A CALCULUS ON LÉVY EXPONENTS AND SELFDECOMPOSABILITY
ON BANACH SPACES

BY

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Abstract. In infinite-dimensional Banach spaces there is no complete
colorization of the Lévy exponents of infinitely divisible probability
measures. Here we propose a calculus on Lévy exponents that is derived
from some random integrals. As a consequence we prove that each selfde-
composable measure can be factorized as another selfdecomposable mea-
sure and its background driving measure that is s-selfdecomposable. This
complements a result from the paper of Iksanov, Jurek and Schreiber in the

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integral.

1. INTRODUCTION

Recall that a Borel probability measure $\mu$, on a real separable Banach space $E$, is
called infinitely divisible if for each natural number $n$ there exists a probability
measure $\mu^n$ such that $\mu^n \ast n = \mu$; the class of all infinitely divisible measures will be
denoted by $ID$. It is well known that their Fourier transforms (the Lévy–Khintchine
formulas) can be written as follows:

\begin{equation}
\hat{\mu}(y) = e^{\Phi(y)}, \quad y \in E',
\end{equation}

and the exponents $\Phi$ are of the form

\begin{equation}
\Phi(y) = i\langle y, a \rangle - \frac{1}{2} \langle y, Ry \rangle + \int_{E \setminus \{0\}} [e^{i\langle y, x \rangle} - 1 - i\langle y, x \rangle 1_B(x)] M(dx),
\end{equation}

where $E'$ denotes the dual Banach space, $\langle \cdot, \cdot \rangle$ is an appropriate bilinear form be-
tween $E'$ and $E$, $a$ is a shift vector, $R$ is a covariance operator corresponding to

the Gaussian part of \( \mu \), and \( M \) is a Lévy spectral measure. There is a one-to-one correspondence between \( \mu \in ID \) and the triples \([a, R, M]\) in its Lévy–Khintchine formula (1.1); cf. Araujo and Giné [1], Chapter 3, Section 6, p. 136. The function \( \Phi(y) \) from (1.1) is called then the Lévy exponent of \( \mu \).

Remark 1.1. (a) If \( E \) is a Hilbert space, then Lévy spectral measures \( M \) are completely characterized by the integrability condition \( \int_E (1 \land \|x\|^2) M(dx) < \infty \), and Gaussian covariance operators \( R \) coincide with the positive trace-class operators; cf. Parthasarathy [12], Chapter VI, Theorem 4.10.

(b) When \( E \) is a Euclidean space, then Lévy exponents are completely characterized as continuous negative-definite functions; cf. Cuppens [4] and Schoenberg’s theorem on p. 80.

Finally, a Lévy process \( Y(t), t \geq 0 \), means a continuous in probability process with stationary and independent increments and \( Y(0) = 0 \). Without loss of generality we may and do assume that it has paths in the Skorokhod space \( D_E[0, \infty) \) of \( E \)-valued càdlàg functions (i.e., right continuous with left-hand limits). There is a one-to-one correspondence between the class \( ID \) and the class of Lévy processes.

The càdlàg paths of a process \( Y \) allow us to define random integrals of the form \( \int_{(a,b]} h(s)Y(r(ds)) \) by the formal formula of integration by parts. Namely,

\[
\int_{(a,b]} h(s)Y(r(ds)) := h(b)Y(r(b)) - h(a)Y(r(a)) - \int_{(a,b]} Y(r(s))dh(s),
\]

where \( h \) is a real-valued function of bounded variation and \( r \) is a monotone and right-continuous function. Furthermore, we have

\[
\left( \mathcal{L}\left( \int_{(a,b]} h(s)Y(r(ds)) \right) \right)(y) = \exp \left[ \int_{(a,b]} \log \left( \mathcal{L}(Y(1)) \right)(h(s)y)r(ds) \right],
\]

where \( \mathcal{L}(\cdot) \) denotes the probability distribution and \( \hat{\mu}(\cdot) \) denotes the Fourier transform of a measure \( \mu \); cf. Jurek and Vervaat [10] or Jurek [7], or Jurek and Mason [9], Section 3.6, p. 116.

2. A CALCULUS ON LÉVY EXPONENTS

Let \( \mathcal{E} \) denote the totality of all functions \( \Phi : E' \to \mathbb{C} \) appearing as the exponent in the Lévy–Khintchine formula (1.1). Hence we have

\[
\mathcal{E} + \mathcal{E} \subset \mathcal{E}, \quad \lambda \cdot \mathcal{E} \subset \mathcal{E} \text{ for all positive } \lambda,
\]

which means that \( \mathcal{E} \) forms a cone in the space of all complex-valued functions defined on \( E' \). Furthermore, if \( \Phi \in \mathcal{E} \), then all dilations \( \Phi(\cdot) \in \mathcal{E} \). These follow from the fact that infinite divisibility is preserved under convolution and under (convolution) powers to positive real numbers.
Here we consider two integral operators acting on \( E \) or its part. Namely,

\[
\mathcal{J} : E \rightarrow E, \quad (\mathcal{J} \Phi)(y) := \int_0^1 \Phi(sy) ds, \quad y \in E';
\]

\[
\mathcal{I} : \text{E}_{\log} \rightarrow \text{E}_{\log}, \quad (\mathcal{I} \Phi)(y) := \int_0^1 \Phi(sy)s^{-1} ds, \quad y \in E'.
\]

Note that \( \mathcal{J} \) is well defined on all of \( E \) since, by (1.3), \( \mathcal{J} \Phi \) is the Lévy exponent of the well-defined integral \( \int_{(0,1)} tdY(t) \), where \( Y(1) \) has the Lévy exponent \( \Phi \); cf. Jurek [7] or [8]. On the other hand, \( \mathcal{I} \) is only defined on \( \text{E}_{\log} \), which corresponds to infinitely divisible measures with finite logarithmic moments, since \( \mathcal{I} \Phi \) is the Lévy exponent of the random integral \( -\int_{(0,1)} tdY(-\ln t) = \int_{(0,\infty)} e^{-s} dY(s) \), where \( \Phi \) is the Lévy exponent of \( Y(1) \) that has finite logarithmic moment; cf. Jurek and Vervaat [10].

Here are the main algebraic properties of the mappings \( \mathcal{J} \) and \( \mathcal{I} \).

**Lemma 2.1.** The operators \( \mathcal{I} \) and \( \mathcal{J} \) acting on appropriate domains (Lévy exponents) have the following basic properties:

(a) \( \mathcal{I}, \mathcal{J} \) are additive and positive homogeneous operators;
(b) \( \mathcal{I}, \mathcal{J} \) commute under the composition and \( \mathcal{J}(\mathcal{I}(\Phi)) = (\mathcal{I} - \mathcal{J})\Phi \).

Other equivalent forms of the last property are:

\[ \mathcal{J}(\mathcal{I} + \mathcal{I}) = \mathcal{I}; \quad \mathcal{I}(\mathcal{I} - \mathcal{J}) = \mathcal{J}; \quad (\mathcal{I} - \mathcal{J})(\mathcal{I} + \mathcal{I}) = \mathcal{I}. \]

**Proof.** Part (a) follows from the fact that \( E \) forms a cone. For part (b) let us note that

\[
(\mathcal{J}(\mathcal{I}(\Phi)))(y) = \int_0^1 (\mathcal{I}(\Phi))(ty) dt = \int_0^1 \int_0^1 \Phi(sty)s^{-1} ds dt
= \int_0^1 \int_0^1 \Phi(ry)r^{-1} dr dt = \int_0^1 \Phi(ry) dt r^{-1} dr
= \int_0^1 \Phi(ry) r^{-1} dr - \int_0^1 \Phi(ry) dr = \mathcal{I}\Phi(y) - \mathcal{J}\Phi(y)
= (\mathcal{I} - \mathcal{J})\Phi(y),
\]

which proves the equality in (b). Note that from the above (the first line of the above argument) we infer also that the operators \( \mathcal{I} \) and \( \mathcal{J} \) commute, which completes the argument. ■

**Lemma 2.2.** The operators \( \mathcal{I} \) and \( \mathcal{J} \), defined by (2.2), have the following additional properties:

(a) \( \mathcal{J} : \text{E}_{\log} \rightarrow \text{E}_{\log} \) and \( \mathcal{I} : \text{E}_{(\log)^2} \rightarrow \text{E}_{\log} \).
(b) If \((I - J)\Phi \in \mathcal{E}\), then the corresponding infinitely divisible measure \(\tilde{\mu}\) with the Lévy exponent \((I - J)\Phi(y), y \in \mathcal{E}'\), has finite logarithmic moment.

(c) \((I - J)\Phi + I(I - J)\Phi = (I - J)\Phi + J\Phi = \Phi\) for all \(\Phi \in \mathcal{E}\).

Proof. (a) Since the function \(E \ni x \rightarrow \log(1 + \|x\|)\) is subadditive, for an infinitely divisible probability measure \(\mu = [a, R, M]\) we have

\[(2.3) \int_{E} \log(1 + \|x\|)\mu(dx) < \infty \text{ if and only if } \int_{\{\|x\| > 1\}} \log \|x\| M(dx) < \infty;\]

cf. Jurek and Mason [9], Proposition 1.8.13. Furthermore, if \(M\) is the spectral Lévy measure appearing in the Lévy exponent \(\Phi\), then \(J\Phi\) has a Lévy spectral measure \(JM\) (we keep that potentially conflicting notation), where

\[(2.4) (JM)(A) := \int_{(0,1)} M(t^{-1}A)dt = \int_{(0,1)} \int_{E} 1_{\{\|x\| > 0\}} M(dx) dt\]

for all Borel subsets \(A\) of \(E \setminus \{0\}\). Hence

\[
\begin{align*}
&\int_{\{\|x\| > 1\}} \log \|x\|(JM)(dx) = \int_{(0,1)} \int_{E} 1_{\{\|x\| > 1\}} (tx) \log(t\|x\|) M(dx) dt \\
&= \int_{(0,1)} \int_{\{\|x\| > t^{-1}\}} \log(t\|x\|) M(dx) dt = \int_{\{\|x\| > 1\}} \log(t\|x\|) M(dx) \\
&= \int_{\{\|x\| > 1\}} \log \|x\|^{-1} M(dx) \\
&= \int_{\{\|x\| > 1\}} \log \|x\| M(dx) - \int_{\{\|x\| > 1\}} [1 - \|x\|^{-1}] M(dx).
\end{align*}
\]

Since the last integral is always finite as we integrate a bounded function with respect to a finite measure, we get the first part of (a). For the second one, let us note that

\[
\int_{\{\|x\| > 1\}} \log \|x\|(IM)(dx) = \int_{0}^{\infty} \int_{\{\|x\| > 1\}} \log \|x\| M(e^t dx) dt \\
= \frac{1}{2} \int_{\{\|x\| > 1\}} \log^2 \|x\| M(dx),
\]

where \(IM\) is the Lévy spectral measure corresponding to the Lévy exponent \(I\Phi\).
For the part (b), note that the assumption made there implies that the measure

\[(2.5) \quad \tilde{M}(A) := M(A) - \int_{(0,1)} M(t^{-1}A)dt \geq 0 \text{ for all Borel sets } A \subset E \setminus \{0\}\]

is the Lévy spectral measure of some \(\tilde{\mu}\). [Note that there is no restriction on the Gaussian part.] In fact, if \(\tilde{M}\) is a nonnegative measure, then it is necessarily a Lévy spectral measure because \(0 \leq \tilde{M} \leq M\) and \(M\) is a Lévy spectral measure; cf. Araujo and Giné [1], Chapter 3, Theorem 4.7, p. 119. To establish the logarithmic moment of \(\tilde{\mu}\) we argue as follows. Observe that for any constant \(k > 1\) we have

\[
0 \leq \int_{\{1 < ||x|| \leq k\}} \log ||x|| \tilde{M}(dx)
\]

\[
= \int_{\{1 < ||x|| \leq k\}} \log ||x|| M(dx) - \int_{(0,1)} \int_{\{1 < ||x|| \leq k\}} \log ||x|| M(t^{-1}dx)dt
\]

\[
= \int_{\{1 < ||x|| \leq k\}} \log ||x|| M(dx) - \int_{(0,1)} \int_{\{t^{-1} < ||x|| \leq kt^{-1}\}} \log(t||x||) M(dx)dt
\]

\[
= \int_{\{1 < ||x|| \leq k\}} \log ||x|| M(dx) - \int_{\{1 < ||x|| \leq k\}} \frac{1}{||x||^{-1}} \int_{\{k < ||x||\}} \log(t||x||) dt M(dx)
\]

\[
= \int_{\{1 < ||x|| \leq k\}} \log ||x|| M(dx) - \int_{\{1 < ||x|| \leq k\}} ||x||^{-1} \int_{1}^{k} \log(w) dw M(dx)
\]

\[
- \int_{\{k < ||x||\}} ||x||^{-1} \int_{1}^{k} \log(w) dw M(dx)
\]

\[
= \int_{\{1 < ||x|| \leq k\}} \log ||x|| M(dx) - \int_{\{1 < ||x|| \leq k\}} ||x||^{-1}(||x|| \log ||x|| - ||x|| + 1) M(dx)
\]

\[
- (k \log k - k + 1) \int_{\{||x|| > k\}} ||x||^{-1} M(dx)
\]

\[
= \int_{\{1 < ||x|| \leq k\}} (1 - ||x||^{-1}) M(dx) - (k \log k - k + 1) \int_{\{||x|| > k\}} ||x||^{-1} M(dx)
\]

\[
\leq M(||x|| > 1) < \infty,
\]

and consequently \(\int_{\{||x|| > 1\}} \log ||x|| \tilde{M}(dx) < \infty\). This with property (2.3) completes the proof of the part (b).

Finally, since \((I - J)\Phi\) is in a domain of definition of the operator \(I\), so the part (c) is a consequence of Lemma 2.1 (b). Thus the proof is complete. \(\blacksquare\)
3. FACTORIZATIONS OF SELFDECOMPOSABLE DISTRIBUTIONS

The classes of limit laws $\mathcal{U}$ and $L$ are obtained by non-linear shrinking transformations and linear transformations (multiplications by scalars), respectively; cf. Jurek [7] and references therein. However, there are many (unexpected) relations between $\mathcal{U}$ and $L$ as was already proved in Jurek [7] and more recently in Iksanov et al. [6]. Furthermore, more recently selfdecomposable distributions are used in modelling real phenomena, in particular in mathematical finance; for instance cf. Bingham [2], Carr et al. [3] or Eberlein and Keller [5]. This motivates further studies on factorizations and other relations between the classes $\mathcal{U}$ and $L$, like those in Theorems 3.1 and 3.2 below.

In this section we will apply the operators $I$ and $J$ to Lévy exponents of selfdecomposable (the class $L$) and s-selfdecomposable (the class $\mathcal{U}$) probability measures. For the convenience of the readers recall here that

\[
\mu \in L \quad \text{iff} \quad \forall (t > 0) \exists \nu_t \quad \mu = T_{e^{-t}} \mu * \nu_t
\]
\[
\quad \text{iff} \quad \mu = \mathcal{L} \left( \int_{(0, \infty)} e^{-t} dY(t) \right), \quad \mathcal{L}(Y(1)) \in ID_{\log};
\]
\[
\mu \in \mathcal{U} \quad \text{iff} \quad \mu = L \left( \int_{(0,1]} t \, dY(t) \right), \quad \mathcal{L}(Y(1)) \in ID.
\]

Measures from the class $\mathcal{U}$ are called s-selfdecomposable; cf Jurek [7], [8]. The corresponding Fourier transforms of measures from $L$ and $\mathcal{U}$ follow easily from (1.2) and (1.3); cf. Jurek and Vervaat [10] or the above references.

**Lemma 3.1.** If $\mu$ is a selfdecomposable probability measure on a Banach space $E$ with characteristic function $\hat{\mu}(y) = \exp[\Phi(y)]$, $y \in E'$, then

\[
\tilde{\Phi}(y) := \Phi(y) - \int_{(0,1]} \Phi(sy) ds = (I - J)\Phi(y), \quad y \in E',
\]

is a Lévy exponent corresponding to an infinitely divisible probability measure with finite logarithmic moment.

Equivalently, if $M$ is the Lévy spectral measure of a selfdecomposable $\mu$, then the measure $\tilde{M}$ given by

\[
\tilde{M}(A) := M(A) - \int_0^1 M(t^{-1}A) dt, \quad A \subset E \setminus \{0\},
\]

is a Lévy spectral measure on $E$ that additionally integrates the logarithmic function on the complement of any neighborhood of zero.

**Proof.** If $\mu = [a, R, M]$ is selfdecomposable (or, in other words, a class $L$ distribution), then we infer that

\[
M(A) - M(e^tA) \geq 0 \quad \text{for all} \quad t > 0 \quad \text{and Borel} \quad A \subset E \setminus \{0\},
\]
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and that there is no restriction on the remaining two parameters (the shift vector and the Gaussian covariance operator) in the Lévy–Khintchine formula (1.1). Multiplying both sides by $e^{-t}$ and then integrating over the positive half-line we conclude that $\tilde{M}$, given by (2.5), is a non-negative measure. Since $\tilde{M} \leq M$ and $M$ is a Lévy spectral measure, so is $\tilde{M}$; cf. Theorem 4.7 in Chapter 3 of Araujo and Giné [1]. Finally, our Lemma 2.2 (b) gives the finiteness of the logarithmic moment. Thus the proof is complete. ■

**Theorem 3.1.** For each selfdecomposable probability measure $\mu$, on a Banach space $E$, there exists a unique $s$-selfdecomposable probability measure $\tilde{\mu}$ with finite logarithmic moment such that

$$\mu = \tilde{\mu} \star I(\tilde{\mu}) \quad \text{and} \quad J(\mu) = I(\tilde{\mu}).$$

In fact, if $\tilde{\mu}(y) = \exp[\Phi(y)]$, then $(\tilde{\mu})(y) = \exp \left[\Phi(y) - \int_{(0,1)} \Phi(ty) dt\right]$,

$$\Phi(y) = \Phi(y) - \int_{(0,1)} \Phi(ty) dt = \left(\Phi - \int_{(0,\infty)} \Phi(t)e^{-t} dt\right)(y),$$

is a Lévy exponent as well, because of Lemma 3.1. Again by Lemma 3.1 (or Lemma 2.2 (b)), a probability measure $\tilde{\mu}$ defined by the Fourier transform $(\tilde{\mu})(y) = \exp(\Phi - \int_{(0,\infty)} \Phi(t)e^{-t} dt)(y)$ has logarithmic moment. Consequently, $I(\tilde{\mu})$ is a well-defined probability measure whose Lévy exponent is equal to $I(I-J)\Phi$. Finally, Lemmas 2.1 (b) and 2.2 (c) give the factorization (3.2).

Since $I(\tilde{\mu}) \in L$ has the property that $\tilde{\mu} \star I(\tilde{\mu})$ is again in $L$, therefore Theorem 1 from Iksanov et al. [6] implies that $\tilde{\mu} \in U$, i.e., it is an $s$-selfdecomposable probability distribution.

To see the second equality in (3.3) one should observe that it is equivalent to the equality $J\Phi = I(I-J)\Phi$ that indeed holds true in view of Lemma 2.1 (b).

Suppose there exists another factorization of the form $\mu = \rho \star I(\rho)$ and let $\Xi(y)$ be the Lévy exponent of $\rho$. Then we see that $\Phi(y) = \Xi(y) + (I \Xi)(y) = (I + I) \Xi(y)$. Hence, applying to both sides $I - J$ we conclude that

$$(I - J)\Phi = ((I - J)(I + I)) \Xi = \Xi,$$

where the last equality is from Lemma 2.1 (b). This proves the uniqueness of $\tilde{\mu}$ in the representation (3.2), and thus the proof of Theorem 3.1 is completed. ■
Remark 3.1. The factorization (3.2) in Theorem 3.1 can be also derived from previous papers as follows:

For each selfdecomposable (or class \(L\)) \(\mu\) there exists a unique \(\rho \in ID_{\log}\) such that \(\mu = \mathcal{I}(\rho)\); Jurek and Vervaat [10]. Since \(\tilde{\mu} := \mathcal{J}(\rho)\) is an \(s\)-selfdecomposable (class \(U\)) with logarithmic moment (cf. Jurek [7]), therefore \(\mathcal{I}(\tilde{\mu}) \ast \tilde{\mu} \in L\) in view of Iksanov et al. [6]. Finally, again by Jurek [7], \(\mathcal{I}(\tilde{\mu}) \ast \tilde{\mu} \in L\) in view of Iksanov et al. [6].

However, the present proof is less involved, more straightforward and, moreover, the result and the proof of finiteness of the logarithmic moment in Lemma 2.2 (b) are completely new. Last but not least, the “calculus” on Lévy exponents, introduced in this note, is of an interest in itself.

Example 3.1. Let \(\Sigma_p\) be a symmetric stable distribution on a Banach space \(E\), with the exponent \(p\). Then its Lévy exponent \(\Phi_p\) is equal to

\[
\Phi_p(y) = -\int_S |\langle y, x \rangle|^p m(dx),
\]

where \(m\) is a finite Borel measure on the unit sphere \(S\) of \(E\); cf. Samorodnitsky and Taqqu [13]. Hence \((I - \mathcal{J})\Phi_p(y) = p/(p + 1)\Phi_p(y)\), which means that in Corollary 3.1, both \(\nu_1\) and \(\nu_2\) are stable with the exponent \(p\) and measures \(m_1 := (p/(p + 1))m\) and \(m_2 := (1/(p + 1))m\), respectively.

Example 3.2. Let \(\eta\) denote the Laplace (double exponential) distribution on the real line \(\mathbb{R}\); cf. Jurek and Yor [11]. Then its Lévy exponent \(\Phi_\eta\) is equal
to \( \Phi_\eta(t) := -\log(1 + t^2) \), \( t \in \mathbb{R} \). Consequently, it follows that \((I - J)\Phi_\eta(t) = 2(\arctan t - t)t^{-1}\) is the Lévy exponent of the class \( U \) probability measure \( \nu_1 \) from Corollary 3.1, and \((2t - \arctan t - t \log(1 + t^2))t^{-1}\) is the Lévy exponent of the class \( Lf \) measure \( \nu_2 \) from Corollary 3.1.

Before we formulate the next result we need to recall that, by (3.1), the class \( U \) is defined here as \( U = J(ID) \). Consequently, by iteration argument we can define

\[
U^{(1)} := U, \quad U^{(k+1)} := J(U^{(k)}) = J^{k+1}(ID), \quad k = 1, 2, \ldots ;
\]

cf. Jurek [8] for other characterization of classes \( U^{(k)} \). Elements from the semi-groups \( U^{(k)} \) are called \( k \)-times s-selfdecomposable probability measures.

**Theorem 3.2.** Let \( n \) be any natural number and \( \mu \) be a selfdecomposable probability measure. Then there exist \( k \)-times s-selfdecomposable probability measures \( \tilde{\mu}_k \), for \( k = 1, 2, \ldots, n \), such that

\[
\mu = \tilde{\mu}_1 * \tilde{\mu}_2 * \ldots * \tilde{\mu}_n * I(\mu_n), \quad J^k(\mu) = I(\tilde{\mu}_k), \quad k = 1, 2, \ldots, n.
\]

In fact, if \( \Phi \) is the exponent of \( \mu \), then \( \tilde{\mu}_k \) has the exponent \( I^{k-1}(I - J)^k \Phi = (I - J)^k \Phi \) and

\[
\Phi = (I - J)^{n-1}(I - J)^n \Phi + (I - J)^{n-2}(I - J)^n \Phi + \ldots + (I - J)(I - J)^{n-1} \Phi + (I - J) \Phi = (I - J)^n \Phi + \Phi.
\]

**Proof.** For \( n = 1 \) the factorization (3.6) and the formula (3.7) are true by Theorem 3.1, with \( \tilde{\mu}_1 := \tilde{\mu} \). Suppose our claim (3.6) is true for \( n \). Since \( \rho := I(\mu_n) \) is selfdecomposable, applying to it Theorem 3.1, we have \( \rho = \tilde{\rho} * I(\tilde{\rho}) \), where \( \tilde{\rho} \) has the Lévy exponent \((I - J)\rho^n = J^n(I - J)\Phi \), and thus it corresponds to an \((n + 1)\)-times s-selfdecomposable probability because, by Theorem 3.1, \((I - J)\Phi \) is already s-selfdecomposable; then we apply \( n \) times the operator \( J \); cf. the definition (3.5). Thus the factorization (3.6) holds for \( n + 1 \), which completes the proof of the first part of the theorem.

Similarly, applying inductively the decomposition (3.3) and using Lemma 2.1 (b), we get the formula (3.6). Thus the proof is complete. \( \square \)

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