CHARLIER AND EDGEWORTH EXPANSIONS FOR DISTRIBUTIONS AND DENSITIES IN TERMS OF BELL POLYNOMIALS

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Abstract. We show that the coefficients of the Charlier differential series for distributions and densities are simply Bell polynomials in the cumulants. The same is true for the Edgeworth expansions of distributions and densities of sample means. We use this to obtain higher order extensions of these well-known series.


Key words and phrases: Bell polynomials; Charlier series; Cornish and Fisher.

1. INTRODUCTION AND SUMMARY

One of the great achievements in the theory of statistics have been the expansions for quantiles of the distribution of an approximately normal estimate given by Cornish and Fisher [4] and later extended by Fisher and Cornish [5]. They built on the expansions of Edgeworth for the distribution and density of a sample mean. These are derived from the ‘Type A’ differential series of Charlier [2]. (A more accessible reference is Stuart and Ord [7].) Later Hill and Davis [6] extended the results of Cornish and Fisher to estimates with non-normal limits. Other extensions were given by Withers [8].

None of these authors realised the central role that Bell polynomials play in these expansions. In Section 2 we show that the coefficients of the Charlier differential series are Bell polynomials in the cumulants. We give these explicitly up to order 14. In Section 3 we show that the coefficients of the Edgeworth expansions for the distribution of a standardised sample mean are also Bell polynomials. We give these up to order 16. The reformulation of these expansions in terms of Bell polynomials does not appear to have been noted by others (at least to the best knowledge of these authors).
Suppose that

\[ S(t) = \sum_{r=1}^{\infty} k_r t^r / r! \]

converges in a neighbourhood of \( t = 0 \) for some sequence of complex numbers \( k = (k_1, k_2, \ldots) \). Then

\[ S(t)^k / k! = \sum_{r=k}^{\infty} B_{rk}(k) t^r / r! \]

for \( k = 0, 1, \ldots \), where, by definition, \( B_{rk}(k) \) is the partial exponential Bell polynomial. So, \( B_{r0}(k) = \delta_{r0} \), \( B_{r1}(k) = k_r \) and \( B_{rr}(k) = k_r^r \). Note \( B_{rk}(k) \) are tabulated on pages 307–308 of Comtet [3] for \( 1 \leq k \leq r \leq 12 \). Recurrence formulas for them are given on page 136. Note also that \( B_{rk}(k) \) are a linear combination of terms of the form \( k_1^{n_1} k_2^{n_2} \ldots \), where \( \sum_j n_j = k \), \( \sum_j j n_j = r \). So,

\[ \kappa_r \equiv \alpha_r \delta^{a+b} \Rightarrow B_{rk}(k) = B_{rk}(\alpha) \delta^{ak+br}. \]

The complete exponential Bell polynomial \( B_r(k) \) is defined by

\[ B_r(k) = \sum_{k=0}^{r} B_{rk}(k) \]

for \( r \geq 0 \). Consequently,

\[ \exp \left( S(t) \right) = \sum_{r=0}^{\infty} t^r B_r(k) / r!. \]


2. THE CHARLIER DIFFERENTIAL SERIES

Let \( X \) be a real absolutely continuous random variable with distribution \( F \), density \( f \), and finite moments and cumulants \( m_r = E X^r, \kappa_r = \kappa_r(X) \).

Let \( N \) be a standard normal random variable with density \( \phi(x) = (2\pi)^{-1/2} \exp(-x^2/2) \).

Let \( H_k = H_k(x) \) be the \( k \)th Hermite polynomial defined by

\[ H_k(x) = \phi(x)^{-1} (-d/dx)^k \phi(x) = E(x + iN)^k \]
for $i = \sqrt{-1}$. See Withers [9]. So, $\{H_k/\sqrt{k!}\}$ form a complete orthonormal set of real functions on $R$ with respect to $\phi(x)$:

(2.1) \[ \int H_j H_k \phi = k! \delta_{jk}, \]

where $\int g = \int g(x) dx$ and $\delta_{jk}$ is the Kronecker delta function, equal to 1 or 0 for $j = k$ or not.

Suppose that

(2.2) \[ \int f^2/\phi < \infty. \]

Then $f/\phi$ lies in $L_2(\phi)$ and has the Fourier expansion

(2.3) \[ f(x)/\phi(x) = \sum_{r=0}^{\infty} B_r H_r(x)/r!, \]

where

(2.4) \[ B_r = \int H_r f = E H_r(X) = E(X + iN)^r = \sum_{0 \leq j \leq r/2} \binom{r}{2j} m_{r-2j} \nu_{2j}, \]

where

\[ \nu_{2j} = E N^{2j} = 1 \cdot 3 \ldots (2j - 1) = (2j)!/(2^j j!), \]

and $N$ is independent of $X$.

Note that (2.3) holds in the sense of convergence in $L_2(\phi)$:

\[ \int \left[ f(x)/\phi(x) - \sum_{r=0}^{K} B_r H_r(x)/r! \right]^2 \phi(x) dx \to 0 \]

as $K \to \infty$. Observe that (2.4) gives the Fourier coefficient $B_k$ in terms of the moments. For example, since $H_0 = 1, H_1 = x, H_2 = x^2 - 1, H_3 = x^3 - 3x, H_4 = x^4 - 6x^2 + 3, H_5 = x^5 - 10x^3 + 15x, \ldots$, we have $B_0 = 1, B_1 = m_1, B_2 = m_2 - 1, B_3 = m_3 - 3m_1, B_4 = m_4 - 6m_2 + 3, B_5 = m_5 - 10m_3 + 15m_1, B_6 = m_6 - 15m_4 + 45m_2 - 15, \ldots$ Section 6.31 of Stuart and Ord [7] gives these up to $B_8$ for the case $m_1 = 0$.

Note that integrals like $\int f^2/\phi$ and $\int f \ln \phi$ are only meaningful if $X$ is dimension-free. Suppose in fact that $X$ is standardised so that $E X = 0$ and $\text{var}(X) = 1$, that is, $m_1 = 0$ and $m_2 = 1$. Then $B_0 = 1, B_1 = B_2 = 0$ and (2.3) can be written as

\[ f(x)/\phi(x) - 1 = \sum_{r=3}^{\infty} B_r H_r(x)/r!. \]
Note that (2.3) is known as the Gram–Charlier series. The expressions for the $B_k$ look simpler if we convert from moments to cumulants: $B_3 = \kappa_3$, $B_4 = \kappa_4$, $B_5 = \kappa_5$, $B_6 = \kappa_6 + 10\kappa_3^2$, $B_7 = \kappa_7 + 35\kappa_3\kappa_4$, $B_8 = \kappa_8 + 56\kappa_3\kappa_5 + 35\kappa_3^2$, as noted in (6.41) of Stuart and Ord [7]. However, this conversion becomes laborious. An alternative derivation that avoids this labour is to use the fact that, for $D = d/dx$, the operator $\exp\left(h(-D)^r/r!\right)$ acting on a density $f$ increases its $r$th cumulant by $h$ but does not change the other cumulants. (This result goes back to Edgeworth. For $r = 1$ this gives Taylor’s expansion. We assume that derivatives of all orders exist.) So, if

$$m_1 = 0, \quad m_2 = 1, \quad \kappa_r = \kappa_r - \delta r_2, \quad S(t) = \sum_{r=1}^{\infty} k_r t^r / r!,$$

then $f = \exp\left(S(-D)\right)\phi$. Consequently, by (1.3), we have the simple formula

$$f(x) = \phi(x) \sum_{r=0}^{\infty} B_r H_r(x) / r!,$$

where $B_r = B_r(k) = B_r(\kappa)_{\kappa_1 = \kappa_2 = 0}$, that is, (2.3) with $B_r = B_r(k)$. Using Comtet’s table, (1.2) immediately gives $B_k$ to $k = 12$:

- $B_0 = \kappa_9 + 84\kappa_3\kappa_6 + 126\kappa_4\kappa_5 + 280\kappa_3^2$,
- $B_{10} = \kappa_{10} + 120\kappa_3\kappa_7 + 210\kappa_4\kappa_6 + 126\kappa_5^2 + 2100\kappa_3^2\kappa_4$,
- $B_{11} = \kappa_{11} + 16\kappa_3\kappa_8 + 330\kappa_4\kappa_7 + 462\kappa_5\kappa_6 + 4620\kappa_3^2\kappa_5 + 5775\kappa_3\kappa_4^2$,
- $B_{12} = \kappa_{12} + 220\kappa_3\kappa_9 + 495\kappa_4\kappa_8 + 792\kappa_5\kappa_7 + 462\kappa_6^2 + 9240\kappa_3^2\kappa_6 + 27720\kappa_3\kappa_4\kappa_5 + 5775\kappa_3^3 + 15400\kappa_3^2$.

Since $k_1 = k_2 = 0$, about two thirds of the terms in (1.2) are zero. An alternative is to calculate $B_k$ as follows:

$$B_r = \sum_{1 \leq k \leq r/3} [r]_{2k} B_{r-2k,k}(\eta)$$

for $r \geq 1$, where $\eta_{r-2} = \kappa_r / r(r - 1)$ and $[r]_{2k} = r! / (r - 2k)!$. Observe that (2.6) follows by noting that $S(t) = t^2 S_1(t)$, where $S_1(t) = \sum_{j=1}^{\infty} \eta_j t^j / j!$. So, $\exp\left(S(t)\right) = \sum_{k=0}^{\infty} t^{2k} S_1(t)^k / k!$. Substitute $S_1(t)^k / k! = \sum_{r=k}^{\infty} B_{rk}(\eta) t^r / r!$ and take the coefficient of $t^r / r!$ to obtain

$$B_{rk}(\eta) / r! = B_{r-2k,k}(\eta) / (r - 2k)!.$$

Note that (2.6) now follows from (1.2). For example, Comtet’s table gives $B_{rk}$ for $r \leq 12$, and so $B_{13} = \kappa_{13} + 13\alpha$ at

$$\alpha = 22\kappa_3\kappa_{10} + 55\kappa_4\kappa_9 + 99\kappa_5\kappa_8 + 132\kappa_6\kappa_7 + 1320\kappa_3^2\kappa_7 + 4620\kappa_3\kappa_4\kappa_6 + 44352\kappa_3\kappa_8 + 3465\kappa_3^2\kappa_5 + 15400\kappa_3^2\kappa_4.$$
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and $B_{14} = \kappa_{14} + 13/3$ at

$$
\beta = 77\kappa_4\kappa_{10} + 154\kappa_5\kappa_9 + 231\kappa_6\kappa_8 + 132\kappa^2 + 14(2\kappa_3\kappa_{11} + 165\kappa_2\kappa_8
+ 660\kappa_4\kappa_7 + 924\kappa_3\kappa_5\kappa_6 + 693\kappa_4\kappa_5^2 + 3080\kappa_3\kappa_5\kappa_8
+ 5775\kappa_2\kappa_6^2)
+ 8085\kappa_4^2\kappa_6.
$$

Charlier and Cramer showed that the Gram–Charlier series converges absolutely and uniformly under stronger conditions than the $L_2$ condition (2.2). For example, Cramer showed that this holds if

$$
\int f^2 / \phi < \infty, \quad f(x) \to \infty \quad \text{as} \quad |x| \to \infty.
$$

Section 6.22 of Stuart and Ord [7] gives this and another theorem of Cramer on its convergence. These theorems do not apply, for example, to the double exponential density as this does not satisfy the $L_2$ condition (2.2).

By (2.1), we have a form of Parseval’s identity:

$$
\int f^2 / \phi = \sum_{k=0}^{\infty} B^2_k / k!.
$$

(2.8)

The integrated form of (2.3) is:

$$
P(X \leq x) = \Phi(x) - \phi(x) \sum_{k=1}^{\infty} B_k H_{k-1}(x) / k!.
$$

(2.9)

3. THE EDGEWORTH EXPANSION

Suppose that $X = X_n$ is a standardised sample mean of sample of size $n$ from a population with $r$th cumulant $l_r$, say $X_n = (n/l_2)^{1/2}(Y - l_1)$. For $r \geq 2$, $X_n$ has $r$th cumulant $\kappa_r = \alpha_r n^{1-r/2} = \alpha_r \epsilon^{r-2}$, where $\epsilon = n^{-1/2}$ and $\alpha_r = l_r / l_2^{r/2}$; $\kappa_3 = \alpha_3 \epsilon$, $\kappa_4 = \alpha_4 \epsilon^2$, $\kappa_5 = \alpha_5 \epsilon^3$, $\kappa_6 = \alpha_6 \epsilon^4$, $\kappa_7 = \alpha_7 \epsilon^5$, $\kappa_8 = \alpha_8 \epsilon^6$, $\kappa_9 = \alpha_9 \epsilon^7$, $\kappa_{10} = \alpha_{10} \epsilon^8$, . . . By (1.1), $B_r(k) = \epsilon^{r-2k} B_r$, where $B_r = B_r(\alpha)$. So, by (1.2), for $r \geq 3$ we have

$$
B_r = \sum_{k=1}^{r} B_r \epsilon^{r-2k}
= \sum_{m=K_r}^{r-2} \{B_r \epsilon^m : k = (r - m)/2, \ r - m \ \text{even}\},
$$

(3.1)

where

$$
K_{3j} = j, \ K_{3j+1} = j + 1, \ K_{3j+2} = j + 2.
$$

(3.2)
In particular,
\[B_3 = B_{31} \epsilon, \quad B_4 = B_{41} \epsilon^2, \quad B_5 = B_{51} \epsilon^3,\]
\[B_6 = B_{62} \epsilon^2 + B_{61} \epsilon^4,\]
\[B_7 = B_{72} \epsilon^3 + B_{71} \epsilon^5,\]
\[B_8 = B_{82} \epsilon^4 + B_{81} \epsilon^6,\]
\[B_9 = B_{93} \epsilon^3 + B_{92} \epsilon^5 + B_{91} \epsilon^7,\]
\[B_{10} = B_{10,3} \epsilon^4 + B_{10,2} \epsilon^6 + B_{10,1} \epsilon^8,\]
\[B_{11} = B_{11,3} \epsilon^5 + B_{11,2} \epsilon^7 + B_{11,1} \epsilon^9,\]
\[B_{12} = B_{12,4} \epsilon^4 + B_{12,3} \epsilon^6 + B_{12,2} \epsilon^8 + B_{12,1} \epsilon^{10},\]
\[B_{13} = B_{13,4} \epsilon^5 + B_{13,3} \epsilon^7 + B_{13,2} \epsilon^9 + B_{13,1} \epsilon^{11},\]
\[B_{14} = B_{14,4} \epsilon^6 + B_{14,3} \epsilon^8 + B_{14,2} \epsilon^{10} + B_{14,1} \epsilon^{12}.\]

From Comtet's table and (2.7) we have immediately
\[B_{r1} = \alpha_r,\]
\[B_{r2} = 10 \alpha_r^3,\]
\[B_{72} = 35 \alpha_4 \alpha_3,\]
\[B_{82} = 56 \alpha_5 \alpha_3 + 35 \alpha_4^2,\]
\[B_{93} = 280 \alpha_3^3, \quad B_{92} = 84 \alpha_6 \alpha_3 + 126 \alpha_5 \alpha_4,\]
\[B_{10,3} = 2100 \alpha_4 \alpha_3^2, \quad B_{10,2} = 120 \alpha_7 \alpha_3 + 210 \alpha_6 \alpha_4 + 126 \alpha_5^2,\]
\[B_{11,3} = 4620 \alpha_5 \alpha_5 + 5775 \alpha_3 \alpha_4^2,\]
\[B_{11,2} = 165 \alpha_3 \alpha_8 + 330 \alpha_4 \alpha_7 + 462 \alpha_5 \alpha_6,\]
\[B_{12,4} = 15400 \alpha_3^3, \quad B_{12,3} = 9240 \alpha_3^2 \alpha_6 + 27720 \alpha_3 \alpha_4 \alpha_5 + 5775 \alpha_4^3,\]
\[B_{12,2} = 220 \alpha_3 \alpha_9 + 495 \alpha_4 \alpha_8 + 792 \alpha_5 \alpha_7 + 462 \alpha_6^2,\]
\[B_{13,4} = 13 \cdot 15400 \alpha_3^3,\]
\[B_{13,3} = 13(1320 \alpha_3^2 \alpha_7 + 4620 \alpha_3 \alpha_4 \alpha_6 + 44352 \alpha_3 \alpha_5 \alpha_4^2 + 3465 \alpha_3 \alpha_4 \alpha_5),\]
\[B_{13,2} = 13(22 \alpha_3 \alpha_10 + 55 \alpha_4 \alpha_9 + 99 \alpha_5 \alpha_8 + 132 \alpha_6 \alpha_7),\]
\[B_{14,4} = 13 \cdot 14(3080 \alpha_3 \alpha_5 + 5775 \alpha_3 \alpha_4^2),\]
\[B_{14,3} = 13[14(165 \alpha_3^2 \alpha_8 + 660 \alpha_3 \alpha_4 \alpha_7 + 924 \alpha_3 \alpha_5 \alpha_6 + 693 \alpha_4 \alpha_5^2) + 8085 \alpha_3 \alpha_6^2],\]
\[B_{14,2} = 13(77 \alpha_4 \alpha_10 + 154 \alpha_5 \alpha_9 + 231 \alpha_6 \alpha_8 + 132 \alpha_7^2 + 28 \alpha_3 \alpha_11).\]

Substituting (3.1) into (2.5) and (2.9) gives the Edgeworth expansions
\[f(x)/\phi(x) = 1 + \sum_{j=1}^{\infty} h_{1j}(x)n^{-j/2}\]
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and

\[(3.4)\]

\[F(x) = \Phi(x) - \phi(x) \sum_{j=1}^{\infty} h_{0j}(x)n^{-j/2},\]

where

\[h_{1j}(x) = \sum_{k=1}^{j} \left\{ B_{jk} H_{r}(x)/r! : r = j + 2k \right\}\]

and

\[h_{0j}(x) = \sum_{k=1}^{j} \left\{ B_{rk} H_{r-1}(x)/r! : r = j + 2k \right\} .\]

In particular,

\[h_{11} = B_{31} H_{3}/3!,\]
\[h_{12} = B_{41} H_{4}/4! + B_{62} H_{6}/6!,\]
\[h_{13} = B_{51} H_{5}/5! + B_{72} H_{7}/7! + B_{93} H_{9}/9!,\]
\[h_{14} = B_{61} H_{6}/6! + B_{82} H_{8}/8! + B_{10,3} H_{10}/10! + B_{12,4} H_{12}/12!,\]
\[h_{15} = B_{71} H_{7}/7! + B_{92} H_{9}/9! + B_{11,3} H_{11}/11! + B_{13,4} H_{13}/13! + B_{15,5} H_{15}/15!,\]

and so on. Note that \(h_{0j}\) is just \(h_{1j}\) with \(H_{r}\) replaced by \(H_{r-1}\). Differentiating \(f\) \(p\) times gives

\[(-1)^p f^{(p)}(x)/\phi(x) = H_{p}(x) + \sum_{j=1}^{\infty} h_{p+1,j}(x)n^{-j/2},\]

where

\[h_{p+1,j}(x) = \sum_{k=1}^{j} \left\{ B_{rk} H_{r+p}(x)/r! : r = j + 2k \right\} .\]

That is, \(h_{p+1,j}\) is just \(h_{1j}\) with \(H_{r}\) replaced by \(H_{r+p}\).

For some exact conditions for the expansions (3.3) and (3.4), see Theorem 19.2 and Corollary 20.4 of Bhattacharya and Rao [1]. For adjustments to (3.4) for lattice random variables see their Theorem 23.1. Similarly, the Parseval identity (2.8) can be rewritten as

\[\int f^{2}/\phi = \sum_{r=0}^{\infty} b_r n^{-r},\]

where

\[b_0 = 1, \quad b_1 = B_{31}^2/3! = \alpha_3^2/6,\]
\[b_2 = B_{41}^2/4! + B_{62}^2/6! = \alpha_4^2/24 + 5\alpha_3^4/36,\]
\[b_3 = B_{51}^2/5! + 2B_{62} B_{64}/6! + B_{72}^2/7! + B_{93}^2/9! = \alpha_5^2/120 + \alpha_3^2\alpha_6/36 + 35\alpha_3^4\alpha_4^2/144 + 35\alpha_3^6/162,\]
Suppose that
\[ b_4 = B_{61}^2/6! + 2B_{71}B_{71}/7! + B_{82}^2/8! + 2B_{93}B_{92}/9! + B_{10,3}^2/10! \]
\[ + B_{12,4}^2/12!, \]
\[ b_5 = B_{71}^2/7! + 2B_{82}B_{81}/8! + (2B_{93}B_{91} + B_{93}^2)/9! + 2B_{10,3}B_{10,2}/10! \]
\[ + B_{11,3}^2/11! + 2B_{12,4}B_{12,3}/12! + B_{13,4}^2/13! + B_{15,5}^2/15!. \]

Note that \( h_{p5} \), \( h_{b5} \) also need \( B_{15,5} \) while \( h_{p6} \), \( b_6 \) need \( B_{16,5} \) and \( B_{18,6} \). These are given by

\[ (3.5) \quad B_{3j,j} = (\alpha_3/6)^j(3j)!/j!, \]
\[ B_{3j+1,j} = (\alpha_3/6)^j-1(\alpha_4/24)(3j+1)!/(j-1)!, \]
\[ B_{3j+2,j} = [(\alpha_3/6)^j-1(\alpha_5/120) + (j-1)(\alpha_3/6)^j-2(\alpha_4/12)^2/8] \]
\[ \times (3j+2)!/(j-1)!. \]

To prove this, set \( \lambda_{r-2} = \alpha_r/r(r-1) \). By (2.7), we have

\[ B_{3j+m,j}/(3j+m)! = B_{j+m,j}(\lambda)/(j+m)! \quad \text{for} \ m = 0, 1, 2. \]

Now use

\[ B_{j,j}(\lambda) = \lambda_1^j, \]
\[ B_{j+1,j}(\lambda) = j(j+1)\lambda_1^{j-1}\lambda_2/2, \]
\[ B_{j+2,j}(\lambda) = j(j+1)(j+2)[\lambda_1^{j-1}\lambda_3/6 + (j-1)\lambda_1^{j-2}\lambda_2^2/8]. \]

So, we obtain (3.5). (A modification of this argument gives \( B_r \sim e^{K_r} \).

Consequently, we have explicit formulas for \( h_{pj} \) for \( j \leq 16 \). These expressions appear to be new, as well as our expressions for \( b_j \). (But \( h_{p17} \) needs the equality \( B_{15,14}/45! = B_{17,14}(\lambda)/17! \).

**Example 3.1.** Suppose that \( X_0 \) is a gamma random variable with mean \( \gamma \). Its \( r \)th cumulant is \((r-1)!\gamma \). Its standardised form \( X = (X_0 - \gamma)/\sqrt{\gamma} \) has \( r \)th cumulant \((r-1)!\gamma^{1-r/2}I(r \geq 2) \). Set \( s = t/\sqrt{\gamma} \). Then

\[ k_r = (r-1)!\gamma^{1-r/2}I(r \geq 3), \]
\[ S(t) = \gamma \sum_{r=3}^{\infty} s^r/r = -\gamma[\ln(1-s) + s + s^2/2], \]
\[ \exp(S(t)) = \exp\left(-\gamma(s + s^2/2)\right)(1-s)^{-\gamma}, \]
\[ B_r = \gamma^{-r/2}r! \sum_{a+2b+c=r} (-\gamma)^a(-\gamma/2)^b[\gamma]_c/\text{abb!}. \]
where \([\gamma]_c = \gamma(\gamma + 1) \ldots (\gamma + c - 1)\). However, it is simpler just to apply the previous results of this section with \(n = \gamma\) and \(l_r\) the \(r\)th cumulant of an exponential distribution with mean 1, that is, \(l_r = \alpha_r = (r - 1)!\). Substituting this we see that \(B_r\) is given by (3.1) with

\[
\begin{align*}
B_{1r} &= (r - 1)!
\end{align*}
\]
\[
B_{62} = 40,
\]
\[
B_{72} = 420,
\]
\[
B_{82} = 3948,
\]
\[
B_{93} = 2240, \quad B_{92} = 38304,
\]
\[
B_{10,3} = 50400, \quad B_{10,2} = 2065824,
\]
\[
B_{11,3} = 859320, \quad B_{11,2} = 4419360,
\]
\[
B_{12,4} = 246400, \quad B_{12,3} = 13665960, \quad B_{12,2} = 53048160,
\]
\[
B_{13,4} = 9609600, \quad B_{13,3} = 839041632, \quad B_{13,2} = 684478080,
\]
\[
B_{14,4} = 258978720, \quad B_{14,3} = 3060393336, \quad B_{14,2} = 9464307840.
\]

Note that \(h_{p5}, b_5, h_{p6}, b_6\) also need

\[
\begin{align*}
B_{15,5} &= 44844800, \quad B_{16,5} = 2690688000 \quad \text{and} \quad B_{18,6} = 12197785600.
\end{align*}
\]

Observe that (3.2) explains the behaviour of \(B_k\) noted in Example 6.3, page 229 of Stuart and Ord [7].

In this section, we have focused on the case of the sample mean. The results could be extended for:

(a) smooth functions of a vector of sample means;

(b) sample quantiles. The classes of statistics, (a) and (b), have been well studied in the recent literature on Edgeworth expansions. Bell polynomials might also be useful in simplifying the explicit computation of higher order Edgeworth expansions for these more general classes of statistics. We hope to address this issue in a future work.

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