} \right).
\]

For \(T\) random we have (see [23])

**Theorem 2.2.** Let \(T \in RV(\nu)\). Then \(B_T + \mu T\) and \(\max_{t \in [0,T]} (B_t + \mu t)\) are tail-equivalent, i.e.
\[
P\left( \max_{t \in [0,T]} (B_t + \mu t) > z \right) \sim P(B_T + \mu T > z) \quad \text{as} \; z \to +\infty.
\]

Now, let us go to consider a diffusion process which is the solution of the SDE (2.1); we obtain the following preliminary results:

**Proposition 2.2.** Let \(T \in RV(\nu)\) and let us suppose that all the assumptions of Theorem 2.1 are satisfied. Then
\[
L(\alpha^{-1}(z))\left(\alpha^{-1}(z)\right)^{-\nu} \leq P(\langle Y \rangle_T > z) \leq L(\beta^{-1}(z))\left(\beta^{-1}(z)\right)^{-\nu},
\]
where \(\langle Y \rangle_t\), \(\alpha(t)\), \(\beta(t)\) are defined in Theorem 2.1.
From the inequalities \( \alpha(t) \leq \langle Y \rangle_t \leq \beta(t) \) we obtain
\[
P(\alpha(T) > z) \leq P(\langle Y \rangle_T > z) \leq P(\beta(T) > z),
\]
i.e.
\[
P(T > \alpha^{-1}(z)) \leq P(\langle Y \rangle_T > z) \leq P(T > \beta^{-1}(z)).
\]
Since \( T \in RV(\nu) \), the assertion (2.8) immediately follows. ■

**Proposition 2.3.** Under the assumptions of Proposition 2.2, if \( \alpha^{-1}(t) \) and \( \beta^{-1}(t) \) are regularly varying at \( +\infty \) with index \( \gamma > 0 \), i.e. there exist constants \( c_\alpha, c_\beta > 0 \) such that \( \alpha^{-1}(t) \sim c_\alpha t^\gamma \) and \( \beta^{-1}(t) \sim c_\beta t^\gamma \) as \( t \to \infty \), then \( \langle Y \rangle_T \in V(\gamma \nu) \). In the case when \( c_\alpha = c_\beta \), we have \( \langle Y \rangle_T \in RV(\gamma \nu) \).

**Proof.** Since \( \alpha^{-1}(t) \) and \( \beta^{-1}(t) \) are regularly varying at \( +\infty \) with index \( \gamma \), it follows that as \( z \to \infty \), \( (\alpha^{-1}(z))^{-\nu} \sim c_\alpha z^{-\gamma \nu} \), \( (\beta^{-1}(z))^{-\nu} \sim c_\beta z^{-\gamma \nu} \), and so \( \langle Y \rangle_T \in V(\gamma \nu) \). The other assertion can be trivially verified. ■

**Remark 2.1.** If \( \langle Y \rangle_t \) is deterministic, then \( \langle Y \rangle_t = \alpha(t) \equiv \beta(t) \), so Proposition 2.2 implies that \( P(\langle Y \rangle_T > z) = L(\alpha^{-1}(z)) (\alpha^{-1}(z))^{-\nu} \), while if \( c = c_\alpha = c_\beta \), Proposition 2.3 implies that \( P(\langle Y \rangle_T > z) \sim c L(z) z^{-\gamma \nu} \).

Now, we can obtain

**Theorem 2.3.** Let \( T \in RV(\nu) \) and let us suppose that all the assumptions of Theorem 2.1 are satisfied. Moreover, let us assume that the functions \( \alpha^{-1}(t) \) and \( \beta^{-1}(t) \) are regularly varying at \( +\infty \) with index \( \gamma > 0 \). Then for \( z \to +\infty \) it follows that
\[
(2.9) \quad L(\alpha^{-1}(z^2)) (\alpha^{-1}(z^2))^{-\nu} \leq P(S_T > z) \leq L(\beta^{-1}(z^2)) (\beta^{-1}(z^2))^{-\nu},
\]
where \( L \) is a function slowly varying at \( \infty \).

**Proof.** We recall from [12] that if \( \tilde{B}_t \) is BM and \( \Lambda \) is a nonnegative random variable such that \( \Lambda \in RV(\mu) \), then as \( z \to \infty \)
\[
(2.10) \quad P(\max_{s \in [0,1]} \tilde{B}_s > z) = P(\sqrt{\Lambda} \max_{s \in [0,1]} \tilde{B}_s > z) \sim E(\max_{s \in [0,1]} \tilde{B}_s)^{2\mu} P(\Lambda > z^2).
\]
Moreover,
\[
E(\max_{s \in [0,1]} \tilde{B}_s)^{2\mu} = \frac{2\mu}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} + \mu\right) \equiv \mathcal{E}(\mu).
\]
Then
\[
P(S_T > z) = P(\max_{s \in [0,T]} Y(s) > u(z)) = P(\max_{s \in [0,T]} \tilde{B}_s > u(z)) = P(\max_{s \in [0,\mu(T)]} \tilde{B}_s > u(z)).
\]
Consequently, if \( \langle Y \rangle_t \) is deterministic, we have \( \langle Y \rangle_T \in RV(\gamma \nu) \). Therefore, by (2.10) with \( \Lambda = \langle Y \rangle_T \) and \( \mu = \gamma \nu \), we get, for \( z \to \infty \),

\[
P(S_T > z) \sim E\left( \max_{s \in [0,1]} \tilde{B}_s \right)^{2\gamma \nu} P(\langle Y \rangle_T > z^2) = \mathcal{E}(\gamma \nu) L(z^2) z^{-2\gamma \nu}.
\]

If \( \langle Y \rangle_t \) is non-deterministic, by the same arguments as those used in the proof of Proposition 2.3, we see that \( a, b > 0 \) exist such that

\[
a E\left( \max_{s \in [0,1]} \tilde{B}_s \right)^{2\gamma \nu} P(\langle Y \rangle_T > z^2) \leq P(S_T > z) \leq b E\left( \max_{s \in [0,1]} \tilde{B}_s \right)^{2\gamma \nu} P(\langle Y \rangle_T > z^2).
\]

Since \( P(T > \alpha^{-1}(z^2)) \leq P(\langle Y \rangle_T > z^2) \leq P(T > \beta^{-1}(z^2)) \), the assertion (2.9) easily follows. \( \blacksquare \)

**Remark 2.2.** A trivial case where \( \langle Y \rangle_t \) is deterministic occurs when \( X(t) \equiv Y(t) = \int_0^t \bar{\sigma}(s) dB_s \), where \( \bar{\sigma}(\cdot) > 0 \) is a deterministic function and \( \langle Y \rangle_t = \int_0^t \bar{\sigma}^2(s) ds \) behaves like \( t^\gamma \), \( t \to \infty \). For such a process, if \( T \) is given and fixed, it follows that

\[
P(S_T > z) = 2 \Psi(z/\sqrt{\int_0^T \bar{\sigma}^2(s) ds}).
\]

Instead, if \( T \in RV(\nu) \), let \( \rho^{-1}(s) \) be the inverse function of \( \rho(t) \equiv \langle Y \rangle_t \) and let us suppose that \( \rho^{-1}(s) \) behaves like \( s^\gamma \) \( (s \to \infty) \). By using also (2.9), for \( z \to \infty \) we obtain

\[
P(S_T > z) = P(\max_{s \in [0,T]} B_{\rho(s)} > z)
= P\left( \max_{s \in [0,\rho(T)]} B_{s} > z \right) \sim \mathcal{E}(\gamma \nu) P(\rho(T) > z^2)
= \mathcal{E}(\gamma \nu) P\left( \int_0^T \bar{\sigma}^2(s) ds > z^2 \right) = \mathcal{E}(\gamma \nu) L(z^2) z^{-2\gamma \nu}.
\]

**THE ARC-SINE LAW**

Let us consider now, for \( T \) given and fixed, the first instant \( \theta \) at which \( X(t) \) attains its maximum value in the interval \([0, T]\), i.e.

\[
X(\theta) = \max_{t \in [0,T]} X(t) = S_T.
\]

Notice that \( \theta \) is not a stopping time. As is well known (see [19]), when \( X(t) \equiv B_t \), the distribution of \( \theta \) follows the arc-sine law, that is:

(2.11) \[
P(\theta \leq t) = \frac{2}{\pi} \arcsin \sqrt{\frac{t}{T}}, \quad 0 < t < T.
\]
Let $X(t)$ be a diffusion process satisfying the assumptions of Theorem 2.1; first we suppose that the quadratic variation $(Y)_t = \rho(t)$ is deterministic. Then

$$u(S_T) = \max_{t \in [0,T]} u(X(t)) = \max_{t \in [0,T]} \tilde{B}_\rho(t) = \max_{s \in [0,\rho(T)]} \tilde{B}_s,$$

and so $\tilde{B}_\rho(\theta) = u(X(\theta)) = \max_{t \in [0,\rho(T)]} \tilde{B}_t$. Therefore, $\rho(\theta)$ follows the arc-sine law, i.e.

$$P(\rho(\theta) \leq t) = \frac{2}{\pi} \arcsin \sqrt{\frac{t}{\rho(T)}}, \quad t \in [0, \rho(T)].$$

Thus,

$$P(\theta \leq t) = P(\rho(\theta) \leq \rho(t)) = \frac{2}{\pi} \arcsin \sqrt{\frac{\rho(t)}{\rho(T)}}, \quad t \in [0, T].$$

In particular, if $X(t)$ is conjugated to BM, then $\rho(t) = t$, and so $\theta$ follows the arc-sine law. If $\rho(t)$ is not deterministic, recalling that $\alpha(t) \leq \rho(t) \leq \beta(t)$, we get $\max_{t \in [0,\alpha(T)]} B_t \leq u(S_T) \leq \max_{t \in [0,\beta(T)]} B_t$. If we denote by $\tilde{\theta}_\alpha$ and $\tilde{\theta}_\beta$ the first instant at which $B_t$ attains its maximum in the interval $[0, \alpha(T)]$ and in the interval $[0, \beta(T)]$, respectively, we obtain $\tilde{\theta}_\alpha \leq \rho(\theta) \leq \tilde{\theta}_\beta$. Thus,

$$\frac{2}{\pi} \arcsin \sqrt{\frac{t}{\beta(T)}} \leq P(\rho(\theta) \leq t) \leq \frac{2}{\pi} \arcsin \sqrt{\frac{t}{\alpha(T)}}, \quad t \in (0, \alpha(T)),$$

and therefore

$$\frac{2}{\pi} \arcsin \sqrt{\frac{\alpha(t)}{\beta(T)}} \leq P(\theta \leq t) \leq \frac{2}{\pi} \arcsin \sqrt{\frac{\beta(t)}{\alpha(T)}}, \quad 0 < t < \beta^{-1}(\alpha(T)).$$

In the special case when $X(t)$ is an integral process with deterministic integrand, by combining the result of [4] and that of [22], we are able to obtain

**Theorem 2.4.** Let $X(t) = \int_0^t \tilde{\sigma}(s) dB_s$, where $\tilde{\sigma}(\cdot)$ is a deterministic positive function, and let us suppose that $\rho(\infty) = \infty$, where $\rho(t) = \int_0^t \tilde{\sigma}^2(s) ds$. Then the following holds:

$$\inf_{\tau \in [0,1]} E[S_T - X(\tau)]^2 = \inf_{\tau \in [0,1]} E[X(\theta) - X(\tau)]^2 = \rho(1) \cdot \inf_{\tau \in [0,1]} \left[ E[\tilde{\theta} - \tau] + \frac{1}{2} \right],$$

where the infimum is taken over all stopping times $\tau \in [0, 1]$ of $X(t)$ and $\tilde{\theta}$ is the first instant at which $\tilde{B}_t$ attains its maximum in the interval $[0, 1]$.

Moreover, let us consider the following two optimal stopping problems:

(i) $\inf_{\tau \in (0,1)} E \left[ X(\theta) - X(\tau) \right]^2$ and (ii) $\inf_{\tau \in (0,1)} E \left[ |\theta - \tau| \right]^2$. 
Then the optimal stopping times in (i) and (ii) are the same and they are equal to

$$
\tau^* = \inf\{0 < t < \rho(1) : \max_{s \in [0,t]} B_s - B_t \geq z_0^* \sqrt{\rho(1) - t}\},
$$

where $z_0^*$ is the unique positive root of the equation $4\Phi(z) - 2z\phi(z) - 3 = 0$, and $\phi(z)$ and $\Phi(z)$ are the density and the distribution function, respectively, of a standard Gaussian variable. If one considers the optimal stopping problem (i) or (ii) in the interval $[0, T]$, in the above formulae $\rho(1)$ has to be replaced with $\rho(T)$.

**Proof.** Let $(\mathcal{F}^B_t)_{t \leq 1}$ be the filtration generated by $\hat{B}_t$. Then by Lemma 1 of [22] which is true for BM, for any $(\mathcal{F}^B_t)$-stopping time $\tau \in [0, 1]$, we have

$$
E[\hat{B}_\tau - B_\tau]^2 = E[|\hat{\theta} - \tau|] + \frac{1}{2}.
$$

By reducing $X(t)$ to BM, we obtain

$$
\inf_{\tau \in [0,1]} E[X(\theta) - X(\tau)]^2 = \inf_{\tau \in [0,1]} E[\hat{B}_\rho(\theta) - \hat{B}_\rho(\tau)]^2 = \inf_{\tau' \in [0,\rho(1)]} E[\hat{B}_{\theta'} - \hat{B}_{\tau'}]^2,
$$

where we have set $\theta' = \rho(\theta)$ and $\tau' = \rho(\tau) \in [0, \rho(1)]$. By the scaling property of BM, $\hat{B}_{\rho(1)\cdot t}/\sqrt{\rho(1)} \equiv W_t$ is also BM. Thus, setting $s = \tau'/\rho(1)$, we can write the last infimum as

$$
\rho(1) \inf_{s \in [0,1]} E[\max_{\tau \in [0,1]} W_{\tau'} - W_{\tau}]^2 = \rho(1) \inf_{\tau \in [0,1]} E[\hat{B}_\theta - \hat{B}_{\tau}]^2,
$$

which implies the main assertion. The other assertions are obtained by using Theorem 2.2 of [4].

**3. SOME EXAMPLES**

**Example 3.1 (Ornstein–Uhlenbeck process).** For $b, \sigma > 0$, let us consider the process $X(t)$ such that $dX(t) = -bX(t)dt + \sigma dB_t$, $X(0) = x_0$. The explicit solution is $X(t) = e^{-bt}U(t)$, where $U(t) = x_0 + \int_0^t \sigma e^{bs} dB_s$. Setting $Y(t) = \int_0^t \sigma e^{bs} dB_s$ and using a random time-change, we obtain $U(t) = x_0 + B_{\rho(t)}$, where the quadratic variation $\langle Y \rangle_t = (\sigma^2/2b)(e^{2bt} - 1)$ is deterministic. Thus, we have

$$
P(\bar{S}_T > z) = P\left( \max_{s \in [0,T]} e^{-bs}U(s) > z \right) \leq P\left( \max_{s \in [0,T]} U(s) > z \right)$$

$$= P\left( \max_{s \in [0,T]} B_{\rho(s)} > z - x_0 \right) = P\left( \max_{t \in [0,\rho(T)]} B_t > z - x_0 \right).$$
Let us consider the process \( X_t \) given and fixed, we obtain

\[
P(S_T > z) \leq 2\Psi\left( \frac{z - x_0}{\sqrt{\rho(T)}} \right).
\]

Instead, if \( T \) is random and \( P(T > z) = L(z)(\rho(z))^{-\nu} (z \to \infty) \), then we have

\[
P(\rho(T) > z) = P(T > \rho^{-1}(z)) = L(\rho^{-1}(z))z^{-\nu}, \text{ and so } \rho(T) \in RV(\nu). \]

By using Theorem 2.3, we obtain

\[
P(\max_{s \in [0,T]} X(s) > z) \leq \frac{2^\nu}{\sqrt{\pi}} \Gamma\left( \frac{1}{2} + \nu \right) L(\rho^{-1}((z - x_0)^2))(z - x_0)^{-2\nu}.
\]

**Example 3.2 (Feller process).** Let us consider the process \( X(t) \in [0, +\infty) \) such that \( dX(t) = \frac{1}{2}dt + \sqrt{X(t)} \dot{W}_t, X(0) = 0 \). The process \( X(t) \) is conjugated to Brownian motion by means of the function \( v(x) = 2\sqrt{x} \). Thus, if \( T \) is given and fixed, we obtain

\[
P(S_T > z) = P(\max_{s \in [0,T]} v(X(s)) > v(z)) = P(\max_{s \in [0,T]} B_s > 2\sqrt{z}) = 2\Psi\left( \frac{2\sqrt{z}}{\sqrt{T}} \right).
\]

If \( T \in RV(\nu) \), for \( z \to \infty \) we get

\[
P(S_T > z) = P(\max_{s \in [0,T]} B_s > 2\sqrt{z}) \sim \frac{2^\nu}{\sqrt{\pi}} \Gamma\left( \frac{1}{2} + \nu \right) P(T > 4z)
= \frac{2^\nu}{\sqrt{\pi}} \Gamma\left( \frac{1}{2} + \nu \right) L(4z)(4z)^{-\nu}.
\]

For what concerns \( \theta \), it follows the arc-sine law, since \( X(t) \) is conjugated to BM.

**Example 3.3 (Geometrical Brownian motion).** For \( b, \sigma > 0 \), let us consider the diffusion equation \( dX(t) = bX(t) + \sigma X(t) dB_t, X(0) = x_0 \). In the framework of the Black–Scholes model in mathematical finance, it describes the time course of the price \( X(t) \) of risky assets. The explicit solution is given by \( X(t) = x_0 \exp\left( (b - \sigma^2/2)t + \sigma B_t \right) \). If \( t \leq \sigma^2/2 \), we get

\[
P(S_T > z) \leq P\left( \max_{s \in [0,T]} \exp(\sigma B_s) > \frac{z}{x_0} \right) = P\left( \max_{s \in [0,T]} B_s > \frac{\log(z/x_0)}{\sigma} \right).
\]

Consequently, if \( T \) is given and fixed, we have \( P(S_T > z) \leq 2\Psi\left( \frac{\log(z/x_0)}{\sigma\sqrt{T}} \right) \); if \( T \in RV(\nu) \), for \( z \to \infty \) we obtain

\[
P(S_t > z) \leq \frac{2^\nu}{\sqrt{\pi}} \Gamma\left( \frac{1}{2} + \nu \right) L\left( \frac{\log^2(z/x_0)}{\sigma^2} \right) \left( \frac{\log(z/x_0)}{\sigma} \right)^{-2\nu}.
\]
EXAMPLE 3.4 (Wright & Fisher-like process). Let $X(t)$ be such that

$$dX(t) = \left(\frac{1}{4} - \frac{1}{2} X(t)\right) dt + \sqrt{X(t)(1 - X(t))} \vee 0 \, dB_t, \quad X(0) = 0.$$ 

It is a particular case of the Wright & Fisher diffusion equation for population genetics, and it is also used in certain diffusion models for neural activity (see [18]). It can be shown (see e.g. [1] and [2]) that $X(t) \in [0, 1]$ for all $t \geq 0$, and it is conjugated to BM by means of the function $v(x) = 2 \arcsin \sqrt{x}$. For $T$ given and fixed, we have

$$P(S_T > z) = 2 \Psi\left(\frac{2 \arcsin \sqrt{z}}{\sqrt{T}}\right).$$

Clearly, $\theta$ follows the arc-sine law.

EXAMPLE 3.5. The usefulness of inequalities (2.5) for $T$ fixed, and (2.9) for $T \in RV(\nu)$, relies on the fact that the function $\alpha(t)$ is close enough to $\beta(t)$. Here we show an example of diffusion satisfying all the assumptions of Section 2, for which this holds. Let $\sigma > 0$ and $\epsilon > 0$, and consider the SDE

$$dX(t) = \frac{\epsilon \sigma^2 \sin(2X(t))}{2(1 + \epsilon \cos X(t))} dt + \sigma dB_t.$$ 

We have

$$u(x) = \frac{1}{1 + \epsilon} \left[ \left(1 + \frac{\epsilon}{2}\right)x + \frac{\epsilon}{4} \sin(2x) \right].$$

As easily seen, $X(t)$ is recurrent, and

$$\langle Y \rangle_t = \int_0^t \frac{(1 + \epsilon \cos^2 X(s))}{(1 + \epsilon)^2} \sigma^2 ds$$

(note that $\langle Y \rangle_t$ is non-deterministic). Thus

$$\alpha(t) \doteq \frac{\sigma^2 t}{(1 + \epsilon)^2} \leq \langle Y \rangle_t \leq \sigma^2 t \doteq \beta(t).$$

Evidently, if $\epsilon \approx 0$, then $\beta(t) \approx \alpha(t)$.

EXAMPLE 3.6 (A temporally non-homogeneous SDE). For $t \in [0, 1]$, let us consider the SDE

$$dZ(t) = -\frac{Z(t)}{1-t} dt + dB_t, \quad Z(0) = Z(1) = 0,$$
whose solution is
\[ Z(t) = (1 - t) \int_0^t \frac{1}{1 - s} dB_s. \]

The diffusion \( Z(t) \) is the Brownian bridge, i.e. BM conditioned to take the value 0 at time \( t = 1 \). Now, set \( X(t) = Z(t)/(1 - t) \). The quadratic variation of \( X(t) \) is \( \langle X \rangle_t = \rho(t) = t/(1 - t) \), \( 0 \leq t \leq 1 \). Consequently, by a random time-change we obtain \( X(t) = \tilde{B}(t/(1 - t)) \) for a suitable BM \( \tilde{B} \). If \( T \in (0, 1) \) is given and fixed and \( z > 0 \), we have
\[
P(S_T > z) = P(\max_{t \in [0, T]} X(t) > z) = 2\Psi \left( \frac{z}{1 - T} \right).
\]

If \( \tau^Z \) denotes the first-passage time of \( Z(t) \) over the straight line \( y = z(1 - t) \), we get \( P(S_T > z) = P(\tau^Z \leq T) \). The distribution of the first instant \( \theta \) at which \( X(t) \) attains its maximum is
\[
P(\theta \leq t) = \frac{2}{\pi} \arcsin \left( \frac{t(1 - T)}{T(1 - t)} \right), \quad 0 \leq t < T < 1.
\]

4. CONCLUSION AND FINAL REMARKS

For a certain class of one-dimensional diffusions \( X(t) \), we have studied the rate of decay of \( P(\max_{s \in [0, T]} X(s) > z) \) as \( z \to \infty \), both in the case when \( T \) is fixed (Theorem 2.1) and when \( T \) is a random variable independent of the process \( X \) (Theorem 2.3). Moreover, we have studied the distribution of the first instant at which \( X(t) \) attains its maximum. By combining a deterministic transformation of the process \( X(t) \) with a random time-change, the quantities under investigation have been related to those regarding a suitable Brownian motion.

It is worthwhile to note that, by using the Donsker’s approximating procedure, it is possible to approximate \( B_t \) by means of a binomial random walk. Precisely, let \( \{U_n\}_{n \geq 1} \) be a sequence of independent identically distributed random variables such that \( P(U_n = -1) = P(U_n = 1) = \frac{1}{2} \). Setting \( V_0 = 0 \) and \( V_n = U_1 + \ldots + U_n \), \( n \geq 1 \), it is well known that the sequence of processes defined by
\[
B^{(n)}_t = \left( V_{[nt]} + (nt - [nt])U_{[nt]+1} \right)/\sqrt{n}
\]
converges in distribution to a standard BM as \( n \to \infty \) (see e.g. [21]).

Now, let \( X(t) \) be a diffusion belonging to the class considered in Theorem 2.1 or Theorem 2.3, for which \( \rho(t) = (Y)_t \) is deterministic. Using the arguments and the notation of Section 2, we are able to infer that the sequence of processes defined by
\[
X^{(n)}(t) = u^{-1}\left( \left( V_{[n\rho(t)]} + (n\rho(t) - [n\rho(t)])U_{[n\rho(t)]+1} \right)/\sqrt{n} \right)
\]
On the maximum of a one-dimensional diffusion

converges in distribution to $X(t)$ as $n \to \infty$ (note, however, that this approximation scheme becomes somehow ambiguous if $\rho(t)$ is a random process). Thus, since quantities of the form $E[f(B_t, \max_{s \in [0,t]} B_s)]$ have been estimated, e.g. in [9], by using the Donsker’s approximating procedure, analogous estimations could be obtained for quantities of the form $E[f(X(t), \max_{s \in [0,t]} X(s))]$, by using $X^{(n)}(t)$.

We conclude noting that the arguments of this paper can be used also to obtain information about the distribution of $s(T) = \min_{t \in [0,T]} X(t)$, by using the distribution of the minimum of BM, that is,

$$P(\min_{t \in [0,T]} B_t \leq x) = 1 \quad \text{if } x \geq 0,$$

$$P(\min_{t \in [0,T]} B_t \leq x) = 2\Phi\left(\frac{x}{\sqrt{T}}\right) \quad \text{if } x < 0.$$

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REFERENCES


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