PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 29, Fasc. 1 (2009), pp. 119–134

A KINGMAN CONVOLUTION APPROACH TO BESSEL PROCESSES*

BY

NGUYEN VAN THU (HCM CITY)

Dedicated to Professor Kazimierz Urbanik,
my former teacher, from whom I learnt
how a man and a mathematician could become to each other

Abstract. In this paper we study Bessel processes in terms of the Kingman convolution method. In particular, we propose a higher dimensional model of the Kingman convolution algebras. We show that every Bessel process started at 0 is induced by a Kingman convolution. Moreover, a new concept of increments of stochastic processes is introduced. It permits to regard Bessel processes as "stationary and independent increments processes".

2000 AMS Mathematics Subject Classification: Primary: 60G48, 60G51, 60G57; Secondary: 60J25, 60J60, 60J99.

Key words and phrases: Kingman convolution, radial characteristic function, independent increment-type processes, Rayleigh distribution, Urbanik convolution algebras.

1. INTRODUCTION, NOTATION AND PRELIMILARIES

This study is inspired by a distinguished part of Bessel processes in financial mathematics for decades. Indeed, for each $n = 1, 2, \ldots$ let

$$\mathbf{W}_t = (W_t^{(1)}, W_t^{(2)}, \ldots, W_t^{(n)})$$

be an n-dimensional Brownian motion $(BM^{(n)})$ and $\rho_t = \|\mathbf{W}_t\|$ its radial part. Consider the following process:

$$\beta_t = \sum_{i=1}^{n} \int_0^t \frac{W_s^{(i)}}{\rho_s} dW_s^{(i)} \tag{1.1}$$

* The paper has been completed during the author's stay at the Department of Mathematics and Informatics, Philipps University, Marburg, July–September 2006, with a support from the Humboldt Foundation.

which, since $\langle\beta,\beta\rangle_t = t$, stands for a linear Brownian motion, i.e. a $BM^{(1)}$. By virtue of Revuz and Yor [12], p. 439, we have

$$\rho_t^2 = \rho_0^2 + 2\int_0^t \rho_s d\beta_s + nt. \tag{1.2}$$

Replacing n by any nonnegative number $\delta \geqslant 0$, we see that the equation (1.2) leads to the following interpolation class of stochastic differential equations (SDE):

$$Z_t = x + 2\int_0^t \sqrt{|Z_s|}d\beta_s + \delta t, \tag{1.3}$$

where $x \geqslant 0$.

Note that (1.3) is a special case of the Cox–Ingersoll–Ross family of diffusions [2] and has a unique solution which is strong, nonnegative and adapted with respect to the natural filtration $\{\mathcal{F}_t\}$ of $\{W_t\}$. Consequently, in the case when $\delta \geqslant 0$, $x \geqslant 0$, the absolute sign in (1.3) can be omitted and $\{Z_t\}$ can be modelled as short term interest rates (cf. Cox et al. [2]).

DEFINITION 1.1 (cf. Revuz and Yor [12], XI). For every $\delta \geqslant 0, x \geqslant 0$, the unique strong solution of the equation (1.3) is called the *square of δ-dimensional Bessel process started at x* and is denoted by $BESQ^\delta(x)$. Further, the square root of $BESQ^\delta(x^2)$ is called the *Bessel process*[1] *of dimension δ started at x* and is denoted by $BES^\delta(x)$.

In the sequel, we study the class of processes $BES^\delta(x), \delta = 2(s+1) \geqslant 1$, via the Kingman convolution method and also use s as a fixed index of the Bessel process.

Let $\mathcal{P}$ denote the class of all probability measures (p.m.'s) on the positive half-line $\mathbb{R}^+$ endowed with the weak convergence, and $*_{1,\delta}, \delta \geqslant 1$, denote the Kingman convolution which was introduced by Kingman [5] in connection with the addition of independent spherically symmetric random vectors (r.vec.'s) in a Euclidean space. Namely, for each continuous bounded function f on $\mathbb{R}^+$ we write

$$\int_0^\infty f(x)\mu *_{1,\delta} \nu(dx) \tag{1.4}$$
$$= \frac{\Gamma(s+1)}{\sqrt{\pi}\Gamma\big(s+\frac{1}{2}\big)} \int_0^\infty\int_0^\infty\int_{-1}^1 f\big((x^2+2uxy+y^2)^{1/2}\big)(1-u^2)^{s-1/2}\mu(dx)\nu(dy)du,$$

[1] If n is replaced by a negative real number, then the corresponding unique strong solution to the equation (1.3) exists, and thus the Bessel process of a negative dimension δ can be defined (cf. Revuz and Yor [12], Exercise 1.33, p. 453).

where $\mu,\nu \in P$ and $\delta = 2(s+1) \geqslant 1$ (cf. Kingman [5] and Urbanik [15]). The algebra $(\mathcal{P}, *_{1,\delta})$ is the most important example of Urbanik convolution algebras (cf. Urbanik [15]). In the language of the Urbanik convolution algebras, the *characteristic measure*, say σ_s, of the Kingman convolution has the Rayleigh density

$$d\sigma_s(y) = \frac{2(s+1)^{s+1}}{\Gamma(s+1)} y^{2s+1} \exp\big(-(s+1)y^2\big)dy \tag{1.5}$$

with the characteristic exponent $\varkappa = 2$ and the kernel Λ_s,

$$\Lambda_s(x) = \Gamma(s+1)J_s(x)/(1/2x)^s, \tag{1.6}$$

where $J_s(x)$ denotes the Bessel function,

$$J_s(x) := \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}. \tag{1.7}$$

It is known (cf. Kingman [5], Theorem 1) that the kernel Λ_s itself is an ordinary characteristic function (ch.f.) of a symmetric p.m., say F_s, defined on the interval $[-1,1]$. Thus, if θ_s denotes a random variable (r.v.) with distribution F_s, then for each $t \in \mathbb{R}^+$

$$\begin{aligned} \Lambda_s(t) &= E\exp(it\theta_s) \\ &= \int_{-1}^{1} \exp(itx)dF_s(x). \end{aligned} \tag{1.8}$$

Suppose that X is a nonnegative r.v. with distribution $\mu \in \mathcal{P}$ and X is independent of θ_s. The *radial characteristic function* (rad.ch.f.) of μ, denoted by $\hat{\mu}(t)$, is defined by

$$\begin{aligned} \hat{\mu}(t) &= E\exp(itX\theta_s) \\ &= \int_{0}^{\infty} \Lambda_s(tx)\mu(dx) \end{aligned} \tag{1.9}$$

for every $t \in \mathbb{R}^+$. In particular, the rad.ch.f. of σ_s is

$$\hat{\sigma}_s(t) = \exp\left(-\frac{t^2}{2}\right), \quad t \in \mathbb{R}^+. \tag{1.10}$$

It should be noted, since the rad.ch.f. is defined uniquely up to the mapping $x \to ax, a > 0, x \in \mathbb{R}^+$, that the representation (1.10) may differ from the one given in Urbanik [15] and Kingman [5] only by a scale parameter.

2. CARTESIAN PRODUCT OF KINGMAN CONVOLUTIONS

Denote by $\mathbb{R}^{+k}, k = 1, 2, \ldots$, the k-dimensional nonnegative cone of $\mathbb{R}^k$ and $\mathcal{P}(\mathbb{R}^{+k})$ the class of all p.m.'s on $\mathbb{R}^{+k}$ equipped with the weak convergence. In the sequel, we will denote the multidimensional vectors and distributions and r.vec.'s by bold letters. For each point z of any set Z let δ_z denote the Dirac measure (the unit mass) at the point z. In particular, if $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^{k+}$, then

$$\delta_{\mathbf{x}} = \delta_{x_1} \times \delta_{x_2} \times \ldots \times \delta_{x_k}, \tag{2.1}$$

where the sign $\times$ denotes the Cartesian product of measures. We put, for $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_k) \in \mathbb{R}^{+k}$,

$$\delta_{\mathbf{x}} \bigcirc_k \delta_{\mathbf{y}} = \{\delta_{x_1} \circ \delta_{y_1}\} \times \{\delta_{x_2} \circ \delta_{y_2}\} \times \ldots \times \{\delta_{x_k} \circ \delta_{y_k}\}; \tag{2.2}$$

for the sake of simplicity, here and somewhere below we denote the Kingman convolution operation $*_{1,\delta}$ simply by $\circ$. Since convex combinations of p.m.'s of the form (2.1) are dense in $\mathcal{P}(\mathbb{R}^{+k})$, the relation (2.2) can be extended to arbitrary p.m.'s $\mathbf{F}, \mathbf{G} \in \mathcal{P}(\mathbb{R}^{+k})$. Namely, we put

$$\mathbf{F} \bigcirc_k \mathbf{G} = \iint_{\mathbb{R}^{+k}} \delta_{\mathbf{x}} \bigcirc_k \delta_{\mathbf{y}} \mathbf{F}(d\mathbf{x})\mathbf{G}(d\mathbf{y}). \tag{2.3}$$

In the sequel, the binary operation $\bigcirc_k$ will be called the *k-times Cartesian product of Kingman convolutions*. It is easy to show that the binary operation $\bigcirc_k$ is continuous in the weak topology, which together with (1.4) and (2.3) implies the following theorem.

THEOREM 2.1. *The pair $\big(\mathcal{P}(\mathbb{R}^{+k}), \bigcirc_k\big)$ is a commutative topological semigroup with $\delta_{\mathbf{0}}$ as the unit element. Moreover, the operation $\bigcirc_k$ is distributive with respect to convex combinations of p.m.'s in $\mathcal{P}(\mathbb{R}^{+k})$.*

In the sequel, the pair $\big(\mathcal{P}(\mathbb{R}^{+k}), \bigcirc_k\big)$ will be called a *k-dimensional Kingman convolution algebra*[2]. For every $\mathbf{F} \in \mathcal{P}(\mathbb{R}^{+k})$ the k-dimensional rad.ch.f. $\hat{\mathbf{F}}(\mathbf{t}), \mathbf{t} = (t_1, t_2, \ldots, t_k) \in \mathbb{R}^{k+}$, is defined by

$$\hat{\mathbf{F}}(\mathbf{t}) = \int_{\mathbb{R}^{+k}} \prod_{j=1}^{k} \Lambda_s(t_j x_j)\mathbf{F}(\mathbf{dx}), \tag{2.4}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^{+k}$.

The k-dimensional Rayleigh distribution, say $\mathbf{\Sigma}_s$, is defined by

$$\mathbf{\Sigma}_s = \sigma_s \times \sigma_s \times \ldots \times \sigma_s \quad (k \text{ times}). \tag{2.5}$$

[2] Higher dimensional Urbanik convolution algebras can be introduced in the same way as here for the Kingman convolution case but this subject will be treated systematically elsewhere.

Furthermore, for any nonnegative numbers $\lambda_r, r = 1, 2, \ldots$, the distribution

$$\mathbf{F} = \{T_{\lambda_1}\sigma_s\} \times \{T_{\lambda_2}\sigma_s\} \times \ldots \times \{T_{\lambda_k}\sigma_s\} \tag{2.6}$$

stands for a *k-dimensional Rayleighian distribution*. Here and in the sequel, if X is an r.v. or an r.vec. with distribution μ and λ is a real number, then we denote by $T_\lambda\mu$ the distribution of λX.

By virtue of formulas (1.10) and (2.4)–(2.6) we have the following

THEOREM 2.2. *Suppose distributions* $\mathbf{\Sigma}$ *and* $\mathbf{F}$ *are of the form* (2.5) *and* (2.6). *Then, for any* $\mathbf{t} \in \mathbb{R}^{+k}$,

$$-\log \hat{\mathbf{\Sigma}}_s(\mathbf{t}) = \frac{1}{2}\sum_{j=1}^{k} t_j^2 \tag{2.7}$$

and

$$-\log \hat{\mathbf{F}}(\mathbf{t}) = \frac{1}{2}\sum_{j=1}^{k} \lambda_j^2 t_j^2. \tag{2.8}$$

Let $\theta, \theta_1, \theta_2, \ldots, \theta_k$ be independent identically distributed (i.i.d.) r.v.'s with common distribution F_s. We set

$$\mathbf{\Theta}_s = (\theta_1, \theta_2, \ldots, \theta_k). \tag{2.9}$$

Assume that $\mathbf{X} = (X_1, X_2, \ldots, X_k)$ is a k-dimensional r.vec. with distribution $\mathbf{F}$ and $\mathbf{X}$ is independent of $\mathbf{\Theta}$. We put

$$[\mathbf{\Theta}, \mathbf{X}] = (\theta_1 X_1, \theta_2 X_2, \ldots, \theta_k X_k). \tag{2.10}$$

Then the following formula is the multidimensional generalization of (1.9) and is equivalent to (2.4):

$$\hat{\mathbf{F}}(\mathbf{t}) = E\exp(i\langle \mathbf{t}, [\mathbf{\Theta}, \mathbf{X}]\rangle), \tag{2.11}$$

where $\mathbf{X}$ and $\mathbf{\Theta}$ are assumed to be independent, $\mathbf{t} = (t_1, t_2, \ldots, t_k) \in \mathbb{R}^{+k}$, and the symbol $\langle\ ,\ \rangle$ denotes the inner product in $\mathbb{R}^k$. In fact, we have

$$\begin{aligned} E\exp\big(i\langle(\theta_1 t_1, \theta_2 t_2, \ldots, \theta_k t_k), \mathbf{X}\rangle\big) &= \int_{\mathbb{R}^{+k}} E\exp\big(i\sum_{j=1}^{k} t_j x_j \theta_j\big)\mathbf{F}(d\mathbf{x}) \\ &= \int_{\mathbb{R}^{+k}} \prod_{j=1}^{k} \Lambda_s(t_j x_j)\mathbf{F}(d\mathbf{x}) = \hat{\mathbf{F}}(\mathbf{t}). \end{aligned} \tag{2.12}$$

As a consequence of the representation (2.11) we have

COROLLARY 2.1. *For each* $\mathbf{F} \in \mathcal{P}(\mathbb{R}^{k+})$ *the rad.ch.f.* $\hat{\mathbf{F}}(\mathbf{t})$ *is also an ordinary k-dimensional ch.f., and hence it is uniformly continuous.*

The following lemma will be used in the representation of k-dimensional infinitely divisible (ID) p.m.'s.

LEMMA 2.1. (i) *For every* $t \geqslant 0$

$$\lim_{x\to 0}\frac{1-\Lambda_s(tx)}{x^2}=\lim_{x\to 0}\frac{1-Ee^{itx\theta}}{x^2}=\frac{t^2}{2}. \tag{2.13}$$

(ii) *For any vectors* $\mathbf{x}=(x_0,x_1,\dots,x_k)$ *and* $\mathbf{t}=(t_0,t_1,\dots,t_k)\in\mathbb{R}^{k+1}$, $k=1,2,\dots$,

$$\lim_{\rho\to 0}\frac{1-\prod_{r=0}^{k}\Lambda_s(t_rx_r)}{\rho^2}=\sum_{r=0}^{k}\lambda_r(\mathbf{x})t_r^2, \tag{2.14}$$

with $\rho=\|\mathbf{x}\|$ *and* $\lambda_r(\mathbf{x}),\ r=0,1,\dots,k$, *given by*

$$\lambda_r(\mathbf{x})=\begin{cases}\frac{1}{2}\cos^2\phi, & r=0,\\ \frac{1}{2}(\sin\phi\sin\phi_1\dots\sin\phi_{r-1}\cos\phi_r)^2, & 1\leqslant r\leqslant k-2,\\ \frac{1}{2}(\sin\phi\sin\phi_1\dots\sin\phi_{k-2}\cos\psi)^2, & r=k-1,\\ \frac{1}{2}(\sin\phi\sin\phi_1\dots\sin\phi_{k-2}\sin\psi)^2, & r=k,\end{cases} \tag{2.15}$$

where $0\leqslant\psi,\ \phi,\ \phi_r\leqslant\pi/2,\ r=1,2,\dots,k-2$, *are angles of* $\mathbf{x}$ *appearing in its polar form.*

Proof. (i) The equation (1.8) together with the l'Hôpital rule implies that

$$\lim_{x\to 0}\frac{1-\Lambda_s(tx)}{x^2}=\lim_{x\to 0}\frac{1-Ee^{it\theta}}{x^2}=\frac{t^2}{2},$$

which proves (2.13).

(ii) In order to prove (2.14) assume that the points $\mathbf{x}=(x_0,x_1,\dots,x_k)\in\mathbb{R}^{k+1}$ are of the polar form

$$x_r=\begin{cases}\rho\cos\phi, & r=0,\\ \rho\sin\phi\sin\phi_1\dots\sin\phi_{r-1}\cos\phi_r, & 1\leqslant r\leqslant k-2,\\ \rho\sin\phi\sin\phi_1\dots\sin\phi_{k-2}\cos\psi, & r=k-1,\\ \rho\sin\phi\sin\phi_1\dots\sin\phi_{k-2}\sin\psi, & r=k,\end{cases} \tag{2.16}$$

where $0\leqslant\psi,\phi,\phi_r\leqslant\pi/2, r=1,2,\dots,k-2$. Putting

$$A(\mathbf{\Theta},\mathbf{t},\mathbf{\Phi})=t_0\theta_0\cos\phi+\sum_{r=1}^{k-2}t_r\theta_r\sin\phi\sin\phi_1\dots\sin\phi_{r-1}\cos\phi_r \tag{2.17}$$
$$+t_{k-1}\theta_{k-1}\sin\phi\sin\phi_1\dots\sin\phi_{k-2}\cos\psi+t_k\theta_k\sin\phi\sin\phi_1\dots\sin\phi_{k-2}\sin\psi$$

and

$$V(\mathbf{\Theta}, \mathbf{t}, \mathbf{\Phi}) = \sum_{r=0}^{k} t_r x_r \theta_r, \tag{2.18}$$

where the $\theta_r, r = 0, 1, 2, \ldots$, are symmetric i.i.d. r.v.'s with distribution σ_s, $\mathbf{\Phi} = (\psi, \phi, \phi_1, \ldots, \phi_k)$ and $\mathbf{\Theta} := (\theta_0, \theta_1, \ldots, \theta_k)$. By virtue of (2.12) and (2.16) and applying l'Hôpital rule, we have

$$\begin{aligned} \lim_{\rho\to 0} \frac{1 - \prod_{r=0}^{k} \Lambda_s(t_r x_r)}{\rho^2} &= \lim_{\rho\to 0} \frac{1 - E\big(\exp\big(i\sum_{r=0}^{k} t_r x_r \theta_r\big)\big)}{\rho^2} \\ &= \frac{(d^2/d\rho^2)\big(1 - E\exp\big(i\rho A(\mathbf{\Theta}, \mathbf{t}, \mathbf{\Phi})\big)\big)}{(d^2/d\rho^2)\rho^2}\Bigg|_{\rho=0} \\ &= \frac{1}{2} E V^2(\mathbf{\Theta}, \mathbf{t}, \mathbf{\Phi}) \exp\big(i\rho V(\mathbf{\Theta}, \mathbf{t}, \mathbf{\Phi})\big)\Big|_{\rho=0}. \end{aligned} \tag{2.19}$$

Since σ_s has expectation zero and variance one, it follows that

$$E V^2(\theta, \mathbf{t}, \phi) = \sum_{j=1}^{k} t_j^2 x_j^2, \tag{2.20}$$

which together with (2.19) implies (2.14). ■

Proceeding successively, we have the following theorem:

THEOREM 2.3. *Every p.m.* $\mathbf{F} \in \mathcal{P}(\mathbb{R}^{+k})$ *is uniquely determined by its k-dimensional rad.ch.f.* $\hat{\mathbf{F}}$ *and the following formula holds*:

$$\widehat{\mathbf{F}_1 \bigcirc_k \mathbf{F}}_2(\mathbf{t}) = \widehat{\mathbf{F}_1}(\mathbf{t})\widehat{\mathbf{F}_2}(\mathbf{t}), \tag{2.21}$$

where $\mathbf{F}_1, \mathbf{F}_2 \in \mathcal{P}(\mathbb{R}^{+k})$ *and* $\mathbf{t} \in \mathbb{R}^{+k}$.

P r o o f. The formula (2.21) follows from (1.4) and (2.3). Next, using the formulas (2.3) and (2.4) and integrating the function $\hat{\mathbf{F}}(t_1 u_1, \ldots, t_k u_k)$ k times with respect to σ_s, we get

$$\begin{aligned} &\int_{\mathbb{R}^{+k}} \hat{\mathbf{F}}(t_1 u_1, \ldots, t_k u_k) \sigma_s(du_1) \ldots \sigma_s(du_k) \\ &\qquad = \int_{\mathbb{R}^+} \ldots \int_{\mathbb{R}^+} \prod_{j=1}^{k} \Lambda_s(t_j x_j u_j) \mathbf{F}(\mathbf{dx}) \sigma_s(du_1) \ldots \sigma_s(du_k) \\ &\qquad = \int_{\mathbb{R}^{+k}} \prod_{j=1}^{k} \exp(-t_j^2 x_j^2) \mathbf{F}(\mathbf{dx}), \end{aligned} \tag{2.22}$$

which, by the change of variables $y_j = x_j^2, j = 1, \ldots, k$, and by the uniqueness of the k-dimensional Laplace transform, implies that $\mathbf{F}$ is uniquely determined by the left-hand side of (2.22). ∎

As a consequence of the formula (2.22) we have the following corollary which is an analogue of the continuity theorem for multidimensional Laplace transforms.

THEOREM 2.4. *Suppose that* $\{\mathbf{F}_n\}$ *is a sequence of distributions on* $\mathbb{R}^{k+}$ *and* $\{\phi_n\}$ *is a sequence of the corresponding rad.ch.f.'s. Then* $\mathbf{F}_n$ *converges weakly to a distribution* $\mathbf{F}$ *if and only if* $\{\phi_n\}$ *converges uniformly on every compact subset of* $\mathbb{R}^{k+}$ *to a rad.ch.f.* ϕ.

For any $\mathbf{x} \in \mathbb{R}^{+k}$ the *generalized translation operators* (g.t.o.'s) $\mathbf{T}^{\mathbf{x}}$ acting on the Banach space $\mathbb{C}_b(\mathbb{R}^{+k})$ of real bounded continuous functions f on $\mathbb{R}^{+k}$ are defined, for each $\mathbf{y} \in \mathbb{R}^{+k}$, by

$$\mathbf{T}^{\mathbf{x}} f(\mathbf{y}) = \int_{\mathbb{R}^{+k}} f(\mathbf{u}) \{\delta_{\mathbf{x}} \bigcirc_k \delta_{\mathbf{y}}\}(d\mathbf{u}). \tag{2.23}$$

In terms of these g.t.o.'s the k-dimensional rad.ch.f. of p.m.'s on $\mathbb{R}^{+k}$ can be characterized as follows (see Vólkovich [17] for the proof):

THEOREM 2.5. *A real bounded continuous function* f *on* $\mathbb{R}^{+k}$ *is a* (k*-dimensional*) *rad.ch.f. of a p.m. if and only if* $f(\mathbf{0}) = 1$ *and* f *is* $\{\mathbf{T}^{\mathbf{x}}\}$*-nonnegative definite in the sense that for any* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in \mathbb{R}^k$ *and* $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{C}$

$$\sum_{i,j=1}^{k} \lambda_i \bar{\lambda}_j \mathbf{T}^{\mathbf{x}_i} f(\mathbf{x}_j) \geqslant 0. \tag{2.24}$$

The k-dimensional ID elements with respect to $\bigcirc_k$ can be defined as follows:

DEFINITION 2.1. A p.m. $\mu \in \mathcal{P}(\mathbb{R}^{+k})$ is called *infinitely divisible* (ID) if for every natural m there exists a p.m. μ_m such that

$$\mu = \mu_m \bigcirc_k \ldots \bigcirc_k \mu_m \quad (m \text{ times}).$$

The simplest but most important example of k-dimensional ID distributions are the k-dimensional Rayleigh distributions. More generally, if $\mathbf{F}$ is a k-dimensional Rayleighian distribution, then it is also ID. Let us denote by $ID(\bigcirc_k)$ the class of all i.d.p.m.'s in $\big(\mathcal{P}(\mathbb{R}^{+k}), \bigcirc_k\big)$. The following theorem, being a generalization of Theorem 7 in Kingman [5], stands for an analogue of the Lévy–Khintchine representation for rad.ch.f.'s of i.d.p.m.'s in the k-dimensional Kingman convolution.

THEOREM 2.6. *A p.m.* $\mu \in ID(\bigcirc_k)$ *if and only if there exists a* σ*-finite measure* M (*a Lévy measure*) *on* $\mathbb{R}^{+k}$ *with the property that* $M(\{\mathbf{0}\}) = 0$, *M is finite* *outside every neighborhood of* $\mathbf{0}$ *and*

$$\int_{\mathbb{R}^{+k}} \frac{\|\mathbf{x}\|^2}{1+\|\mathbf{x}\|^2} M(d\mathbf{x}) < \infty \tag{2.25}$$

and, for each $\mathbf{t} = (t_1, \ldots, t_k) \in \mathbb{R}^{k+}$,

$$-\log \hat{\mu}(\mathbf{t}) = \int_{\mathbb{R}^{+k}} \Big\{1 - \prod_{j=1}^{k} \Lambda_s(t_j x_j)\Big\} \frac{1+\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2} M(d\mathbf{x}), \tag{2.26}$$

where, at the origin $\mathbf{0}$, *the integrand on the right-hand side of* (2.26) *is assumed to be of the form*

$$\sum_{j=1}^{k} \lambda_j(\mathbf{x}) t_j^2 = \lim_{\|\mathbf{x}\|\to 0} \Big\{1 - \prod_{j=1}^{k} \Lambda_s(t_j x_j)\Big\} \frac{1+\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2} \tag{2.27}$$

for nonnegative $\lambda_j(\mathbf{x}), j = 1, 2, \ldots, k$, *and* $\mathbf{x} \in \mathbb{R}^{k+}$, *given by equations* (2.15). *In particular, if* M *tends to the measure* $\mathbf{0}$, *then* μ *becomes a Rayleighian distribution with the rad.ch.f.*

$$-\log \hat{\mu}(\mathbf{t}) = \frac{1}{2} \sum_{j=1}^{k} \lambda_j t_j^2, \qquad \mathbf{t} \in \mathbb{R}^{k+}, \tag{2.28}$$

for some nonnegative $\lambda_j, j = 1, \ldots, k$.

Moreover, the representation (2.26) *is unique.*

P r o o f. The proof is carried out in several steps.

(i) *If* ϕ *is a* k*-dimensional ID rad.ch.f., then it does not vanish on* $\mathbb{R}^{k+}$.

Indeed, denote by Φ_k the totality of k-dimensional ID rad.ch.f.'s (of the fixed index s). Then, we have

$$\Phi_k = \bigcap_{n=1}^{\infty} \{\phi : \phi^{1/n} \in \Phi_n\}, \tag{2.29}$$

which, together with (2.12) and (2.21), implies that every k-dimensional ID rad.ch.f. is a symmetric ordinary ID ch.f. and, consequently, it does not vanish on $\mathbb{R}^{k+}$.

(ii) *Any* $\nu \in ID(\bigcirc_k)$ *with rad.ch.f.* $\hat{\nu} = \psi \in \Phi_k$ *can be expressed in the form* (2.26).

Accordingly, for every n there exists $\psi_n \in \Phi_k$ such that $\psi = \psi_n^n$. By virtue of (i), $\psi(\mathbf{t}) > 0$ for each $\mathbf{t}$. Therefore,

$$\log \psi(\mathbf{t}) = \lim_{n\to\infty} n\{\psi_n(\mathbf{t}) - 1\}. \tag{2.30}$$

Let H_n be a p.m. such that

$$\psi_n(\mathbf{t}) = \int\limits_{\mathbb{R}^{k+}} \prod_{j=1}^{k} \Lambda_s(t_j x_j)\mathbf{H}_n(d\mathbf{x}), \quad \mathbf{t} \in \mathbb{R}^{k+}. \tag{2.31}$$

Putting

$$\mathbf{G}_n(A) = n \int\limits_{A} \frac{\|\mathbf{x}\|^2}{1+\|\mathbf{x}\|^2}\mathbf{H}_n(d\mathbf{x}) \tag{2.32}$$

and taking into account the equations (2.30) and (2.31) we get

$$-\log\psi(\mathbf{t}) = \lim_{n\to\infty} \int\limits_{\mathbb{R}^{k+}} \big\{1 - \prod_{j=1}^{k} \Lambda_s(t_j x_j)\big\} \frac{1+\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2}\mathbf{G}_n(d\mathbf{x}), \tag{2.33}$$

which can be rewritten as

$$-\log\psi(\mathbf{t}) = \lim_{n\to\infty} \int\limits_{\mathbb{R}^{k+}} \big\{1 - \prod_{j=1}^{k} \Lambda_s(t_j x_j)\big\}\mathbf{K}_n(d\mathbf{x}), \tag{2.34}$$

where $\mathbf{K}_n$ are finite measures vanishing at $\mathbf{0}$ defined by

$$\mathbf{K}_n(d\mathbf{x}) := \frac{1+\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2}\mathbf{G}_n(d\mathbf{x}) \quad (n = 1, 2, \ldots).$$

Replacing $\mathbf{t}$ in the equation (2.34) by $[\mathbf{t}, \mathbf{u}], \mathbf{t}, \mathbf{u} \in \mathbb{R}^{k+}$, and integrating with respect to $\sigma_s \times \ldots \times \sigma_s(d\mathbf{u})$, we obtain

$$\begin{aligned}
&- \int\limits_{\mathbb{R}^{k+}} \log\psi([\mathbf{t}, \mathbf{u}])\sigma_s \times \ldots \times \sigma_s(d\mathbf{u}) \\
&\qquad = \int\limits_{\mathbb{R}^{k+}} \lim_{n\to\infty} \int\limits_{\mathbb{R}^{k+}} \big\{1 - \prod_{j=1}^{k} \Lambda_s(t_j u_j x_j)\big\}\mathbf{K}_n(d\mathbf{x})\sigma_s \times \ldots \times \sigma_s(d\mathbf{u}) \\
&\qquad = \lim_{n\to\infty} \int\limits_{\mathbb{R}^{k+}} \big\{1 - \prod_{j=1}^{k} \exp(-t_j^2 x_j^2)\big\}\mathbf{K}_n(d\mathbf{x}),
\end{aligned}$$

which, by changing variables $x_j^2 \to u_j, j = 1, 2, \ldots, k$, and applying the continuity theorem for the classical infinitely divisible Laplace transforms on $\mathbb{R}^{k+}$, implies that there exists a finite measure $\mathbf{K}$ vanishing at $\mathbf{0}$ and a subsequence $\{\mathbf{K}_{m_r}\}$ which converges to $\mathbf{K}$ in the sense that for any bounded continuous function f from $\mathbb{R}^{k+}$ to $\mathbb{R}$ vanishing on a neighborhood of $\mathbf{0}$ and

$$\lim_{r\to\infty} \int\limits_{\mathbb{R}^{k+}} f(\mathbf{x})\mathbf{K}_{m_r}(d\mathbf{x}) = \int\limits_{\mathbb{R}^{k+}} f(\mathbf{x})\mathbf{K}(d\mathbf{x}).$$

This together with (2.33) and (2.14) implies that every ψ is of the form (2.26) for a Lévy measure $\mathbf{M}$.

(iii) Now, if $\mathbf{M}$ tends to the zero measure, it follows that, at the origin $\mathbf{0}$, the integrand on the right-hand side of (2.26) is determined by (2.1), which is a consequence of Lemma 2.1.

(iv) Conversely, the uniqueness of the formula (2.26) can be proved in the same way as in the classical case (cf. Sato [13], Theorems 8.1 and 8.7). ∎

3. CONVOLUTION STRUCTURE OF BESSEL PROCESSES

Given a p.m. $\mu \in \mathcal{P}$ and $n = 1, 2, \ldots$ we put, for any $x \in \mathbb{R}^+, B \in \mathcal{B}(\mathbb{R}^+)$, where $\mathcal{B}(\mathbb{R}^+)$ denotes the Borel σ-field of $\mathbb{R}^+$,

$$P_n(x, E) = \delta_x \circ \mu^{\circ n}(E), \tag{3.1}$$

where the power is taken in the convolution $\circ$ sense. Using the rad.ch.f. one can show that $\{P_n(x, E)\}$ satisfies the Chapman–Kolmogorov equation, and therefore there exists a nonnegative homogeneous Markov sequence, say $\{S_n^x\}$, $n = 0, 1, 2, \ldots$, with transition probability $\{P_n(x, E)\}$.

In what follows we will discuss the case of Bessel processes which stand for a continuous counterpart of the above symmetric random walks. Namely, suppose that μ is ID with respect to the Kingman convolution $\circ$. We put

$$q(t, x, E) := \mu^{\circ t} \circ \delta_x(E) \tag{3.2}$$

and take into account the fact that the family $\{q(t, x, \cdot)\}$ of distributions satisfies the Chapman–Kolmogorov equation, and therefore it stands for a transition probability of a homogeneous strong Markov Feller process, say $\{X_t^x\}, t, x \in \mathbb{R}^+$, and, moreover, $\{X_t^x\}$ is stochastically continuous and has a cadlag version (cf. Nguyen [7], Theorem 2.6).

DEFINITION 3.1. A stochastic process $\{X_t^x\}$ is called a *Lévy-type* (or $\circ$-*Lévy*) *process* if

(i) $X_0^x = x$ with probability 1;

(ii) $\{X_t^x\}$ is a strong Markov Feller process with transition probability of the form (3.2);

(iii) $\{X_t^x\}$ is a stochastically continuous process having cadlag realizations with probability 1.

It is evident that all Lévy processes are $*$-Lévy ones. The simplest example of Lévy-type but non-Lévy processes is the absolute value of the linear Brownian motion. Similarly, the following theorem shows that Bessel processes started from 0 stand for Lévy-type processes induced by the Kingman convolution.

THEOREM 3.1. *Let $\{B_t^\delta\}$ denote a Lévy-type process which has transition probability* (3.2) *with $x = 0$ and $\mu = \sigma_s$. Then, up to a scale change, $\{B_t^\delta\}$ and $BES^\delta(0)$ have the same distribution. Consequently, they are induced by the Kingman convolution.*

P r o o f. Let P_x^δ denote the law of $BES^\delta(x), \delta \geqslant 0, x \geqslant 0$, on $C(\mathbb{R}^+, \mathbb{R})$ (cf. [12], XI, p. 446) which entails that the density $p_t^\delta(0, y)$ of the Bessel semigroup is

$$p_t^\delta(0, y) = 2^{-s} t^{-(s+1)} \Gamma(s+1)^{-1} y^{2s+1} \exp(-y^2/2t). \tag{3.3}$$

It should be noted that functions (3.3) are Rayleigh functions of y. In addition, if $t = 2$, we get $P_2^\delta(0, \cdot) = \sigma_s$. Next, by (1.10), we have

$$\widehat{\sigma_s^{\circ t}}(u) = \exp\big(- tu^2/4(s+1)\big), \qquad u \geqslant 0. \tag{3.4}$$

Our further aim is to prove that, up to a scale change, the rad.ch.f. of $\sigma_s^{\circ t}$ is equal to the rad.ch.f. of $P_t^\delta(0, y)$. Accordingly, integrating the kernel $\Lambda_s(uz)$ with respect to $P_t^\delta(0, z)$ we see, by (1.4), (1.6) and (3.3), that the rad.ch.f. of $P_t^\delta(0, y)$ is given, for each $u \geqslant 0$, by

$$\begin{aligned} \widehat{P_t^\delta}(0, y)(u) &= \int_0^\infty \Lambda_s(uz) P_t^\delta(0, z) dz \\ &= 2^{-s} t^{-(s+1)} \Gamma(s+1)^{-1} \int_0^\infty z^{2s+1} \Lambda_s(uz) \exp(-z^2/2t) dz. \end{aligned} \tag{3.5}$$

Hence and by virtue of the Weber integral[3] for $u \geqslant 0$ we have

$$\begin{aligned} &\widehat{q_t^\delta}(0, y)(u) \\ &\quad = \{2^{-s} t^{-(s+1)} \Gamma(s+1)^{-1}\}\{2^{-1} 2^{s+1} t^{s+1} \Gamma(s+1) \exp(-tu^2/2)\} \\ &\quad = \widehat{\sigma_s^{\circ t}}(u), \end{aligned} \tag{3.6}$$

which shows that $q_t^\delta(0) = \sigma_s^{\circ t}$. ∎

---

[3] From Watson ([18], p. 394) we have, for $s \geqslant -1/2, a \geqslant 0, p > 0$,

$$\int_0^\infty t^{s+1} J_s(at) \exp(-p^2 t^2) dt = a^s (2p^2)^{-s-1} \exp(-a^2/4p^2),$$

which may be written as

$$\int_0^\infty t^{2s+1} \Lambda_s(at) \exp(-p^2 t^2) dt = \frac{1}{2} \Gamma(s+1) p^{-2(s+1)} \exp\big(-a^2/(4p^2)\big).$$

4. BESSEL PROCESSES AS STATIONARY INDEPENDENT "INCREMENTS" PROCESSES

Suppose that $X_j, j = 1, 2, \ldots$, are nonnegative independent r.v.'s with the corresponding distributions $F_{X_j}, j = 1, 2, \ldots$, and $\theta, \theta_1, \theta_2, \ldots$ are i.i.d. r.v.'s with the common distribution F_s and the r.v.'s $X_j, j = 1, 2, \ldots, \theta, \theta_1, \theta_2, \ldots$ are independent. Following Kingman ([5], formula (10)) we say, for a fixed $s \geqslant -1/2$, that any one of the equivalent r.v.'s

$$X_1 \oplus X_2 := \sqrt{X_1^2 + X_2^2 + 2X_1X_2\theta_1} \tag{4.1}$$

is a *radial sum* of the two independent nonnegative r.v.'s X_1, X_2. By induction, the *radial sum* $X_1 \oplus X_2 \oplus \ldots \oplus X_k$ is defined for any finite $k = 2, 3, \ldots$ It should be noted ([5], formula (12)) that the operation $\oplus$ is associative.

DEFINITION 4.1. Let $\mathcal{B}_b$ be the ring of subsets of a non-empty bounded Borel subsets of $\mathbb{R}^+$. A function

$$M : \mathcal{B}_b \to L^+, \tag{4.2}$$

where $L^+ = L^+(\Omega, \mathcal{F}, P)$ denotes the class of all nonnegative r.v.'s on the probability space $(\Omega, \mathcal{F}, P)$, is said to be an $\circ$*-scattered random measure* if

(i) $M(\emptyset) = 0$ with probability 1;

(ii) for any $A, B \in \mathcal{B}_b$, $A \cap B = \emptyset$; then $M(A)$ and $M(B)$ are independent and

$$M(A \cup B) \stackrel{d}{=} M(A) \oplus M(B); \tag{4.3}$$

(iii) for any pairwise disjoint sets $A_1, A_2, \ldots \in \mathcal{B}_b$ with the union in $\mathcal{B}_b$, the r.v.'s $M(A_1), M(A_2), \ldots$ are independent and

$$M\big(\bigcup_{j=1}^{\infty} A_j\big) \stackrel{d}{=} \bigoplus_{j=1}^{\infty} M(A_j). \tag{4.4}$$

It is well known that if $\{W(t)\}, t \in \mathbb{R}^+$, is a Brownian motion process, then there exists a Gaussian stochastic measure $M(A), A \in \mathcal{B}_0$, where $\mathcal{B}_0$ is the ring of bounded Borel subsets of $\mathbb{R}^+$ with the property that, for every $t \geqslant 0$, we have $W(t) = M\big((0, t]\big)$. The same is also true for Bessel processes. Namely, we get

THEOREM 4.1. *Let* $\{B_t^\delta\}$ *denote a Bessel process started at* 0. *Then there exists a unique* (*up to finite-dimensional distributions*) $\circ$*-scattered random measure* $B(A), A \in \mathcal{B}_b$, *with the Lebesgue measure as its control measure such that for each* $t \geqslant s \geqslant 0$ *we have*

$$B([0, t]) = B_s^\delta \oplus B\big((s, t]\big). \tag{4.5}$$

Moreover, the control measure associated with B *is the Lebesgue measure.*

We proceed the proof of the theorem by showing the following lemma.

LEMMA 4.1. *Let* $\pi := \{0 = t_0 < t_1 < t_2 < \ldots\}$ *be a subdivision of* $\mathbb{R}^+$. *Then there exist independent r.v.'s* $X_1, X_2, \ldots$ *such that*

$$X_k \stackrel{d}{=} \sigma_s^{t_k - t_{k-1}}, \quad k = 0, 1, 2, \ldots \tag{4.6}$$

Moreover, *we have*

$$B_{t_n}^{\delta} \stackrel{d}{=} X_1 \oplus X_2 \oplus \ldots \oplus X_n, \quad n = 2, 3, \ldots, \tag{4.7}$$

and

$$B\big((t_n, t_{(n+r]})\big) \stackrel{d}{=} \sigma_s^{t_{n+r} - t_n}. \tag{4.8}$$

Proof. Following the idea of Kingman ([5], p. 20) let us take as a sample space Ω the Cartesian product of countably many intervals $\mathbb{R}^+$ with countably many intervals $[-1, 1]$. The probability measure is defined on Ω as the product of the distributions $\sigma_s^{t_k - t_{k-1}}$, $k = 1, 2, \ldots$, on each of the first set of $\mathbb{R}^+$ together with the distribution F_s (cf. (1.8)) on each of the second set. If the typical point $\omega \in \Omega$ has components

$$X_1(\omega), X_2(\omega), \ldots; \eta_1(\omega), \eta_2(\omega), \ldots, \tag{4.9}$$

define $S_m(\omega)$ inductively by

$$S_0 = 0, \tag{4.10}$$

$$S_{m+1}(\omega) = \{S_m^2(\omega) + X_{m+1}^2(\omega) + 2\eta_m(\omega) S_m(\omega) X_{m+1}(\omega)\}^{1/2}. \tag{4.11}$$

Thus, we have

$$S_{m+1} = S_m \oplus X_{m+1}, \tag{4.12}$$

which, by virtue of the associativity of $\oplus$, implies that for each $m = 2, 3, \ldots$

$$S_m = X_1 \oplus X_2 \oplus \ldots \oplus X_m. \tag{4.13}$$

Moreover, since $X_k, k = 2, 3, \ldots$, are independent, it follows that

$$S_m \stackrel{d}{=} \sigma^{t_m} \stackrel{d}{=} B(t_m). \tag{4.14}$$

Now, since the operation $\oplus$ is associative (cf. Kingman [5], Theorem 1), one can show that

$$\begin{gathered} S_{m+r} = S_m \oplus S_r^m, \\ S_0^m = 0, \quad S_{r+1}^m X = S_r^m \oplus X_{m+r+1}. \end{gathered} \tag{4.15}$$

Note, by (4.14) and (4.15), $\sigma^{t_{m+r} - t_m} \stackrel{d}{=} S_r^m \stackrel{d}{=} \big(X_m \oplus \ldots \oplus X_{m+r}\big)$, which entails (4.6), (4.7) and (4.8). ∎

P r o o f o f T h e o r e m 4.1. Let $\mathbb{B}_0$ denote the class of finite unions of disjoint finite intervals $(a, b]$, i.e.

$$\bigcup_{j=1}^{k} I_j, I_j = (t_{2j}, t_{2j+1}], \quad j = 0, 1, \ldots, \ k = 1, 2, \ldots \tag{4.16}$$

We put

$$B\Big(\bigcup_{j=1}^{k} I_j\Big) = \bigoplus_{j=1}^{k} B(I_j). \tag{4.17}$$

Using the transfinite induction, Lemma 4.1, and the usual extension method of random interval functions one can get an $\circ$-random measure $B(\cdot)$ on $\mathcal{B}_b$ with the required properties. ∎

DEFINITION 4.2. For every $0 \leqslant a \leqslant b$ the quantity $M\big((a, b]\big)$ is called the *increment-type* of the Bessel processes $BES^{\delta}(0)$.

Moreover, from (3.1) and (4.1) we have

THEOREM 4.2. *Every Bessel process which starts at* 0 *has a modification as a process with stationary and increments-type process.*

Acknowledgements. The author would like to express his sincere thanks to René Schilling and Andrea E. Kyprianou for their fruitful discussions on the subject. Thanks are also due to the referee for very careful readings which brought essential improvement to the paper.

REFERENCES

[1] N. H. Bingham, *Random walks on spheres*, Z. Wahrsch. Verw. Gebiete 22 (1973), pp. 169–172.

[2] J. C. Cox, J. E. Ingersoll Jr. and S. A. Ross, *A theory of the term structure of interest rates*, Econometrica 53 (2) (1985).

[3] K. Itô and H. P. Mckean Jr., *Diffusion Processes and Their Sample Paths*, Springer, Berlin–Heidelberg–New York 1996.

[4] O. Kalenberg, *Random Measures*, 3rd edition, Academic Press, New York 1983.

[5] J. F. C. Kingman, *Random walks with spherical symmetry*, Acta Math. 109 (1963), pp. 11–53.

[6] B. M. Levitan, *Generalized Translation Operators and some of Their Applications*, Israel Program for Scientific Translations, Jerusalem 1962.

[7] V. T. Nguyen, *Generalized independent increments processes*, Nagoya Math. J. 133 (1994), pp. 155–175.

[8] V. T. Nguyen, *Generalized translation operators and Markov processes*, Demonstratio Math. 34 (2) (2001), pp. 295–304.

[9] V. T. Nguyen, *Double-indexes Bessel diffusions*, in: *Proceedings of the International Symposium on "Abstract and Applied Analysis"*, World Scientific 2004, pp. 563–567.

[10] V. T. Nguyen, S. Ogawa and M. Yamazato, *A convolution approach to mutivariate Bessel processes*, in: *Proceedings of the 6th Ritsumeikan International Symposium on

"Stochastic Processes and Applications to Mathematical Finance", J. Akahori, S. Ogawa and S. Watanabe (Eds.), World Scientific 2006, pp. 233–244.

[11] B. S. Rajput and J. Rosiński, *Spectral representation of infinitely divisible processes*, Probab. Theory Related Fields 82 (1989), pp. 451–487.

[12] D. Revuz and M. Yor, *Continuous Martingals and Brownian Motion*, Springer, Berlin–Heidelberg 1991.

[13] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, 1999.

[14] T. Shiga and S. Watanabe, *Bessel diffusions as a one-parameter family of diffusion processes*, Z. Warsch. Verw. Gebiete 27 (1973), pp. 34–46.

[15] K. Urbanik, *Generalized convolutions*, Studia Math. 23 (1964), pp. 217–245.

[16] K. Urbanik, *Cramér property of generalized convolutions*, Bull. Polish Acad. Sci. Math. 37 (16) (1989), pp. 213–218.

[17] V. E. Vólkovich, *On symmetric stochastic convolutions*, J. Theoret. Probab. 5 (3) (1992), pp. 417–430.

[18] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge 1944.

[19] M. Yor, *Some Aspects of Brownian Motion*, Part I: *Some Special Functionals*, Lecture Notes in Math., ETH Zurich, Birkhäuser Verlag, Basel 1992.

Department of Mathematics
International University, HCMC
Quarter Nr. 6, LinhTrung Ward, Thu Duc Distr.
HoChiMinh City, Vietnam
E-mail: nvthu@hcmiu.edu.vn

Received on 27.8.2006;
revised version on 16.7.2008