PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 28, Fasc. 1 (2008), pp. 143–162

ON BESOV REGULARITY OF BROWNIAN MOTIONS IN INFINITE DIMENSIONS

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Abstract. We extend to the vector-valued situation some earlier work of Ciesielski and Roynette on the Besov regularity of the paths of the classical Brownian motion. We also consider a Brownian motion as a Besov space valued random variable. It turns out that a Brownian motion, in this interpretation, is a Gaussian random variable with some pathological properties. We prove estimates for the first moment of the Besov norm of a Brownian motion. To obtain such results we estimate expressions of the form $\mathbb{E}\sup_{n\geqslant 1}\|\xi_n\|$, where ξ_n are independent centered Gaussian random variables with values in a Banach space. Using isoperimetric inequalities we obtain two-sided inequalities in terms of the first moments and the weak variances of ξ_n.

2000 AMS Mathematics Subject Classification: Primary: 60J65; Secondary: 28C20, 46E40, 60G17.

Key words and phrases: Gaussian random variable, maximal estimates, Besov–Orlicz norm, non-separable Banach space, sample path.

1. INTRODUCTION

Let $(\Omega,\mathcal{A},\mathbb{P})$ be a complete probability space. Let $W\colon [0,1]\times\Omega\to\mathbb{R}$ be a standard Brownian motion. Since W has continuous paths, it is easy to check that $W\colon\Omega\to C([0,1])$ is a $C([0,1])$-valued Gaussian random variable. Moreover, since W is α-Hölder continuous for all $\alpha\in(0,1/2)$, one can also show that, for all $0<\alpha<1/2$, $W\colon\Omega\to C^\alpha([0,1])$ is a Gaussian random variable. In this way one obtains results like

$$\mathbb{E}\exp(\varepsilon\|W\|^2_{C^\alpha([0,1])})<\infty \quad \text{for some } \varepsilon>0.$$

In [2] and [3] Ciesielski has improved the Hölder continuity results of Brownian motion using Besov spaces. He has proved that almost all paths of W are

* Supported by the Academy of Finland (grant 114374).

** Supported by the Netherlands Organisation for Scientific Research (NWO) 639.032.201 and by the Research Training Network MRTN-CT-2004-511953.

in the Besov space $B^{1/2}_{p,\infty}(0,1)$ for all $p \in [1,\infty)$ or even in the Besov–Orlicz space $B^{1/2}_{\Phi_2,\infty}(0,1)$, where $\Phi_2(x) = \exp(x^2) - 1$ (for the definition we refer to Section 2). In [11] Roynette has characterized the set of indices α, p, q for which the paths of Brownian motion belong to the Besov spaces $B^{\alpha}_{p,q}(0,1)$.

The proofs of the above results are based on certain coordinate expansions of the Brownian motion and descriptions of the Besov norms in terms of the corresponding expansion coefficients of a function. We will give more direct proofs of these results which employ the usual modulus-of-continuity definition of the Besov norms. Our methods also carry over to the vector-valued situation.

Let X be a real Banach space. We will write $a \lesssim b$ if there exists a universal constant $C > 0$ such that $a \leqslant Cb$, and $a \eqsim b$ if $a \lesssim b \lesssim a$. If the constant C is allowed to depend on some parameter t, we write $a \lesssim_t b$ and $a \eqsim_t b$ instead. Let $(l^{\Theta}, \|\cdot\|_{\Theta})$ denote the Orlicz sequence space with $\Theta(x) = x^2 \exp(-1/x^2)$. Let $(\xi_n)_{n\geqslant 1}$ be independent centered X-valued Gaussian random variables with weak variances $(\sigma_n)_{n\geqslant 1}$ and $m = \sup_{n\geqslant 1} \mathbb{E}\|\xi_n\|$. In Section 3 we will show that

$$\mathbb{E} \sup_{n\geqslant 1} \|\xi_n\| \eqsim m + \|(\sigma_n)_{n\geqslant 1}\|_{\Theta}. \tag{1.1}$$

As a consequence of the Kahane–Khinchine inequalities a similar estimate holds for $(\mathbb{E}\sup_{n\geqslant 1} \|\xi_n\|^p)^{1/p}$ for all $p \in [1,\infty)$ as well, at the cost of replacing $\eqsim$ by $\eqsim_p$. The proof of (1.1) is based on isoperimetric inequalities for Gaussian random variables (cf. [9]).

In Section 4 we obtain regularity properties of X-valued Brownian motions W. In particular, we show that for the paths of an X-valued Brownian motion W we have $W \in B^{1/2}_{p,\infty}(0,1;X)$ for all $p \in [1,\infty)$ or even $W \in B^{1/2}_{\Phi_2,\infty}(0,1;X)$. Thus we can consider the mappings

$$W\colon \Omega \to B^{1/2}_{p,\infty}(0,1;X) \quad \text{and} \quad W\colon \Omega \to B^{1/2}_{\Phi_2,\infty}(0,1;X).$$

A natural question is whether W is a Gaussian random variable with values in one of these spaces. To answer this question some problems have to be solved, because the Banach spaces $B^{1/2}_{p,\infty}(0,1)$ and $B^{1/2}_{\Phi_2,\infty}(0,1)$ are non-separable. It will be shown in Section 5 that W is indeed a Gaussian random variable, but it has some peculiar properties. For instance, we find that there exists an $\varepsilon > 0$ such that

$$\mathbb{P}(\|W\|_{B^{1/2}_{p,\infty}(0,1;X)} \leqslant \varepsilon) = \mathbb{P}(\|W\|_{B^{1/2}_{\Phi_2,\infty}(0,1;X)} \leqslant \varepsilon) = 0$$

which is rather counterintuitive for a centered Gaussian random variable. It implies in particular that W is not Radon. In the last Section 6 we apply the results from Section 3 to obtain explicit estimates for $\mathbb{E}\|W\|_{B^{1/2}_{p,\infty}(0,1;X)}$ and $\mathbb{E}\|W\|_{B^{1/2}_{\Phi_2,\infty}(0,1;X)}$.

2. PRELIMINARIES

2.1. Orlicz spaces. We briefly recall the definition of Orlicz spaces. More details can be found in [7], [10] and [14].

Let (S, Σ, μ) be a σ-finite measure space and let X be a Banach space. Let $\Phi : \mathbb{R} \to \mathbb{R}_+$ be an even convex function with $\Phi(0) = 0$ and $\lim_{x\to\infty} \Phi(x) = \infty$. The Orlicz space $L^\Phi(S; X)$ is defined as the set of all strongly measurable functions $f : S \to X$ (identifying functions which are equal μ-a.e.) with the property that there exists a $\delta > 0$ such that

$$M_\Phi(f/\delta) := \int_S \Phi\big(\|f(s)\|/\delta\big)\, d\mu(s) < \infty.$$

This space is a vector space and we define

$$\rho_\Phi(f) = \inf\{\delta > 0 : M_\Phi(f/\delta) \leqslant 1\}.$$

The mapping ρ_Φ defines a norm on $L^\Phi(S; X)$ and it turns $L^\Phi(S; X)$ into a Banach space. It is usually referred to as the *Luxemburg norm*.

For $f \in L^\Phi(S; X)$ we also define the *Orlicz norm*

$$\|f\|_\Phi = \inf_{\delta>0} \left\{ \frac{1}{\delta}\big(1 + M_\Phi(\delta f)\big) \right\}.$$

The Orlicz norm is usually defined in a different way using duality, but the above norm gives exactly the same number (cf. [10], Theorem III.13).

The two norms are equivalent, as shown in the following:

LEMMA 2.1. *For all $f \in L^\Phi(S; X)$ we have*

$$\rho_\Phi(f) \leqslant \|f\|_\Phi \leqslant 2\rho_\Phi(f).$$

Proof. Let $\delta > 0$ be such that $M_\Phi(f\delta) \leqslant 1$. Then

$$\frac{1}{\delta}\big(1 + M_\Phi(\delta f)\big) \leqslant \frac{2}{\delta}.$$

Taking the infimum over all $\delta > 0$ such that $M_\Phi(f\delta) \leqslant 1$ gives the second inequality.

For the first inequality, choose $\alpha > \|f\|_\Phi$. Then there exists a $\delta > 0$ such that

$$\frac{1}{\delta}\big(1 + M_\Phi(\delta f)\big) \leqslant \alpha.$$

Since $\Phi(0) = 0$ and Φ is convex, we have $\Phi(x/\beta) \leqslant \Phi(x)/\beta$ for all $x \in \mathbb{R}$ and $\beta \geqslant 1$. Noting that $\alpha\delta \geqslant 1$ it follows that

$$M_\Phi(f/\alpha) = M_\Phi\left(\frac{\delta f}{\delta\alpha}\right) \leqslant \frac{M_\Phi(\delta f)}{\delta\alpha} \leqslant 1.$$

Since $\rho_\Phi(f)$ is the infimum over all $\alpha > 0$ for which the previous inequality holds, and it holds for every $\alpha > \|f\|_\Phi$, we conclude that $\rho_\Phi(f) \leqslant \|f\|_\Phi$. ■

It is clear from the proof that the lemma holds for all functions $\Phi : \mathbb{R}_+ \to \mathbb{R}$ that satisfy $\Phi(0) = 0$ and $\Phi(x/\beta) \leqslant \Phi(x)/\beta$ for all $x \in \mathbb{R}_+$ and $\beta \geqslant 1$. An interesting example of a non-convex function that satisfies the above properties is $\Phi(x) = x\exp(-1/x^2)$.

2.2. The Orlicz sequence space l^Θ. We next present a particular Orlicz space which plays an important role in our studies. The underlying measure space is now $\mathbb{Z}_+$ with the counting measure, and we will consider the function $\Theta\colon \mathbb{R} \to \mathbb{R}_+$ defined by

$$\Theta(x) = x^2 \exp\Big(-\frac{1}{2x^2}\Big). \tag{2.1}$$

This function satisfies the assumptions in Subsection 2.1 and we can associate an Orlicz sequence space l^Θ to it. Thus l^Θ consists of all sequences $a := (a_n)_{n\geqslant 1}$ for which

$$\rho_\Theta(a) := \inf\Big\{\delta > 0\colon \sum_{n\geqslant 1} \frac{a_n^2}{\delta^2}\exp\Big(-\frac{\delta^2}{2a_n^2}\Big) \leqslant 1\Big\} < \infty.$$

The following example illustrates the behaviour of $\rho_\Theta(a)$, but also plays a role later on.

EXAMPLE 2.1. If $a_n = \alpha^n$, where $\alpha \in [1/2, 1)$, then

$$\rho_\Theta(a) \eqsim \sqrt{\log(1-\alpha)^{-1}}.$$

This may be compared with $\|a\|_p \eqsim (1-\alpha)^{-1/p}$, again for $\alpha \in [1/2, 1)$, and $p \in [1, \infty]$.

Proof. We consider the equivalent Orlicz norm $\|a\|_\Theta$. On the one hand,

$$\begin{aligned}\sum_{n\geqslant 1} \lambda^2\alpha^{2n}\exp\Big(-\frac{1}{2\lambda^2\alpha^{2n}}\Big) &\leqslant \sum_{n\geqslant 1} \lambda^2\alpha^{2n}\exp\Big(-\frac{1}{2\lambda^2\alpha^2}\Big)\\ &= \frac{\lambda^2\alpha^2}{1-\alpha^2}\exp\Big(-\frac{1}{2\lambda^2\alpha^2}\Big) \leqslant \frac{\lambda^2}{1-\alpha}\exp\Big(-\frac{1}{2\lambda^2}\Big).\end{aligned}$$

On the other hand, let $N \in \mathbb{Z}_+$ be such that $\alpha^{2N} \leqslant 1/2 < \alpha^{2(N-1)}$. Then

$$\begin{aligned}\sum_{n\geqslant 1} \lambda^2\alpha^{2n}\exp\Big(-\frac{1}{2\lambda^2\alpha^{2n}}\Big) &\geqslant \sum_{n=1}^{N} \lambda^2\alpha^{2n}\exp\Big(-\frac{1}{2\lambda^2\alpha^{2N}}\Big)\\ &\geqslant \lambda^2\alpha^2\frac{1-\alpha^{2N}}{1-\alpha^2}\exp\Big(-\frac{1}{\lambda^2\alpha^2}\Big) \geqslant \frac{\lambda^2}{12(1-\alpha)}\exp\Big(-\frac{4}{\lambda^2}\Big).\end{aligned}$$

Consequently, we obtain

$$\|a\|_\Theta = \inf_{\lambda>0} \frac{1}{\lambda}\big(1+M_\Theta(\lambda a)\big) \eqsim \inf_{\lambda>0} \frac{1}{\lambda}\bigg(1+\frac{\lambda^2}{1-\alpha}\exp(-1/2\lambda^2)\bigg) =: \inf_{\lambda>0} F(\lambda).$$

The differentiable function F tends to ∞ as $\lambda \to 0$ or $\lambda \to \infty$, so its infimum is attained at a point where $F'(\lambda)=0$. Since

$$F'(\lambda) = -\lambda^{-2} + (1-\alpha)^{-1}\exp(-1/2\lambda^2) + (1-\alpha)^{-1}\exp(-1/2\lambda^2)\lambda^{-2},$$

where the middle-term is always positive, $F'(\lambda)=0$ can only happen if

$$(1-\alpha)^{-1}\exp(-1/2\lambda^2) \leqslant 1, \quad \text{i.e.,} \quad \lambda^{-1} \geqslant \lambda_0^{-1} := \sqrt{2\log(1-\alpha)^{-1}}.$$

But $1/\lambda$ is the first term in $F(\lambda)$, so we have proved that $F(\lambda) \gtrsim \sqrt{\log(1-\alpha)^{-1}}$ whenever $0<\lambda\leqslant\lambda_0$. Moreover, $F(\lambda_0) \eqsim \sqrt{\log(1-\alpha)^{-1}}$, which completes the proof. ■

2.3. Besov spaces. We recall the definition of the vector-valued Besov spaces. For the real case we refer to [12] and for the vector-valued Besov space we will give the treatise from [6].

Let X be a real Banach space and let $I=(0,1)$. For $\alpha\in(0,1)$, $p,q\in[1,\infty]$ the *vector-valued Besov space* $B^\alpha_{p,q}(I;X)$ is defined as the space of all functions $f\in L^p(I;X)$ for which the seminorm (with the usual modification for $q=\infty$)

$$\bigg(\int_0^1 \big(t^{-\alpha}\omega_p(f,t)\big)^q \frac{dt}{t}\bigg)^{1/q}$$

is finite. Here

$$\omega_p(f,t) = \sup_{|h|\leqslant t} \|s\mapsto f(s+h)-f(s)\|_{L^p(I(h);X)}$$

with $I(h)=\{s\in I : s+h\in I\}$. The sum of the L^p-norm and this seminorm turn $B^\alpha_{p,q}(I;X)$ into a Banach space. By a dyadic approximation argument (see [6], Corollary 3.b.9) one can show that the above seminorm is equivalent to

$$\|f\|_{p,q,\alpha} := \Big(\sum_{n\geqslant 0} \big(2^{n\alpha}\|s\mapsto f(s+2^{-n})-f(s)\|_{L^p(I(2^{-n});X)}\big)^q\Big)^{1/q}$$

For the purposes below it will be convenient to take

$$\|f\|_{B^\alpha_{p,q}(I;X)} = \|f\|_{L^p(I;X)} + \|f\|_{p,q,\alpha}$$

as a Banach space norm on $B^\alpha_{p,q}(I;X)$.

For $0 < \beta < \infty$, we also introduce the exponential Orlicz and Orlicz–Besov (semi)norms:

$$\|f\|_{\mathfrak{L}^{\Phi_\beta}(I;X)} := \sup_{p\geqslant 1} p^{-1/\beta}\|f\|_{L^p(I;X)},$$
$$\|f\|_{\Phi_\beta,\infty,\alpha} := \sup_{n\geqslant 1} 2^{\alpha n}\|f - f(\cdot - 2^{-n})\|_{\mathfrak{L}^{\Phi_\beta}(I(2^{-n});X)} = \sup_{p\geqslant 1} p^{-1/\beta}\|f\|_{p,\infty,\alpha},$$

and finally the Orlicz–Besov norm:

$$\|f\|_{B^\alpha_{\Phi_\beta,\infty}(I;X)} := \sup_{p\geqslant 1} p^{-1/\beta}\|f\|_{B^\alpha_{p,\infty}(I;X)} \eqsim \|f\|_{\mathfrak{L}^{\Phi_\beta}(I;X)} + \|f\|_{\Phi_\beta,\infty,\alpha}.$$

Because of the inequalities between different L^p-norms, it is immediate that we have equivalent norms above, whether we understand $p \geqslant 1$ as $p \in [1,\infty)$ or $p \in \{1,2,\dots\}$. For definiteness and later convenience, we choose the latter.

The above-given norm of $\mathfrak{L}^{\Phi_\beta}(I;X)$ is equivalent to the usual norm of the Orlicz space $L^{\Phi_\beta}(I;X)$ from Subsection 2.1 where $\Phi_\beta(x) = \exp(|x|^\beta) - 1$ for $\beta \geqslant 1$. For $0 < \beta < 1$, the function Φ_β must be defined in a slightly different way, but it is still essentially $\exp(|x|^\beta)$; see [3].

For $\beta \in \mathbb{Z}_+ \setminus \{0\}$ one can show in the same way as in [3],Theorem 3.4, that

$$\|f\|_{\mathfrak{L}^{\Phi_\beta}(I;X)} \leqslant \|f\|_{L^{\Phi_\beta}(I;X)}. \tag{2.2}$$

2.4. Gaussian random variables. Let $(\Omega, \mathcal{A}, \mathbb{P})$ denote a complete probability space. As in [9] let X be a Banach space with the following property: there exists a sequence $(x_n^*)_{n\geqslant 1}$ in X^* such that $\|x_n^*\| \leqslant 1$ and $\|x\| = \sup_{n\geqslant 1} |x_n^*(x)|$. Such a Banach space will be said to *admit a norming sequence of functionals*. Examples of such Banach spaces are all separable Banach spaces, but also spaces like l^∞. As in [9] a mapping $\xi : \Omega \to X$ will be called a *centered Gaussian* if for all $x^* \in \operatorname{span}\{x_n^* : n \geqslant 1\}$ the random variable $\langle \xi, x^*\rangle$ is a centered Gaussian. For a centered Gaussian random variable we define

$$\sigma(\xi) = \sup_{n\geqslant 1}(\mathbb{E}|\langle \xi, x_n^*\rangle|^2)^{1/2}. \tag{2.3}$$

In [9] it is proved that

$$\lim_{t\to\infty} \frac{1}{t^2} \log \mathbb{P}(\|X\| > t) = -\frac{1}{2\sigma^2},$$

so that the value of σ is independent of the norming sequence $(x_n^*)_{n\geqslant 1}$.

We make some comment on the above definition of a Gaussian random variable. We do not assume that ξ is a Borel measurable mapping. The only obvious fact we will use is that the mapping $\omega \mapsto \|\xi(\omega)\|$ is measurable. If ξ is a Gaussian random variable that takes values in a separable subspace of X, then ξ is Borel measurable, and consequently $\langle \xi, x^* \rangle$ is a centered Gaussian random variable for all $x^* \in X^*$.

A random variable $\xi : \Omega \to X$ is called *tight* if the measure $\mathbb{P} \circ \xi^{-1}$ is tight, and it is called *Radon* if $\mathbb{P} \circ \xi^{-1}$ is Radon. If X is a separable Banach space, then every Borel measurable random variable $\xi : \Omega \to E$ is Radon, and in particular tight. Conversely, if a Gaussian random variable $\xi : \Omega \to X$ is tight, then it almost surely takes values in a separable subspace of X. The next result is well known; a short proof can be found in [9], p. 61.

PROPOSITION 2.1. *Let X be a Banach space and let $\xi : \Omega \to X$ be a centered Gaussian. If ξ is tight, then $\mathbb{P}(\|\xi\| < r) > 0$ for all $r > 0$.*

3. MAXIMAL ESTIMATES FOR SEQUENCES OF GAUSSIAN RANDOM VARIABLES

The next proposition together with Theorem 3.1 may be considered as the vector-valued extension of a result in [4].

PROPOSITION 3.1. *Let X be a Banach space which admits a norming sequence of functionals $(x_n^*)_{n \geqslant 1}$. Let Θ be as in (2.1). Let $(\xi_n)_{n \geqslant 1}$ be X-valued centered Gaussian random variables with first moments and weak variances*

$$m_n = \mathbb{E}\|\xi_n\|, \qquad \sigma_n = \sup_{m \geqslant 1} (\mathbb{E}|\langle \xi_n, x_m^* \rangle|^2)^{1/2}.$$

Then

$$\mathbb{E} \sup_{n \geqslant 1} \|\xi_n\| \leqslant m + 3\rho_\Theta\big((\sigma_n)_{n \geqslant 1}\big), \qquad \textit{where } m = \sup_{n \geqslant 1} m_n.$$

Moreover, if any linear combination of the $(\xi_n)_{n \geqslant 1}$ is a Gaussian random variable and if $\mathbb{E}\sup_{n \geqslant 1} \|\xi_n\| < \infty$, then $\xi := (\xi_n)_{n \geqslant 1}$ is an $l^\infty(X)$-valued Gaussian random variable.

By the Kahane–Khinchine inequalities (cf. [8], Corollary 3.4.1) one obtains a similar estimate for the p-th moments of $\sup_{n \geqslant 1} \|\xi_n\|$. However, this also follows by extending the proof below.

Proof. We may write

$$\mathbb{E} \sup_{n \geqslant 1} \|\xi_n\| \leqslant \mathbb{E} \sup_{n \geqslant 1} \big| \|\xi_n\| - m_n \big| + \sup_{n \geqslant 1} m_n.$$

By (3.2) in [9], for all $t > 0$ we have

$$\mathbb{P}\big(\big|\|\xi_n\| - m_n\big| > t\big) \leqslant 2 \exp\Big(-\frac{t^2}{2\sigma_n^2}\Big). \tag{3.1}$$

For each $\delta > 0$ it follows that

$$
\begin{aligned}
(3.2)\qquad \mathbb{E}\sup_{n\geqslant 1}\big|\|\xi_n\| - m_n\big| &= \int_0^\infty \mathbb{P}\big(\sup_{n\geqslant 1}\big|\|\xi_n\| - m_n\big| > t\big)\,dt \\
&\leqslant \delta + \int_\delta^\infty \mathbb{P}\big(\sup_{n\geqslant 1}\big|\|\xi_n\| - m_n\big| > t\big)\,dt \leqslant \delta + \sum_{n\geqslant 1}\int_\delta^\infty \mathbb{P}\big(\big|\|\xi_n\| - m_n\big| > t\big)\,dt \\
&\leqslant \delta + \sum_{n\geqslant 1} 2\int_\delta^\infty \exp\Big(-\frac{t^2}{2\sigma_n^2}\Big)\,dt = \delta + \sum_{n\geqslant 1} 2\int_{\delta/\sigma_n}^\infty \sigma_n \exp\Big(-\frac{t^2}{2}\Big)\,dt \\
&\leqslant \delta + 2\sum_{n\geqslant 1}\frac{\sigma_n^2}{\delta}\exp\Big(-\frac{\delta^2}{2\sigma_n^2}\Big) = \delta\Big[1 + 2\sum_{n\geqslant 1}\frac{\sigma_n^2}{\delta^2}\exp\Big(-\frac{\delta^2}{2\sigma_n^2}\Big)\Big],
\end{aligned}
$$

where we used the standard estimate

$$\int_\delta^\infty \exp(-t^2/2)\,dt \leqslant \frac{1}{\delta}\exp(-\delta^2/2).$$

If $\delta > 0$ is chosen so that the last series sums up to at most 1, then we have shown that $\mathbb{E}\sup_{n\geqslant 1}\big|\|\xi_n\| - m_n\big| \leqslant 3\delta$. Taking the infimum over all such δ, we obtain the result.

The final assertion follows from the definition of a Gaussian random variable using the norming sequence of functionals $(e_m \otimes x_n^*)_{m,n\geqslant 1}$. ■

REMARK 3.1. The infimum appearing in Proposition 3.1 is dominated by

$$\Big[\Big(\frac{p-1}{e}\Big)^{(p-1)/2}\sum_{n\geqslant 1}\sigma_n^{p+1}\Big]^{1/(p+1)}$$

for any $p \in [1,\infty[$. (Interpret $0^0 = 1$ for $p = 1$.) This follows from the elementary estimate $\exp(-x^2/2) \leqslant [(p-1)/e]^{(p-1)/2}x^{1-p}$ applied to $x = \delta/\sigma_n$.

For an X-valued random variable ξ we take a median M such that

$$\mathbb{P}(\|\xi\| \leqslant M) \geqslant 1/2 \quad \text{and} \quad \mathbb{P}(\|\xi\| \geqslant M) \geqslant 1/2.$$

For convenience we will take $M = M(\xi)$ to be the smallest possible M. Notice that, for all $p \in (0,\infty)$, $\mathbb{E}\|\xi\|^p \geqslant M^p/2$.

Alternatively, we could have replaced the estimate (3.1) in the above proof by

$$\mathbb{P}\big(\big|\|\xi\| - M\big| > t\big) \leqslant \exp\Big(-\frac{t^2}{2\sigma^2}\Big)$$

(see [9], Lemma 3.1) to obtain

PROPOSITION 3.2. *Let X be a Banach space which admits a norming sequence of functionals $(x_n^*)_{n\geqslant 1}$. Let Θ be as in* (2.1). *Let $(\xi_n)_{n\geqslant 1}$ be X-valued centered Gaussian random variables with medians M_n and weak variances*

$$\sigma_n = \sup_{m\geqslant 1} (\mathbb{E}|\langle \xi_n, x_m^* \rangle|^2)^{1/2}.$$

Then

$$\mathbb{E} \sup_{n\geqslant 1} \|\xi_n\| \leqslant M + 2\rho_\Theta\big((\sigma_n)_{n\geqslant 1}\big), \quad \textit{where } M = \sup_{n\geqslant 1} M_n.$$

If the ξ_n are independent Gaussian random variables, then the converse to Proposition 3.1 holds.

THEOREM 3.1. *Let X be a Banach space which admits a norming sequence of functionals. Let Θ be as in* (2.1). *Let $(\xi_n)_{n\geqslant 1}$ be X-valued independent centered Gaussian random variables with first moments $(m_n)_{n\geqslant 1}$ and weak variances $(\sigma_n)_{n\geqslant 1}$. Let $m = \sup_{n\geqslant 1} m_n$. Then*

$$\mathbb{E} \sup_{n\geqslant 1} \|\xi_n\| \eqsim m + \rho_\Theta\big((\sigma_n)_{n\geqslant 1}\big) \eqsim m + \|(\sigma_n)_{n\geqslant 1}\|_\Theta.$$

Moreover, if one of these expressions is finite, then $\xi := (\xi_n)_{n\geqslant 1}$ is an $l^\infty(X)$-valued Gaussian random variable.

Recall from Subsection 2.1 and the definition of Θ that

$$\|(\sigma_n)_{n\geqslant 1}\|_\Theta = \inf_{\delta>0} \left\{ \frac{1}{\delta} \left[1 + \sum_{n\geqslant 1} \delta^2 \sigma_n^2 \exp\left(-\frac{1}{2\delta^2\sigma_n^2} \right) \right] \right\}.$$

Proof. The second two-sided estimate follows from Lemma 2.1.

The estimate $\lesssim$ in the first comparison has been obtained in Proposition 3.1. To prove $\gtrsim$, let us note that $\mathbb{E}\sup_{n\geqslant 1}\|\xi_n\| \geqslant m$ is clear. As for the estimate for $\rho_\Theta\big((\sigma_n)_{n\geqslant 1}\big)$, by scaling we may assume that $\mathbb{E}\sup_{n\geqslant 1}\|\xi_n\| = 1$. Then we have $\mathbb{P}(\sup_{n\geqslant 1}\|\xi_n\| > 3) \leqslant 1/3$, and therefore

$$\begin{aligned} 1/3 \leqslant \mathbb{P}(\sup_{n\geqslant 1} \|\xi_n\| \leqslant 3) &= \prod_{n\geqslant 1} \mathbb{P}(\|\xi_n\| \leqslant 3) = \prod_{n\geqslant 1} \big(1 - \mathbb{P}(\|\xi_n\| > 3)\big) \\ &\leqslant \prod_{n\geqslant 1} \exp\big(-\mathbb{P}(\|\xi_n\| > 3)\big). \end{aligned}$$

It follows that

$$\log 3 \geqslant \sum_{n\geqslant 1} \mathbb{P}(\|\xi_n\| > 3).$$

Let $\varepsilon \in (0,1)$ be an arbitrary number. If for each $n \geqslant 1$ we choose k_n such that $(\mathbb{E}\langle \xi_n, x^*_{k_n}\rangle^2)^{1/2} \geqslant \sigma_n(1-\varepsilon)$, then we obtain

$$\begin{aligned}\log 3 &\geqslant \sum_{n\geqslant 1} \mathbb{P}(\|\xi_n\| > 3) \geqslant \sum_{n\geqslant 1} \mathbb{P}(|\langle \xi_n, x^*_{k_n}\rangle| > 3) \\ &\geqslant \sqrt{\frac{2}{\pi}} \sum_{n\geqslant 1} \frac{3\sigma_n(1-\varepsilon)}{\sigma_n^2(1-\varepsilon)^2+9} \exp\Big(-\frac{9}{2\sigma_n^2(1-\varepsilon)^2}\Big),\end{aligned}$$

where we used

$$\int_a^\infty \exp(-t^2/2)\,dt \geqslant \frac{a}{1+a^2}\exp(-a^2/2).$$

Next, we have

$$\sigma_n^2 = \sup_{m\geqslant 1} \mathbb{E}\langle \xi_n, x^*_m\rangle^2 = \frac{\pi}{2}\sup_{m\geqslant 1} \mathbb{E}|\langle \xi_n, x^*_m\rangle| \leqslant \frac{\pi}{2}\mathbb{E}\|\xi_n\| \leqslant \frac{\pi}{2},$$

and hence $\sigma_n^2(1-\varepsilon)^2 + 9 \leqslant \pi/2 + 9 < 11$ and $\sqrt{2/\pi}\cdot\sigma_n \geqslant 2/\pi\cdot\sigma_n^2$. Thus

$$\log 3 \geqslant \frac{6}{11\pi}\sum_{n\geqslant 1} \sigma_n^2(1-\varepsilon)\exp\Big(-\frac{9}{2\sigma_n^2(1-\varepsilon)^2}\Big).$$

This being true for all $\varepsilon > 0$, it follows in the limit that

$$\sum_{n\geqslant 1}\Big(\frac{\sigma_n}{3}\Big)^2 \exp\Big(-\frac{9}{2\sigma_n^2}\Big) \leqslant \log 3 \cdot \frac{11\pi}{6\cdot 9} < 1.$$

Therefore, $\rho_\Theta\big((\sigma_n)_{n\geqslant 1}\big) \leqslant 3$.

The last assertion follows as in Proposition 3.1. ∎

From the proof of Theorem 3.1 we actually see that

$$\mathbb{E}\sup_{n\geqslant 1}\|\xi_n\| \geqslant \max\Big\{\frac{1}{3}\rho_\Theta\big((\sigma_n)_{n\geqslant 1}\big), m\Big\}.$$

REMARK 3.2. A similar proof as presented above shows that the function Θ in Theorem 3.1 can be replaced by the (non-convex) function Φ defined in Subsection 2.1. Since we prefer to have an Orlicz space, we use the convex function Θ.

In the real-valued case, m is not needed in the estimate of Theorem 3.1. This is due to the fact that it can be estimated by $\sup_{n\geqslant 1}\sigma_n$. The following simple example shows that in the infinite-dimensional setting this is not the case. We shall also encounter the same phenomenon in a more serious example in the proof of Theorem 6.1.

EXAMPLE 3.1. Let $p \in [1,\infty]$ and let $X = l^p$ with the standard unit vectors denoted by e_n. Let $(\sigma_n)_{n\geqslant 1}$ be a sequence of positive real numbers with

$$m_p := \Big(\sum_{n\geqslant 1} \sigma_n^p\Big)^{1/p} < \infty \quad \text{if } p < \infty,$$
$$m_\infty := \rho_\Theta\big((\sigma_n)_{n\geqslant 1}\big) < \infty \quad \text{if } p = \infty.$$

Let $(\gamma_n)_{n\geqslant 1}$ be a sequence of independent standard Gaussian random variables. Then $\xi = \sum_{n\geqslant 1} \sigma_n \gamma_n e_n$ defines an X-valued Gaussian random variable with $m(\xi) = \mathbb{E}\|\xi\| \eqsim_p m_p$ and

$$\sigma(\xi) = \begin{cases} \sup_{n\geqslant 1} \sigma_n, & p \in [2,\infty], \\ \big(\sum_{n\geqslant 1} \sigma_n^r\big)^{1/r}, & p \in [1,2), \end{cases}$$

where $r = 2p/(2-p)$.

4. BESOV REGULARITY OF BROWNIAN PATHS

We say that an X-valued process $\big(W(t)\big)_{t\in[0,1]}$ is a *Brownian motion* if it is strongly measurable and, for all $x^* \in E^*$, $\big(\langle W(t), x^*\rangle\big)_{t\in[0,1]}$ is a real Brownian motion starting at zero. Let Q be the covariance of $W(1)$. For the process W we have:

1. $W(0) = 0$;
2. W has a version with continuous paths;
3. W has independent increments;
4. for all $0 \leqslant s < t < \infty$, $W(t) - W(s)$ has distribution $\mathcal{N}\big(0, (t-s)Q\big)$.

In this situation we say that W is a *Brownian motion with covariance* Q. Notice that every process W that satisfies 3 and 4 has a pathwise continuous version (cf. [5], Theorem 3.23).

In the next result we obtain a Besov regularity result for Brownian motions. The case of real-valued Brownian motions has been considered in [2], [3] and [11]. But even in the real-valued case we believe the proof is new and more direct.

THEOREM 4.1. *Let X be a Banach space and let $p, q \in [1,\infty)$. For an X-valued non-zero Brownian motion W we have*

$$W \in B^{1/2}_{\Phi_2,\infty}(0,1;X) \subset B^{1/2}_{p,\infty}(0,1;X) \quad a.s.,$$
$$W \notin B^{1/2}_{p,q}(0,1;X) \quad a.s.$$

Proof. Define

$$Y_{n,p} := 2^{n/2}\|W(\cdot + 2^{-n}) - W\|_{L^p(I(2^{-n});X)}.$$

We may write

$$
\begin{aligned}
Y_{n,p}^p &= \int_0^{1-2^{-n}} 2^{np/2}\|W(t+2^{-n})-W(t)\|^p\,dt \\
&= \sum_{m=1}^{2^n-1} \int_{(m-1)2^{-n}}^{m2^{-n}} 2^{np/2}\|W(t+2^{-n})-W(t)\|^p\,dt \\
&= \sum_{m=1}^{2^n-1} 2^{-n}\int_0^1 2^{np/2}\big\|W\big((s+m)2^{-n}\big)-W\big((s+m-1)2^{-n}\big)\big\|^p\,ds \\
&= \int_0^1 2^{-n}\sum_{m=1}^{2^n-1}\|\gamma_{n,m,s}\|^p\,ds.
\end{aligned}
$$

Here $\gamma_{n,m,s} = 2^{n/2}\Big(W\big((s+m)2^{-n}\big)-W\big((s+m-1)2^{-n}\big)\Big)$. For fixed $s\in(0,1)$ and $n\geqslant 1$, $(\gamma_{n,m,s})_{m\geqslant 1}$ is a sequence of independent random variables distributed as $W(1)$. Write $c_p = \big(\mathbb{E}\|W(1)\|^p\big)^{1/p}$. If we take second moments, we may use Jensen's inequality to obtain

$$
\begin{aligned}
\mathbb{E}(Y_{n,p}^p - c_p^p)^2 &= \mathbb{E}\Big|\int_0^1\big[2^{-n}\sum_{m=1}^{2^n-1}\|\gamma_{n,m,s}\|^p - c_p^p\big]\,ds\Big|^2 \\
&\leqslant \int_0^1 \mathbb{E}\Big|2^{-n}\sum_{m=1}^{2^n-1}(\|\gamma_{n,m,s}\|^p - c_p^p) - 2^{-n}c_p^p\Big|^2\,ds \\
&= \int_0^1[2^{-2n}(2^n-1)(c_{2p}^{2p}-c_p^{2p}) + 2^{-2n}c_p^{2p}]ds \\
&= 2^{-n}[(1-2^{-n})c_{2p}^{2p} - (1-2^{1-n})c_p^{2p}].
\end{aligned}
$$

It follows that for a fixed $\varepsilon>0$ we have

$$
\sum_{n\geqslant 1}\mathbb{P}(|Y_{n,p}^p - c_p^p|>\varepsilon) \leqslant \frac{1}{\varepsilon^2}\sum_{n\geqslant 1}\mathbb{E}(Y_{n,p}^p-c_p^p)^2 < \infty,
$$

which implies, by the Borel–Cantelli lemma, that

$$
\mathbb{P}(|Y_{n,p}^p - c_p^p|>\varepsilon \text{ infinitely often}) = 0.
$$

This in turn gives

$$
\lim_{n\to\infty} 2^{n/2}\|W(\cdot+2^{-n})-W\|_{L^p(I(2^{-n});X)} = \big(\mathbb{E}\|W(1)\|^p\big)^{1/p} \text{ a.s.} \tag{4.1}
$$

This shows immediately that the paths are a.s. in $B^{1/2}_{p,\infty}(0,1;X)$. From the above calculation it is also clear that $W\notin B^{1/2}_{p,q}(0,1;X)$ a.s. for $q\in[1,\infty)$. Next we show that the paths are in $B^{1/2}_{\Phi_2,\infty}(0,1;X)$ a.s. Note that $\big(\mathbb{E}\|W(1)\|^p\big)^{1/p} \eqsim p^{1/2}$

as $p \to \infty$. The upper estimate $\lesssim$ is a consequence of Fernique's theorem (which states that $\|W(1)\|^2$ is exponentially integrable, since $W(1)$ is a non-zero X-valued Gaussian random variable), whereas $\gtrsim$ follows from the corresponding estimate for real Gaussians after applying a functional. We proved that $\mathbb{E}(Y_{n,p}^p - c_p^p)^2 \leqslant c_{2p}^{2p} 2^{-n}$. Therefore,

$$\mathbb{E}(Y_{n,p}^p c_p^{-p} - 1)^2 \leqslant C 2^{-n} c_{2p}^{2p} c_p^{-2p} \leqslant C 2^{-n} K^{2p},$$

where $K \geqslant 1$ is some constant. Hence for all $\lambda > 1$

$$\mathbb{P}(Y_{n,p} c_p^{-1} > \lambda) \leqslant \mathbb{P}(|Y_{n,p}^p c_p^{-p} - 1| > \lambda^p - 1) \leqslant C 2^{-n} K^{2p} (\lambda^p - 1)^{-2},$$

and thus for $\lambda = 2K$

$$\sum_{n,p=1}^{\infty} \mathbb{P}(Y_{n,p} c_p^{-1} > \lambda) \leqslant C \lambda^{-2} \sum_{n=1}^{\infty} 2^{-n} \sum_{p=1}^{\infty} K^{2p} (\lambda^p - 1)^{-2} < \infty,$$

so that by the Borel–Cantelli lemma

$$\mathbb{P}\big(Y_{n,p} c_p^{-1} > \lambda \text{ for infitely many pairs } (n,p)\big) = 0.$$

Since $c_p \eqsim p^{1/2}$, this means that a.s.

$$\sup_{n,p} 2^{n/2} \|W(\cdot + 2^{-n}) - W\|_{L^p(I(2^{-n});X)} p^{-1/2} < \infty. \quad \blacksquare$$

5. BROWNIAN MOTIONS AS RANDOM VARIABLES IN BESOV SPACES

From the pathwise properties of W studied in the previous section we know that we have a function $W\colon \Omega \to B^{1/2}_{p,\infty}$. We now go into the measurability issues in order to promote it to a random variable.

THEOREM 5.1. *Let X be a Banach space and let $p \in [1,\infty)$. Then an X-valued Brownian motion W is a $B^{1/2}_{p,\infty}(0,1;X)$-valued, and even $B^{1/2}_{\Phi_2,\infty}(0,1;X)$-valued, Gaussian random variable. In particular, there exists an $\varepsilon > 0$ such that*

$$\mathbb{E}\exp(\varepsilon \|W\|^2_{B^{1/2}_{\Phi_2,\infty}(0,1;X)}) < \infty.$$

If the Brownian motion W is non-zero, then the random variables

$$W\colon \Omega \to B^{1/2}_{p,\infty}(0,1;X) \quad \text{and} \quad W\colon \Omega \to B^{1/2}_{\Phi_2,\infty}(0,1;X)$$

are not tight. In fact,

$$\tau_1 := \inf\{\lambda \geqslant 0 : \mathbb{P}(\|W\|_{B^{1/2}_{p,\infty}(0,1;X)} \leqslant \lambda) > 0\} \geqslant \big(\mathbb{E}\|W(1)\|^p\big)^{1/p},$$

and, consequently, also

$$\tau_2 := \inf\{\lambda \geqslant 0 : \mathbb{P}(\|W\|_{B^{1/2}_{\Phi_2,\infty}(0,1;X)} \leqslant \lambda) > 0\} > 0.$$

There is some interest in the numbers τ_1 and τ_2. For general theory we refer the reader to [9], Chapter 3.

For the proof we need the following easy lemma.

LEMMA 5.1. *Let X be a Banach space which admits a norming sequence, let $0 < \alpha < 1$ and $0 < \beta < \infty$. Then for all $p \in [1, \infty)$ there exist*

$$(\Lambda_{pjk})_{j\geqslant 0,k\geqslant 1} \subset B^{\alpha}_{p,\infty}(0,1;X)^* \subset B^{\alpha}_{\Phi_\beta,\infty}(0,1;X)^*,$$

$$(f_{pjk})_{j\geqslant 0,k\geqslant 1} \subset C^{\infty}([0,1];X^*)$$

such that: for all $\phi \in B^{\alpha}_{p,\infty}(0,1;X)$ there are the representations

$$\langle \phi, \Lambda_{p0k}\rangle = \int_0^1 \langle \phi(t), f_{p0k}(t)\rangle\, dt, \qquad k \geqslant 1,$$

$$\langle \phi, \Lambda_{pjk}\rangle = \int_0^{1-2^{-j}} 2^{j\alpha}\langle \phi(t+2^{-j}) - \phi(t), f_{pjk}(t)\rangle\, dt, \qquad j,k \geqslant 1;$$

we have the upper norm bounds

$$p^{-1/\beta}\|\Lambda_{pjk}\|_{B^{\alpha}_{\Phi_\beta,\infty}(0,1;X)^*} \leqslant \|\Lambda_{pjk}\|_{B^{\alpha}_{p,\infty}(0,1;X)^*} \leqslant 1, \qquad k \geqslant 1;$$

and finally the sequences are norming in the following sense:

$$\|\phi\|_{B^{\alpha}_{p,\infty}(0,1;X)} = \sup_{j\geqslant 0,k\geqslant 1} |\langle \phi, \Lambda_{pjk}\rangle|,$$

$$\|\phi\|_{B^{\alpha}_{\Phi_\beta,\infty}(0,1;X)} = \sup_{p\geqslant 1, j\geqslant 0,k\geqslant 1} p^{-1/\beta}|\langle \phi, \Lambda_{pjk}\rangle|.$$

Proof. Let $(x_n^*)_{n\geqslant 1}$ be a norming sequence for X. Let $I = [a,b]$. First observe that there exists a sequence $(F_k)_{k\geqslant 1}$ in $L^{p'}(I;X^*)$, with norm smaller than or equal to one, which is norming for $L^p(I;X)$. Such a sequence is easily constructed using the $(x_n^*)_{n\geqslant 1}$ and standard duality arguments. By an approximation argument we can even take the $(F_k)_{k\geqslant 1}$ in $C^{\infty}(I;X^*)$.

To prove the lemma, let first $a = 0$ and $b = 1$, and let $(f_{p0k})_{k\geqslant 1}$ be the above-constructed sequence $(F_k)_{k\geqslant 1}$. Next we fix $j \geqslant 1$, let $a = 0$ and $b = 1 - 2^{-j+1}$, and let $(f_{pjk})_{k\geqslant 1}$ be the above-constructed sequence for this interval. Let Λ_{pjk} be the elements in $B^{\alpha}_{p,\infty}(0,1;X)^*$ defined as in the statement of the lemma. It is easily checked that this sequence satisfies the required properties. ■

Proof of Theorem 5.1. Since W is strongly measurable as an X-valued process, we may assume that X is separable and therefore that it admits a norming sequence. In Theorem 4.1 it has been shown that the paths of W are a.s. in $B^{1/2}_{\Phi_2,\infty}(0,1;X) \subset B^{1/2}_{p,\infty}(0,1;X)$ for all $p \in [1,\infty)$. It follows from Lemma 5.1 that there exists a norming sequence of functionals $(\Lambda_n)_{n\geqslant 1}$ for $B^{1/2}_{\Phi_2,\infty}(0,1;X)$, as well as in each $B^{1/2}_{p,\infty}(0,1;X)$, such that $\langle W,\Lambda\rangle$ is a centered Gaussian random variable for all $\Lambda \in \operatorname{span}\{\Lambda_n, n \geqslant 1\}$. Therefore, by definition it follows that W is a centered Gaussian random variable. The exponential integrability follows from Corollary 3.2 in [9].

The last assertion follows from (4.1). This also shows that W is not tight since, by Proposition 2.1, for centered Gaussian measures which are tight it follows that $\tau = 0$. ∎

6. MOMENT ESTIMATES FOR BROWNIAN MOTIONS IN BESOV SPACES

Since now we know that

$$\mathbb{E}\|W\|_{B^{1/2}_{p,\infty}(0,1;X)} < \infty \quad \text{and} \quad \mathbb{E}\|W\|_{B^{1/2}_{\Phi_2,\infty}(0,1;X)} < \infty,$$

it seems interesting to estimate these quantities. For this we need a convenient representation of X-valued Brownian motions.

Recall that a family $W_H = \big(W_H(t)\big)_{t\in\mathbb{R}_+}$ of bounded linear operators from H to $L^2(\Omega)$ is called an *H-cylindrical Brownian motion* if

1. $W_H h = \big(W_H(t)h\big)_{t\in\mathbb{R}_+}$ is a real-valued Brownian motion for each $h \in H$,
2. $\mathbb{E}\big(W_H(s)g \cdot W_H(t)h\big) = (s\wedge t)\,[g,h]_H$ for all $s,t\in\mathbb{R}_+,\ g,h\in H$.

We always assume that the H-cylindrical Brownian motion W_H is adapted to a given filtration $\mathcal{F}$, i.e., the Brownian motions $W_H h$ are adapted to $\mathcal{F}$ for all $h \in H$. Notice that if $(h_n)_{n\geqslant 1}$ is an orthonormal basis for H, then $(W_H h_n)_{n\geqslant 1}$ are independent standard real-valued Brownian motions.

Let $W\colon \mathbb{R}_+\times\Omega \to E$ be an E-valued Brownian motion and let $Q \in \mathcal{L}(E^*,E)$ be its covariance operator. Let H_Q be the reproducing kernel Hilbert space or Cameron–Martin space (cf. [1], [13]) associated with Q and let $i_W\colon H_Q \hookrightarrow E$ be the inclusion operator. Then the mappings

$$W_{H_Q}(t) : i_W^* x^* \mapsto \langle W(t), x^*\rangle$$

uniquely extend to an H_Q-cylindrical Brownian motion W_{H_Q}, so that in particular

$$\langle W(t), x^*\rangle = W_{H_Q}(t) i_W^* x^*. \tag{6.1}$$

LEMMA 6.1. *For all $p\in[1,\infty)$ we have*

$$\|i_W\| = \sigma\big(W(1)\big) \lesssim \frac{1}{\sqrt{p}}\big(\mathbb{E}\|W(1)\|^p\big)^{1/p}.$$

P r o o f. Note first that, since $\langle W(t),x^*\rangle$ is a real-valued Gaussian random variable, its moments satisfy

$$\big(\mathbb{E}|\langle W(t),x^*\rangle|^p\big)^{1/p} = \gamma_p\big(\mathbb{E}|\langle W(t),x^*\rangle|^2\big)^{1/2}, \tag{6.2}$$

where γ_p are universal constants behaving like $\gamma_p \eqsim \sqrt{p}$ for $p\in[1,\infty)$.

On the other hand, by (6.1) and the definition of cylindrical Brownian motion,

$$\big(\mathbb{E}|\langle W(t), x^*\rangle|^2\big)^{1/2} = \sqrt{t}\|i_W^* x^*\|.$$

For $t = 1$, taking the supremum over all $x^* \in X^*$ of unit norm, and recalling that $\|i_W\| = \|i_W^*\|$, we prove then the first equality in the assertion. The second one then follows from (6.2) and the obvious estimate

$$\big(\mathbb{E}|\langle W(t), x^*\rangle|^p\big)^{1/p} \leqslant \big(\mathbb{E}\|W(t)\|^p\big)^{1/p} \quad \text{for } \|x^*\| \leqslant 1. \quad \blacksquare$$

LEMMA 6.2. *Let $c > 0$, and $J \subset \mathbb{R}_+$ be an interval of length $|J| \geqslant c$. Consider $W(\cdot + c) - W$ as an $L^p(J, X)$-valued Gaussian random variable. Then*

$$\sigma\big(W(\cdot + c) - W\big) \eqsim c^{1/2+1/p}\|i_W\|.$$

Proof. To prove the claim take $f \in L^{p'}(J; X^*)$. We also use the same symbol for its extension to $\mathbb{R}$ with zero fill. The representation (6.1), the stochastic Fubini theorem, and the Itô isometry yield

$$\begin{aligned}
&\Big(\mathbb{E}\Big|\int_J \big\langle \big(W(t+c) - W(t)\big), f(t)\big\rangle\, dt\Big|^2\Big)^{1/2}\\
&\qquad = \Big(\mathbb{E}\Big|\int_J \big(W_H(t+c) - W_H(t)\big) i_W^* f(t)\, dt\Big|^2\Big)^{1/2}\\
&\qquad = \Big(\mathbb{E}\Big|\int_{\mathbb{R}}\int_{\mathbb{R}_+} \mathbf{1}_{[t,t+c]}(s) i_W^* f(t)\, dW_H(s)\, dt\Big|^2\Big)^{1/2}\\
&\qquad = \Big(\mathbb{E}\Big|\int_{\mathbb{R}_+} \mathbf{1}_{[0,c]} * (i_W^* f)(s)\, dW_H(s)\Big|^2\Big)^{1/2}\\
&\qquad = \Big(\int_{\mathbb{R}} \|\mathbf{1}_{[0,c]} * (i_W^* f)(s)\|_H^2\, ds\Big)^{1/2}.
\end{aligned}$$

Taking the supremum over all $f \in L^{p'}(J; X^*)$ of unit norm, we find that

$$\sigma\big(W(\cdot + c) - W\big) = \|(\mathbf{1}_{[0,c]} *) \otimes i_W^*\|_{L^{p'}(J;X^*) \to L^2(\mathbb{R};H)}.$$

By Young's inequality with $1 + 1/2 = 1/p' + 1/r$ it follows that the operator norm is dominated by

$$\|\mathbf{1}_{[0,c]}\|_{L^r}\|i_W^*\|_{X^* \to H} = c^{1/p+1/2}\|i_W\|.$$

On the other hand, if we test with the functions $f = \mathbf{1}_I \otimes x^* \in L^{p'}(J; X^*)$, where $I \subseteq J$ has length c, we obtain

$$\begin{aligned}
&\|\mathbf{1}_{[0,c]} * (i_W^* f)\|_{L^2(H)} = \|\mathbf{1}_{[0,c]} * \mathbf{1}_I\|_{L^2}\|i_W^* x^*\|_H\\
&\qquad = (2/3)^{1/2} c^{3/2}\|i_W^* x^*\|_H \eqsim c^{1/2+1/p}\frac{\|i_W^* x^*\|_H}{\|x^*\|_{X^*}}\|f\|_{L^{p'}(X^*)}.
\end{aligned}$$

Taking the supremum over $x^* \in X^* \setminus \{0\}$ we get the other side of the asserted norm equivalence. ∎

COROLLARY 6.1. *Let $c \in (0, e^{-1/2}]$, and $J \subset \mathbb{R}_+$ be an interval of length $|J| \geqslant c$. Consider $W(\cdot + c) - W$ as an $\mathfrak{L}^{\Phi_2}(J;X)$-valued Gaussian random variable. Then*

$$\sigma\big(W(\cdot + c) - W\big) \eqsim (\log c^{-1})^{-1/2} c^{1/2} \|i_W\|.$$

P r o o f. Note that the functionals $p^{-1/2}\Lambda_{p0k}$ from Lemma 5.1 (with $\beta = 2$) provide a norming sequence for $\mathfrak{L}^{\Phi_2}(0,1;X)$, and the same construction can be adapted to another interval. Hence

$$\begin{aligned}
&\sigma_{\mathfrak{L}^{\Phi_2}(J;X)}\big(W(\cdot + c) - W\big) \\
&\qquad = \sup_{p \geqslant 1} p^{-1/2} \sup_{k \geqslant 1} \Big(\mathbb{E}\Big| \int_J \big\langle \big(W(t+c) - W(t)\big), f_{p0k}(t) \big\rangle \, dt \Big|^2 \Big)^{1/2} \\
&\qquad = \sup_{p \geqslant 1} p^{-1/2} \sigma_{L^p(J;X)}\big(W(\cdot + c) - W\big) \\
&\qquad \eqsim \sup_{p \geqslant 1} p^{-1/2} c^{1/2 + 1/p} \|i_W\| \eqsim (\log c^{-1})^{-1/2} c^{1/2} \|i_W\|,
\end{aligned}$$

where an elementary maximum value problem was solved in the last step. ∎

THEOREM 6.1. *Let X be a Banach space. Let $p \in [1,\infty)$. For an X-valued Brownian motion W we have*

$$\mathbb{E}\|W\|_{B^{1/2}_{p,\infty}(0,1;X)} \eqsim \big(\mathbb{E}\|W(1)\|^p\big)^{1/p}, \tag{6.3}$$

$$\mathbb{E}\|W\|_{B^{1/2}_{\Phi_2,\infty}(0,1;X)} \eqsim \mathbb{E}\|W(1)\|. \tag{6.4}$$

REMARK 6.1. By Corollary 3.2 in [9], the estimate (6.3) implies that

$$\mathbb{E}\|W\|_{B^{1/2}_{p,\infty}(0,1;X)} \lesssim \sqrt{p}\, \mathbb{E}\|W(1)\|,$$

but we do not know if there is a two-sided comparison here. The above estimate is also an immediate consequence of (6.4) and the definition of the various norms.

P r o o f o f T h e o r e m 6.1. As in Theorem 5.1 we may assume that X admits a norming sequence.

The estimate $\gtrsim$ in (6.3) follows from (4.1). Let us then consider the other direction. Clearly,

$$\mathbb{E}\|W\|_{L^p(0,1;X)} \leqslant \big(\mathbb{E}\|W\|^2_{L^\infty(0,1;X)}\big)^{1/2} \leqslant 2\big(\mathbb{E}\|W(1)\|^2\big)^{1/2} \lesssim \mathbb{E}\|W(1)\|$$

by Doob's maximal inequality and the equivalence of Gaussian moments. Next we consider

$$\mathbb{E}\sup_{j\geqslant 1} 2^{j/2}\|W(\cdot+2^{-j})-W\|_{L^p(0,1-2^{-j};X)}. \tag{6.5}$$

This can be estimated using Proposition 3.1 with the $L^p(0,1;X)$-valued Gaussian random variables $\xi_j = 2^{j/2}[W(\cdot+2^{-j})-W]\mathbf{1}_{[0,1-2^{-j}]}$:

$$\mathbb{E}\sup_{j\geqslant 1}\|\xi_j\| \lesssim \sup_{j\geqslant 1}\mathbb{E}\|\xi_j\| + \|(\sigma_j)_{j\geqslant 1}\|_{\Theta}.$$

The first term is clearly smaller than $\big(\mathbb{E}\|W(1)\|^p\big)^{1/p}$. By Lemma 6.2 and Example 2.1, the Orlicz norm can be computed as

$$\begin{aligned}\|(\sigma_j)_{j\geqslant 1}\|_{\Theta} \eqsim \|i_W\|\,\|(2^{-j/p})_{j\geqslant 1}\|_{\Theta} &\eqsim \|i_W\|\sqrt{\log(1-2^{-1/p})^{-1}}\\ &\eqsim (1+\sqrt{\log p})\|i_W\|.\end{aligned}$$

By Lemma 6.2, this is smaller than $\big(\mathbb{E}\|W(1)\|^p\big)^{1/p}$; indeed, it is much smaller when $p\to\infty$. Thus, just like in Example 3.1, we are in a situation where the m term totally dominates in the estimate (1.1). The proof of (6.3) is complete.

Next, we show (6.4). The lower estimate follows trivially from (6.3). For the upper estimate we write

$$\begin{aligned}&\mathbb{E}\|W\|_{B^{1/2}_{\Phi_2,\infty}(0,1;X)}\\ &\quad\leqslant \mathbb{E}\|W\|_{\mathfrak{L}^{\Phi_2}(0,1;X)} + \mathbb{E}\sup_{j\geqslant 1} 2^{j/2}\|W(\cdot+2^{-j})-W\|_{\mathfrak{L}^{\Phi_2}(0,1-2^{-j};X)}.\end{aligned}$$

The first term can again be estimated using Doob's maximal inequality, since

$$\mathbb{E}\|W\|_{\mathfrak{L}^{\Phi_2}(0,1;X)} \leqslant \mathbb{E}\|W\|_{L^\infty(0,1;X)}.$$

The second term can be treated using Proposition 3.1 with the $\mathfrak{L}^{\Phi_2}(0,1;X)$-valued Gaussian random variables $\xi_j = 2^{j/2}[W(\cdot+2^{-j})-W]\mathbf{1}_{[0,1-2^{-j}]}$. Combining Proposition 3.1 with Remark 3.1, we have

$$\mathbb{E}\sup_{j\geqslant 1}\|\xi_j\| \lesssim \sup_{j\geqslant 1}\mathbb{E}\|\xi_j\| + \Big(\sum_{j\geqslant 1}\sigma_j^4\Big)^{1/4}.$$

From Corollary 6.1 we get

$$\sigma_j \lesssim (\log 2^j)^{-1/2}\|i_W\| \eqsim j^{-1/2}\|i_W\|,$$

so that the series sums up to $\big(\sum_{j\geqslant 1}\sigma_j^4\big)^{1/4} \lesssim \|i_W\| \lesssim \mathbb{E}\|W(1)\|$.

We then estimate $\mathbb{E}\|\xi_j\|$. By (2.2), we have

$$\|f\|_{\mathfrak{L}^{\Phi_2}(0,1-2^{-j};X)} \leqslant \|f\|_{\mathfrak{L}^{\Phi_2}(0,1;X)} \leqslant \|f\|_{L^{\Phi_2}(0,1;X)} = \inf_{\lambda>0} \frac{1}{\lambda} \int_0^1 \exp\left(\lambda^2\|f(t)\|^2\right) dt.$$

Therefore,

$$\mathbb{E}\|\xi_j\| \leqslant \inf_{\lambda>0} \frac{1}{\lambda} \int_0^1 \mathbb{E} \exp\left(\lambda^2 2^j \|W(t+2^{-j}) - W(t)\|^2\right) dt = \inf_{\lambda>0} \frac{1}{\lambda} \mathbb{E} \exp\left(\lambda^2\|W(1)\|^2\right).$$

This may be estimated by expanding into power series:

$$\frac{1}{\lambda} \sum_{k\geqslant 0} \frac{\lambda^{2k}}{k!} \mathbb{E}\|W(1)\|^{2k} \leqslant \frac{1}{\lambda}\left[1 + \sum_{k\geqslant 1} \frac{\lambda^{2k}}{k!} \left(K\sqrt{2k}\,\mathbb{E}\|W(1)\|\right)^{2k}\right] \leqslant \frac{1}{\lambda}\left[1 + \sum_{k\geqslant 1} \left(2e[\lambda K \mathbb{E}\|W(1)\|]^2\right)^k\right],$$

where K is an absolute constant from the Gaussian norm comparison result (see [9], Corollary 3.2), and we used $k^k/k! \leqslant e^k$. Choosing $\lambda = \left(2eK\mathbb{E}\|W(1)\|\right)^{-1}$, we find that $\mathbb{E}\|\xi_j\| \lesssim \mathbb{E}\|W(1)\|$. ■

Acknowledgements. The authors thank Jan van Neerven for some helpful comments.

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Received on 28.2.2007;
revised version on 4.10.2007