\textbf{Abstract.} The probability measures on separable Banach space \(X\) which are stable under the group \(\mathcal{U}(X)\) of all invertible linear operators on \(X\) are investigated. The characterization of such measures on finite-dimensional spaces was obtained by Parthasarathy in [6]. In this paper some generalizations of Parthasarathy's result are given.

1. Parthasarathy proved in [6] (see also [3] and [7]) that the probability measures on the at least two-dimensional Euclidean space \(\mathbb{R}^n\) which are stable under the group \(\mathcal{U}(\mathbb{R}^n)\) of all invertible linear transformations of \(\mathbb{R}^n\) onto itself are precisely the full Gaussian and the degenerate probability measures. Our aim is to discuss \(\mathcal{U}(X)\)-stable distributions on a real separable Banach space \(X\). The main results of this paper are the following: For a Hilbert space \(X\) it is shown that a non-degenerate Gaussian measure is \(\mathcal{U}(X)\)-stable if and only if it is full and completely stable in the sense of paper [4]. Hence we obtain a characterization of \(\mathcal{U}(X)\)-stable Gaussian measures on Hilbert spaces in terms of proper values of their covariance operators. Further, it is shown that a non-degenerate \(\mathcal{U}(X)\)-stable probability measure on \(X\) (dim \(X \geq 2\)) is a full Gaussian measure if and only if there exists a non-trivial finite-dimensional projector in its decomposability semigroup. As an application of this we obtain a Parthasarathy-type theorem for probability distributions on \(X\), which are stable under \(\mathcal{U}(X)\) in some "enough regular" way (for so-called strongly \(\mathcal{U}(X)\)-stable distributions). The question whether in the infinite-dimensional case there are non-Gaussian \(\mathcal{U}(X)\)-stable probability measures is still open.

Further in this section we shall introduce basic notions and facts.

Let \(\|\cdot\|\) be the norm of a real separable Banach space \(X\) and let \(X^*\) be the topological dual of \(X\).

For a set \(A \subset X\) its norm closure is denoted by \(\overline{A}\).

By \(\mathcal{P}(X)\) we denote the set of all probability measures defined on the class of Borel subsets of \(X\). The set \(\mathcal{P}(X)\) with the topology of weak convergence
and multiplication defined by the convolution becomes a topological semigroup. We denote the convolution of two measures \( \lambda \) and \( \mu \) by \( \lambda \ast \mu \), and by \( \mu^{**} \) the \( n \)-th power in the sense of convolution. If a sequence \( \{ \mu_n \} \) of probability measures on \( X \) converges weakly to a \( \mu \in \mathcal{P}(X) \), we write \( \mu_n \rightharpoonup \mu \). Moreover, by \( \delta_x \) (\( x \in X \)) we denote the probability measure concentrated at the point \( x \).

The characteristic functional of \( \mu \) is defined on \( X^* \) by the formula
\[
\hat{\mu}(x^*) = \int_X e^{i x^*(x)} \, d\mu(x) \quad (x^* \in X^*).
\]

Given \( \mu \in \mathcal{P}(X) \), we define \( \overline{\mu} \) by
\[
\overline{\mu}(E) = \mu(-E), \quad \text{where} \quad -E = \{ -x : x \in E \}.
\]

The mapping \( \mu \to \overline{\mu} \) is a continuous automorphism of \( \mathcal{P}(X) \). For any \( \mu \in \mathcal{P}(X) \), \( \mu^e = \mu \ast \overline{\mu} \) is called the symmetrization of \( \mu \).

A probability measure \( \mu \) is said to be symmetric if \( \mu = \overline{\mu} \).

A measure \( \mu \) from \( \mathcal{P}(X) \) is said to be full if its support is not contained in any proper hyperplane of \( X \). It is clear that \( \mu \) is full if and only if \( \mu^e \) is full.

Further, \( \mathcal{B}(X) \) will denote the algebra of continuous linear operators on \( X \) with the norm topology. By \( \mathcal{U}(X) \) we shall denote the group of all invertible operators from \( X \) onto \( X \).

The image and the kernel of \( A \in \mathcal{B}(X) \) will be denoted by \( \text{Im} A \) and \( \text{Ker} A \), respectively. An operator \( A \) is said to be \( n \)-dimensional if \( \dim \text{Im} A = n \) (\( n = 1, 2, \ldots \)). By a projector we mean an operator \( P \) from \( \mathcal{B}(X) \) with the property \( P^2 = P \). The zero operator will be denoted by 0, and the unit operator by \( I \).

For any \( A \in \mathcal{B}(X) \) and \( \mu \in \mathcal{P}(X) \) let \( A\mu \) denote the measure defined by the formula \( A\mu(E) = \mu(A^{-1}(E)) \) for all Borel subsets \( E \) of \( X \). For all \( A \in \mathcal{B}(X) \) and \( \mu, \nu \in \mathcal{P}(X) \) we can easily check the equations
\[
A(\mu \ast \nu) = A\mu \ast A\nu, \quad (A\mu)^\wedge(x^*) = \hat{\mu}(A^*x^*),
\]
where \( A^* \) denotes the adjoint operator. Moreover, it is clear that the mapping \( \langle A, \mu \rangle \to A\mu \) from \( \mathcal{B}(X) \times \mathcal{P}(X) \) onto \( \mathcal{P}(X) \) is jointly sequentially continuous even if \( \mathcal{B}(X) \) is provided with a strong operator topology. Consequently, if a sequence \( \{ A_n \} \) of linear operators is sequentially strongly compact, then for every \( \mu \in \mathcal{P}(X) \) the sequence \( \{ A_n\mu \} \) is compact in \( \mathcal{P}(X) \).

For full measures on finite-dimensional spaces the converse implication is also true. Namely, if the sequence \( \{ A_n\mu \} \) is compact in \( \mathcal{P}(X) \), where \( \mu \) is full and \( A_n \in \mathcal{B}(X) \) (\( n = 1, 2, \ldots \)), then the sequence \( \{ A_n \} \) is compact in \( \mathcal{B}(X) \) (see [11], p. 120).

In the study of limit probability distributions Urbanik [11] introduced the concept of decomposability semigroups \( \mathcal{D}(\mu) \) of linear operators associated
with the probability measure $\mu$. Namely, $\mathcal{D}(\mu)$ consists of all operators $A$ from $\mathcal{B}(X)$ for which $\mu = A\mu \ast v$ holds for a certain probability measure $v$. It is clear that $\mathcal{D}(\mu)$ is a semigroup under multiplication of operators and $\mathcal{D}(\mu)$ always contains the operators 0 and $I$. Moreover, $\mathcal{D}(\mu)$ is closed in $\mathcal{B}(X)$. It has been shown that some probabilistic properties of measures correspond to algebraic and topological properties of their decomposability semigroups (see, e.g., [12]).

The Tortrat representation of infinitely divisible laws on Banach spaces is an important step in our considerations. We recall that for any bounded non-negative Borel measure $F$ on $X$ vanishing at 0 the Poisson measure $e(F)$ associated with $F$ is defined as

$$e(F) = e^{-F(X)} \sum_{n=0}^{\infty} \frac{1}{n!} F^* F^* \cdots F^*,$$

where $F^* F^* \cdots F^*$ is a matrix of size $n$. Let $M$ be a not necessarily bounded Borel measure vanishing at 0. If there exists a representation

$$M = \sup_n F_n,$$

where $F_n$ are bounded and the sequence $\{e(F_n)\}$ of associated Poisson measures is shift compact, then each cluster point of $\{e(F_n)\}$ will be called a generalized Poisson measure and denoted by $\tilde{e}(M)$. The measure $\tilde{e}(M)$ is uniquely defined up to a shift transformation, i.e. for two cluster points, say $\mu_1$ and $\mu_2$, of translates of $\{e(F_n)\}$ there exists an element $x \in X$ such that $\mu_1 = \mu_2 \ast \delta_x$.

Clearly, $\tilde{e}(M_1) = \tilde{e}(M_2)$ implies $M_1 = M_2$. Moreover, if $\tilde{e}(M)$ is full, then so is $M$.

For each pair $\tilde{e}(M)$, $\tilde{e}(N)$ of generalized Poisson measures and each $A \in \mathcal{B}(X)$ we can easily check the equations

$$\tilde{e}(M) \ast \tilde{e}(N) = \tilde{e}(M + N), \quad A\tilde{e}(M) = \tilde{e}(A\tilde{M}),$$

where $A\tilde{M}(S) = AM(S)$ for all Borel subsets $S$ of $X \setminus \{0\}$.

By a Gaussian measure on $X$ we mean a measure $\varrho$ such that, for every $x^* \in X^*$, the induced measure $x^* \varrho$ on $R$ is Gaussian.

Tortrat proved in [10], p. 311, the following analogue of the Lévy-Khintchine representation: $\mu \in \mathcal{D}(X)$ is infinitely divisible, i.e. for every positive integer $n$ there exists a probability measure $\mu_n$ on $X$ such that $\mu_n^* = \mu$ if and only if $\mu$ has a unique representation $\mu = \varrho \ast \tilde{e}(M)$, where $\varrho$ is a symmetric Gaussian measure on $X$ and $\tilde{e}(M)$ is a generalized Poisson measure.

If $\varrho$ is a Gaussian measure on $X$, then the characteristic functional of $\varrho$ takes the form

$$\hat{\varrho}(x^*) = \exp \{i x^* (x) - \frac{1}{2} x^* (R x^*)\} \quad (x^* \in X^*),$$

where $x$ is the mean value of $\varrho$ and $R$ is its covariance operator, i.e. a nuclear operator from $X^*$ into $X$ with the following properties: $x^* (R x^*) = \mathbb{E} x^* x$.
\[
B. \text{Mincer} = \xi_f (R \xi_f) \quad \text{for all } \xi_f, \xi_f \in X^* \text{ (symmetry) and } \xi^* (R \xi^*) \geq 0 \text{ (non-negativity)} \quad \text{(see [1])}. 
\]
Clearly, if \( R \) is the covariance operator of \( \varrho \) and \( A \in \mathcal{B} (X) \), then \( A R A^* \) is the covariance operator of \( A \varrho \). Moreover, for any Gaussian measures \( \varrho_1 \) and \( \varrho_2 \) with the covariances \( R_1 \) and \( R_2 \), respectively, the covariance of \( \varrho_1 \ast \varrho_2 \) is equal to \( R_1 + R_2 \).

We note that \( \varrho \) is full if and only if its covariance operator \( R \) is one-to-one.

If \( X \) is a Hilbert space, then every Gaussian covariance is a positive definite, nuclear operator on \( X \).

2. Let \( \mu \) be a probability measure on a real separable Banach space \( X \) and let \( \mathcal{A} \) be a subgroup of \( \mathcal{U} (X) \). We say that \( \mu \) is \( \mathcal{A} \)-stable if for any \( A, B \in \mathcal{A} \) there exist a \( C \in \mathcal{A} \) and a point \( x \in X \) such that

\[
A \mu * B \mu = C \mu * \delta_x. 
\]

The notion of stability with respect to arbitrary groups of automorphisms in the case of probability measures on locally compact groups was introduced by Parthasarathy and Schmidt in [7]. Earlier (in [6]) Parthasarathy investigated probability distributions on \( R^n \) which are stable under the group of all invertible linear transformations of \( R^n \) onto itself. The theory of \( \mathcal{A} \)-stable measures on Euclidean spaces has been presented in [8]. In particular, [8] contains the Lévy-Khintchine formula for such measures.

It is easy to see that if \( \mu \) is \( \mathcal{A} \)-stable, then for any finite set \( A_1, \ldots, A_n \) of elements of \( \mathcal{A} \) there exist a \( C_n \in \mathcal{A} \) and a point \( x \in X \) for which

\[
A_1 \mu * \ldots * A_n \mu = C_n \mu * \delta_x. 
\]

Consequently, for each positive integer \( n \) there are \( C_n \in \mathcal{A} \) and \( x \in X \) such that \( \mu^{*n} = C_n \mu * \delta_x \), e.g. \( \mu \) is operator-stable in the sense of Sharpe (see [9]). For full measures on \( R^n \) the converse is also true. Namely, a full probability measure \( \mu \) on \( R^n \) is operator-stable if and only if \( \mu \) is stable under a one-parameter subgroup of \( \mathcal{U} (R^n) \) (see [9], Theorem 2).

PROPOSITION 2.1. Let \( \mathcal{A} \) be a subgroup of \( \mathcal{U} (X) \) and let \( \mu \) be a probability measure on \( X \) which is stable under \( \mathcal{A} \). Then \( \mu \) is infinitely divisible. Let \( \mu = \varrho + \varepsilon (M) \) be the decomposition of \( \mu \) into its symmetric Gaussian part \( \varrho \) and its generalized Poisson part \( \varepsilon (M) \). Then both \( \varrho \) and \( \varepsilon (M) \) are stable under \( \mathcal{A} \).

Proof. The proposition follows immediately from the operator stability of \( \mu \) and from the uniqueness of the Tortrat representation.

We note that a generalized Poisson measure \( \varepsilon (M) \) associated with the measure \( M \) is stable under \( \mathcal{A} \) if and only if for any pair \( A, B \in \mathcal{A} \) there exists a \( C \in \mathcal{A} \) such that

\[
AM + BM = CM. 
\]

(2.1)

Let \( \varrho \) be a Gaussian measure with the covariance operator \( R \). It is clear that \( \varrho \) is \( \mathcal{A} \)-stable if and only if for any \( A, B \in \mathcal{A} \) there is a \( C \in \mathcal{A} \) for which the equality \( A R A^* + B R B^* = C R C^* \) holds.
Let $\mu$ be a probability measure on $X$ which is stable under a subgroup $\mathcal{A}$ of $\mathcal{U}(X)$ and assume that $\lambda$ is a real number from $(-1, 1)$ such that $\lambda I$ belongs to $\mathcal{D}(\mu)$. We infer this from the following

**Proposition 2.2.** Let $\mu$ be an $\mathcal{A}$-stable probability measure on $X$ and assume that $A$ is an operator from $\mathcal{A}$ such that

(i) the sequence $\{A^n\}$ converges strongly to 0;

(ii) $A$ commutes with every element of $\mathcal{A}$.

Then $A$ belongs to $\mathcal{D}(\mu)$.

**Proof.** Since $\mu$ is $\mathcal{A}$-stable and $A \in \mathcal{A}$, there exist a sequence $\{C_n\}$ in $\mathcal{A}$ and a sequence $\{x_n\}$ in $X$ such that

$$A\mu * \delta_{x_1} * \ldots * A^n \mu = C_n \mu * \delta_{x_n}$$

for all $n$.

Hence we get the formula

$$C_n^{-1} A \mu * C_n^{-1} A^2 \mu * \ldots * C_n^{-1} A^n \mu = \mu * \delta_{C_n^{-1} x_n} \quad (n = 1, 2, \ldots) \quad (2.2)$$

Since $A$ commutes with $C_n^{-1}$, it is easy to obtain from (2.2) the following equation:

$$\mu * \delta_{C_n^{-1} x_n} * C_n^{-1} A^{n+1} \mu = C_n^{-1} A \mu * A \mu * \delta_{A C_n^{-1} x_n} \quad (n = 1, 2, \ldots) \quad (2.3)$$

Moreover, from (2.2) and Theorem 5.1, Chapter III in [5], we infer that the sequence $\{C_n^{-1} A \mu\}$ is shift compact and, consequently (by (i) $A^n \to 0$ strongly), there exists a sequence $\{y_n\}$ in $X$ for which

$$\mu * \delta_{y_n} = A^n C_n^{-1} A \mu * \delta_{y_n} \Rightarrow \delta_0.$$

Thus, setting $v_n = C_n^{-1} A^{n+1} \mu * \delta_{y_n}$ and $\mu_n = C_n^{-1} A \mu * \delta_{A C_n^{-1} x_n + y_n - C_n^{-1} x_n}$ ($n = 1, 2, \ldots$), by (2.3) we have the formula $\mu * v_n = A \mu * \mu_n$, where $v_n \Rightarrow \delta_0$. Since in this case the sequence $\{\mu_n\}$ must be compact (see Theorem 2.1, Chapter III in [5]), it is then clear that there exists a $v \in \mathcal{P}(X)$ for which $\mu = A \mu * v$. This completes the proof.

We say that a probability measure $\mu \in \mathcal{P}(X)$ is self-decomposable if the inclusion $\{\lambda I: \lambda \in [0, 1]\} \subset \mathcal{D}(\mu)$ holds (see [2]). Thus, if $\mu$ is $\mathcal{U}(X)$-stable, then $\mu$ is self-decomposable. Moreover, for any $\lambda \in (-1, 0)$ we have also $\lambda I \in \mathcal{D}(\mu)$, and since $\mathcal{D}(\mu)$ is closed, the same is true for $\lambda = -1$. But it is easy to prove that $-I \in \mathcal{D}(\mu)$ if and only if $\mu$ is a translation of a symmetric probability measure. Thus, every $\mathcal{U}(X)$-stable measure is in addition a translation of a symmetric one.

Moreover, the following statement is true:

**Proposition 2.3.** Let $\mu$ be a symmetric $\mathcal{A}$-stable probability measure on $X$. Then for any finite group $\mathcal{G} \subset \mathcal{A}$ there exists a $T \in \mathcal{A}$ such that $(T^{-1} AT) \mu = \mu$ for all $A \in \mathcal{G}$.

For the proof see [6], Lemma 3.
Further, we shall need the following lemma:

**Lemma 2.1.** Let $N$ be an infinite-dimensional subspace of $X$. Then there exists an $A \in \mathcal{H}(X)$ such that the linear manifold $A(N) + N$ is dense in $X$.

**Proof.** Let $\{x_n\}$ be a countable dense subset of $X$. Since $N$ is infinite dimensional, we can find a pair of sequences $\{y_n\}$ in $N$ and $\{y^*_n\}$ in $X^*$ such that $y^*_n(y_m) = \delta_{n,m}$ ($n, m = 1, 2, \ldots$). Choose a sequence $\{a_n\}$ of positive numbers such that
\[
\left| \sum_{n=1}^x a_n y^*_n(x) x_n \right| < \|x\| \quad \text{for all } x \neq 0.
\]

If we define the operator $S$ by
\[
Sx = \sum_{n=1}^x a_n y^*_n(x) x_n \quad (x \in X),
\]
then $\|S\| < 1$ and $Sa_n^{-1} y_n = x_n$ for all $n$. Consequently, $S(N) \supset \{x_n\}$. Put $A = I + S$. Obviously, $A$ is invertible and $A(N) + N \supset \{x_n\}$. Thus the lemma is proved.

An application of Lemma 2.1 leads to the following

**Theorem 2.1.** Let $\mu$ be a non-degenerate $\mathcal{H}(X)$-stable probability measure on a real separable Banach space $X$. Then $\mu$ is full.

**Proof.** Denote by $N$ the smallest closed subspace of $X$ for which there exists an element $x_0$ such that $\mu$ is concentrated on the hyperplane $N + x_0$. Since $\mu$ is $\mathcal{H}(X)$-stable, for any $A \in \mathcal{H}(X)$ there exists an operator $B \in \mathcal{H}(X)$ such that the formula
\[
A(N) + N = B(N) = B(N)
\]
holds. Under our assumption $N \neq \{0\}$. We shall prove that $N = X$. Indeed, if $N \neq X$, we can choose an operator $A$ from $\mathcal{H}(X)$ in such a way that $A(N) \neq N$. Then from (2.4) it follows that $N$ (and $X$) must be infinite dimensional. But in this case $A(N) + N$ is dense in $X$ for some $A \in \mathcal{H}(X)$ (Lemma 2.1). Hence and from (2.4) we infer that there exists an operator $B \in \mathcal{H}(X)$ such that $B(N) = X$, which contradicts the assumption $N \neq X$. Thus the theorem is proved.

A probability measure $\mu$ on $X$ is said to be completely stable if for any pair $A, B \in \mathcal{A}(X)$ there exist $C \in \mathcal{A}(X)$ and $x \in X$ such that
\[
A \mu * B \mu = C \mu * \delta_x.
\]

We note that any non-degenerate completely stable distribution $\mu$ on the Euclidean space $\mathbb{R}^n$ is full. Consequently, if $A, B \in \mathcal{H}(\mathbb{R}^n)$, then $C \mu$, where $C$ satisfies (2.5), is also full. Hence $C(\mathbb{R}^n) = \mathbb{R}^n$, i.e. the operator $C$ is invertible. Thus, every completely stable measure on $\mathbb{R}^n$ is $\mathcal{H}(\mathbb{R}^n)$-stable. Since any full Gaussian measure on $\mathbb{R}^n$ is completely stable, the converse is also true.
Completely stable measures on the infinite-dimensional Hilbert space were investigated in [4]. In particular, in [4] it is shown that in this case there exist even completely stable distributions which are not infinitely divisible. Moreover, a characterization of completely stable Gaussian measures in terms of proper values of their covariance operators is given. Namely, it is shown (see [4], Theorem 3) that a non-degenerate Gaussian measure \( \varrho \) on the infinite-dimensional separable Hilbert space \( H \) is completely stable if and only if its covariance operator has infinitely many positive eigenvalues \( a_1 \geq a_2 \geq \ldots \) (i.e. \( \varrho \) is not concentrated on a finite-dimensional hyperplane of \( H \)) and the sequence \( a_n a_{2n} (n=1,2,\ldots) \) is bounded.

Further, by \( H \) we denote a real separable Hilbert space with the inner product \( \langle \cdot, \cdot \rangle \). We shall prove that the non-degenerate Gaussian measures on \( H \) which are \( \mathcal{H}(H) \)-stable are precisely the full Gaussian completely stable measures. The problem of characterization of \( \mathcal{H}(X) \)-stable Gaussian measures on an arbitrary Banach space is still open.

The following propositions are true:

**Proposition 2.4.** Let \( a_1 \geq a_2 \geq \ldots \) and \( b_1 \geq b_2 \geq \ldots \) be the sequences of eigenvalues of covariance operators \( S \) and \( ASA^* \) (\( A \in \mathcal{B}(H) \)), respectively. Then the inequality \( b_n \leq \|A\|^2 a_n (n=1,2,\ldots) \) holds. In particular, if \( A \in \mathcal{H}(H) \), we also have \( a_n \leq \|A^{-1}\|^2 b_n \) for all \( n \).

For the proof see [4], Lemma 2.

**Proposition 2.5.** Let \( S_1 \) and \( S_2 \) be one-to-one covariance operators with the corresponding sequences of eigenvalues \( a_1 \geq a_2 \geq \ldots \), respectively. Then \( S_1 = AS_2 A^* \) for some \( A \in \mathcal{H}(H) \) if and only if the sequence \( \{\max \{a_n b_n, b_n a_n\}\} \) is bounded.

**Proof.** The necessity follows from Proposition 2.4. To prove the sufficiency we assume that \( f_1, f_2, \ldots (f_1, f_2, \ldots) \) is an orthonormal basis of eigenvectors of \( S_1 (S_2) \) corresponding to the eigenvalues \( a_1, a_2, \ldots (b_1, b_2, \ldots) \), respectively. Further, let \( U \) be the unitary operator on \( H \) such that \( Uf_n = e_n (U^{-1} e_n = f_n) \) for all \( n \). Put

\[
H_0 x = \sum_{n=1}^{\infty} \frac{a_n}{b_n} \langle x, f_n \rangle f_n \quad (x \in H).
\]

Since the sequence \( \{\max \{a_n/b_n, b_n/a_n\}\} \) is bounded, \( H_0 \) is a well-defined linear operator from \( \mathcal{H}(H) \). Obviously, \( H_0 \) is a Hermitian operator. Consequently, setting \( A = UH_0 \) we have \( A \in \mathcal{H}(H) \) and \( A^* = HU^{-1} \). Now it is easy to verify the equation \( AS_2 A^* e_n = S_1 e_n (n=1,2,\ldots) \), which shows that \( AS_2 A^* = S_1 \). Thus the proposition is proved.

**Proposition 2.6.** Let \( S \) be a one-to-one covariance operator on \( H \) and \( A, B \in \mathcal{H}(H) \). If there exists an operator \( C \) from \( \mathcal{B}(H) \) such that \( ASA^* + BSB^* = CSC^* \), then we can find an invertible operator with the same property.
Proof. Put \( S_1 = ASA^* + BSB^* \). Obviously, the covariance operator \( S_1 \) is also one-to-one. By our assumption, \( S_1 = CSC^* \). Hence, by Proposition 2.4, we have the formula
\[
\tag{2.6} b_n \leq \|C\|^2 a_n \quad (n = 1, 2, \ldots),
\]
where \( a_1 \geq a_2 \geq \ldots \) and \( b_1 \geq b_2 \geq \ldots \) are the sequences of eigenvalues \( S \) and \( S_1 \), respectively. We note that \( \langle S_1 x, x \rangle \geq \langle ASA^* x, x \rangle \) and \( \langle S_1 x, x \rangle \geq \langle BSB^* x, x \rangle \) for all \( x \). Consequently, if \( c_1 \geq c_2 \geq \ldots \) and \( d_1 \geq d_2 \geq \ldots \) are the sequences of eigenvalues of \( ASA^* \) and \( BSB^* \), respectively, then
\[
\tag{2.7} b_n \geq \max \{c_n, d_n\} \quad (n = 1, 2, \ldots).
\]
But by Proposition 2.4 we get
\[
\tag{2.8} c_n \geq \|A^{-1}\|^2 a_n, \quad d_n \geq \|B^{-1}\|^2 a_n
\]
for all \( n \). From (2.7) and (2.8) we obtain the inequality
\[
b_n \geq \max \{\|A^{-1}\|^2 a_n, \|B^{-1}\|^2 a_n\} \quad (n = 1, 2, \ldots)
\]
which, together with (2.6) and Proposition 2.5, completes the proof.

Theorem 2.2. Let \( \varrho \) be a non-degenerate Gaussian measure on a real separable Hilbert space \( H \). Then \( \varrho \) is \( \mathcal{U}(H) \)-stable if and only if it is a full completely stable measure.

Proof. It follows from Proposition 2.6 that every full Gaussian completely stable probability measure on \( H \) is \( \mathcal{U}(H) \)-stable. Conversely, suppose that \( \varrho \) is a non-degenerate Gaussian measure on \( H \) which is \( \mathcal{U}(H) \)-stable. By Theorem 2.1, \( \varrho \) is full. Obviously, we may assume that \( H \) is infinite dimensional. Let \( S \) denote the covariance operator of \( \varrho \) and let \( a_1 \geq a_2 \geq \ldots \) be the sequence of eigenvalues of \( S \). Using the same arguments as in the proof of Theorem 1 in [4], we infer from \( \mathcal{U}(H) \)-stability of \( \varrho \) that the sequence \( \{a_n/a_{2n}\} \) is bounded. But in this case \( \varrho \) is completely stable (Theorem 3 in [4]). Thus the theorem is proved.

Remark 2.1. Let \( \varrho \) be a full completely stable Gaussian measure on an infinite-dimensional Hilbert space \( H \). Then the sequence \( a_1 \geq a_2 \geq \ldots \) of eigenvalues of its covariance operator fulfills the condition \( \sup_n a_n/a_{2n} < \infty \). Put
\[
a = \sup_n a_n/a_{2n}.
\]

Let \( A, B \in \mathcal{B}(H) \). Then there exists an operator \( C \) such that
\[
A\varrho \ast B\varrho = C\varrho \ast \delta_x
\]
for some \( x \in X \) and \( \|C\| \leq a(\|A\| + \|B\|) \). Namely, the operator \( C \) constructed in the proof of Theorem 2 in [4] has this property. Moreover, the proof of Proposition 2.6 shows that for \( A, B \in \mathcal{U}(H) \) we can find an invertible operator with the same property.
3. Let \( \mu \) be a probability measure on \( X \) and suppose that \( P \) is a projector belonging to \( \mathcal{D}(\mu) \). Then, by Proposition 1.5 in [11], \( I - P \) also belongs to \( \mathcal{D}(\mu) \) and the equality

\[
\mu = P\mu * (I - P)\mu
\]

holds.

Let \( \mu \) be in addition infinitely divisible. Then from (3.1) and the uniqueness of the Tortrat representation of \( \mu \) one can obtain the following

**Proposition 3.1.** Suppose that \( \mu = \varrho * \mathcal{E}(M) \), where \( \varrho \) is a symmetric Gaussian measure with the covariance operator \( R \) and \( \mathcal{E}(M) \) is a generalized Poisson measure. Moreover, let \( P \) be a projector on \( X \). Then:

(i) \( P \in \mathcal{D}(\mu) \) if and only if \( P \in \mathcal{D}(\mu) \cap \mathcal{D}(\mathcal{E}(M)) \);

(ii) \( P \in \mathcal{D}(\varrho) \) if and only if \( R(\text{Im} \ P^*) \subseteq \text{Im} \ P \) and \( R(\text{Ker} \ P^*) \subseteq \text{Ker} \ P \);

(iii) \( P \in \mathcal{D}(\mathcal{E}(M)) \) if and only if the measure \( M \) is concentrated on \( \text{Im} \ P \cup \text{Ker} \ P \).

The following lemma is a crucial step in our considerations:

**Lemma 3.1.** Let \( X \) be a real separable Banach space of dimension at least two and let \( \mathcal{E}(M) \) be a non-degenerate generalized Poisson measure on \( X \). If there exists a non-trivial finite-dimensional projector \( P \) belonging to \( \mathcal{D}(\mathcal{E}(M)) \), then \( \mathcal{E}(M) \) is not \( \mathcal{U}(X) \)-stable.

**Proof.** Clearly, by Theorem 2.1 it is sufficient to prove the lemma under the assumption that \( \mathcal{E}(M) \) is full. Then the measure \( M \) is also full.

Suppose that \( P_1 \) and \( P_2 \) are projectors from \( \mathcal{D}(\mathcal{E}(M)) \). We infer from Proposition 3.1 that \( M \) is concentrated on the set

\[
(\text{Im} \ P_1 \cup \text{Ker} \ P_1) \cap (\text{Im} \ P_2 \cup \text{Ker} \ P_2).
\]

Consequently, the restrictions \( M|_{\text{Im} \ P_1} \) and \( M|_{\text{Ker} \ P_1} \) are concentrated on the unions

\[
(\text{Im} \ P_1 \cap \text{Im} \ P_2) \cup (\text{Im} \ P_1 \cap \text{Ker} \ P_2), \quad (\text{Ker} \ P_1 \cap \text{Im} \ P_2) \cup (\text{Ker} \ P_1 \cap \text{Ker} \ P_2),
\]

respectively. Hence, since \( M \) is full, \( \text{Im} \ P_1 \) is equal to the direct sum of \( \text{Im} \ P_1 \cap \text{Im} \ P_2 \) and \( \text{Im} \ P_1 \cap \text{Ker} \ P_2 \), and \( \text{Ker} \ P_1 \) is equal to the direct sum of \( \text{Ker} \ P_1 \cap \text{Im} \ P_2 \) and \( \text{Ker} \ P_1 \cap \text{Ker} \ P_2 \). Consequently, \( P_1 P_2 = P_2 P_1 \).

Let \( k \) be the least positive integer for which there exists a \( k \)-dimensional projector belonging to \( \mathcal{D}(\mathcal{E}(M)) \) and let \( \mathcal{J}_k \) denote the set of all \( k \)-dimensional projectors from \( \mathcal{D}(\mathcal{E}(M)) \). Consider \( P_1, P_2 \in \mathcal{J}_k, P_1 \neq P_2 \). Since \( P_1 \) commutes with \( P_2 \), the operator \( P_1 P_2 \) is a projector from \( \mathcal{D}(\mathcal{E}(M)) \). Moreover, the dimension of \( P_1 P_2 \) is less than \( k \). By the definition of \( k \), this implies \( P_1 P_2 = P_2 P_1 = 0 \).

Hence, in particular, we infer (note that, for any \( P \in \mathcal{J}_k \), \( M(\text{Im} \ P) > 0 \) and \( M \) is \( \sigma \)-finite) that \( \mathcal{J}_k \) is a countable set.
Contrary to the assertion of the lemma, suppose that $\mathcal{E}(M)$ is $\mathcal{U}(X)$-stable. Then, by (2.1), given an arbitrary $A \in \mathcal{U}(X)$ we can find $B \in \mathcal{U}(X)$ such that

$$AM + M = BM.$$  

Now let $P$ be a projector from $\mathcal{G}(\mathcal{E}(M))$. Then the projector $BPB^{-1}$ belongs to $\mathcal{G}(B\mathcal{E}(M))$. Consequently, since $AM \leq BM$ and $M \leq BM$ by (3.2), Proposition 3.1 implies $BPB^{-1} \in \mathcal{G}(A\mathcal{E}(M)) \cap \mathcal{G}(\mathcal{E}(M))$ (note that $\mathcal{E}(BM) = B\mathcal{E}(M)$ and $\mathcal{E}(AM) = A\mathcal{E}(M)$). Hence $BPB^{-1}, A^{-1}BPB^{-1} A \in \mathcal{G}(\mathcal{E}(M))$.

Obviously, if $P \in \mathcal{G}_k$, then also $BPB^{-1}, A^{-1}BPB^{-1} A \in \mathcal{G}_k$. Thus the set $\mathcal{G}_k$ has the following property: for any $A \in \mathcal{U}(X)$ there exists a $P_A \in \mathcal{G}_k$ such that $A^{-1}P_A A \in \mathcal{G}_k$. Fix $P_0 \in \mathcal{G}_k$. Let $A$ be an arbitrary operator from $\mathcal{U}(X)$ and let $A^{-1}P_A A \in \mathcal{G}_k$ for $P_A \in \mathcal{G}_k$. If $P_A \neq P_0$, then $P_A P_0 = 0$ and, consequently, $A^{-1}P_A A^{-1} P_0 A = 0$. Hence, since $A^{-1}P_A A \in \mathcal{G}_k$, we obtain

$$A^{-1}(\text{Im } P_0) \subset \bigcup_{P \in \mathcal{G}_k} \ker P.$$

If $P_A = P_0$, then $A^{-1}P_0 A \in \mathcal{G}_k$ and, consequently,

$$A^{-1}(\text{Im } P_0) \subset \bigcup_{P \in \mathcal{G}_k} \text{Im } P.$$

Since $A$ is arbitrary, (3.3) and (3.4) together imply that

$$\bigcup_{P \in \mathcal{G}_k} (\ker P \cup \text{Im } P) = X,$$

which contradicts the fact that $\mathcal{G}_k$ is a countable set. The lemma is thus proved.

We note that for any Gaussian measure $\mathcal{G}$ on $X (\dim X \geq 2)$ there are non-trivial finite-dimensional projectors in $\mathcal{G}(\mathcal{G})$. For instance, if $\mathcal{G}$ is a full Gaussian measure with the covariance operator $R$, then every projector of the form $x^* x/Rx^* (Rx^*)$ ($x^* \in X^*$) belongs to $\mathcal{G}(\mathcal{G})$ (this follows, by a simple computation, from Proposition 3.1). Thus, combining Proposition 2.1 and Lemma 3.1 and taking into account Theorem 2.1, we obtain

**Theorem 3.1.** Let $X$ be a real separable Banach space of dimension at least two. Then a non-degenerate $\mathcal{U}(X)$-stable probability measure on $X$ is a full Gaussian measure if and only if there exists a non-trivial finite-dimensional projector in its decomposability semigroup.

4. Before proceeding to state and prove the main results of this paper we shall establish auxiliary propositions.

**Proposition 4.1.** Suppose that, for $n = 1, 2, \ldots, \mu_n \in \mathcal{P}(X)$ and $\{A_n\}, \{B_n\}$ are two sequences of linear operators on $X$. If the sequences $\{A_n \mu_n\}$ and $\{B_n \mu_n\}$ are conditionally compact, then so are the sequences $\{(A_n + B_n) \mu_n\}$ and $\{(A_n - B_n) \mu_n\}$. Moreover, if $A_n \mu_n \Rightarrow v$ for some $v \in \mathcal{P}(X)$ and $B_n \mu_n \Rightarrow \delta_0$, then $(A_n + B_n) \mu_n \Rightarrow v$.
Proof. Since \( \{A_n, \mu_n\} \) and \( \{B_n, \mu_n\} \) are conditionally compact, it follows from Theorem 6.7, Chapter II in [5], that given \( \epsilon > 0 \) there exist compact sets \( K^1_\epsilon \) and \( K^2_\epsilon \) such that

\[
\mu_n \{x: A_n x \in K^1_\epsilon\} > 1 - \epsilon/2, \quad \mu_n \{x: B_n x \in K^2_\epsilon\} > 1 - \epsilon/2
\]

for all \( n \). Then we have

\[
\mu_n \{x: A_n x \in K^1_\epsilon \text{ and } B_n x \in K^2_\epsilon\} > 1 - \epsilon.
\]

We note that

\[
\mu_n \{x: (A_n + B_n) x \in K^1_\epsilon + K^2_\epsilon\} > 1 - \epsilon
\]

Since \( K^1_\epsilon + K^2_\epsilon \) is compact and \( \epsilon \) is arbitrary, it follows once more from Theorem 6.7, Chapter II in [5], that \( \{(A_n + B_n, \mu_n)\} \) is compact.

If \( A \) is an operator from \( \mathcal{B}(X) \), then \( (-A) \mu = A \mu \) for any \( \mu \in \mathcal{P}(X) \). Hence and from the continuity of the operation \( - \) we infer that the compactness of \( \{B_n, \mu_n\} \) implies the compactness of \( \{-B_n, \mu_n\} \). Consequently, the sequence \( \{(A_n - B_n, \mu_n)\} \) is also conditionally compact.

Let \( A_n \mu_n \Rightarrow \nu \) and \( B_n \mu_n \Rightarrow \delta_0 \). By the inequality

\[
\left| \left( (A_n + B_n) \mu_n\right)^*(x^*) - \left( A_n \mu_n\right)^*(x^*) \right| = \left| \hat{\mu}_n \left( x^* + B_n^* x^* \right) - \hat{\mu}_n \left( A_n^* x^* \right) \right|
\]

we get

\[
\lim_{n} \left( (A_n + B_n) \mu_n\right)^*(x^*) = \lim_{n} \left( A_n \mu_n\right)^*(x^*) = \delta(x^*) \quad \text{for all } x^* \in X^*.
\]

Moreover, by the first part of the proposition, the sequence \( \{(A_n + B_n, \mu_n)\} \) is compact. Hence \( (A_n + B_n) \mu_n \Rightarrow \nu \). The proposition is thus proved.

**Proposition 4.2.** Let \( \{T_n\} \) be a sequence of one-dimensional operators from \( \mathcal{B}(X) \), let \( \mu_n \in \mathcal{P}(X) \) \( (n = 1, 2, \ldots) \), and assume that

(i) \( T_n \mu_n \Rightarrow \mu \in \mathcal{P}(X), \mu \neq \delta_0 \),

(ii) the sequence of norms \( \{|\|T_n\|\| \} \) is bounded.

Then the sequence \( \{T_n\} \) is sequentially compact in the strong operator topology.

Proof. Let \( T_n \) be given for each \( n \) by the formula \( T_n = x^*_n(\cdot)x_n \), where \( x^*_n \in X^* \) and \( x_n \) is the element of \( X \) with \( \|x_n\| = 1 \). From (ii) it follows that the sequence \( \{x^*_n\} \) is bounded and, consequently \( (X) \) is separable, \( \sigma(X^*, X) \)-compact. Thus, to prove our statement it is enough to verify that the sequence \( \{x^*_n\} \) is compact.
There exist \( a > 0 \) and \( b > 0 \) such that for sufficiently large \( n \) we have
\[
(4.1) \quad \mu_n \{ x : \| T_n x \| \geq b \} > a.
\]
Indeed, if no such \( a \) and \( b \) exist, then for any \( b > 0 \) there exists a subsequence of \( \{ T_n \} \), say \( \{ T_{n_k} \} \), such that
\[
\lim_{k} \mu_n \{ x : \| T_{n_k} x \| \geq b \} = \lim_{k} T_{n_k} \mu_n \{ x : \| x \| \geq b \} = 0.
\]
But, by (i), \( T_{n_k} \mu_n = \mu \). As \( \mu \neq \delta_0 \), this leads to a contradiction. Moreover, since \( \{ T_n \mu_n \} \) is conditionally compact, it follows from Theorem 6.7, Chapter II in \([5]\), that given \( \epsilon > 0 \) there exists a compact set \( K_\epsilon \) such that
\[
(4.2) \quad T_n \mu_n (K_\epsilon) = \mu_n \{ x : T_n x \in K_\epsilon \} > 1 - \epsilon
\]
for all \( n \). By (4.1) and (4.2) we can find a compact set \( K \) and \( b > 0 \) such that for sufficiently large \( n \), say \( n \geq N \), we have
\[
\mu_n \{ x : \| T_n x \| \geq b \text{ and } T_n x \in K \} > 0.
\]
Consequently, since \( \| T_n x \| = \| x^n (x) \| (n = 1, 2, \ldots) \), there exists a sequence \( \{ y_n \} \) in \( X \) with \( \| x^n (y_n) \| > b \) and \( x_n (y_n) x_n \in K \) for \( n \geq N \). But this implies the compactness of \( \{ x_n \} \). The proposition is thus proved.

**Proposition 4.3.** Let \( \mu \) be a full measure on \( X \), let \( P \) be a one-dimensional projector from \( \mathcal{B}(X) \), and \( C_n \in \mathcal{B}(X) (n = 1, 2, \ldots) \). If the sequence \( \{ C_n P \mu \} \) is compact in \( \mathcal{P}(X) \), then the sequence \( \{ C_n P \} \) is compact in \( \mathcal{B}(X) \). In particular, if \( C_n P \mu = \delta_0 \), then \( \| C_n P \| \to 0 \).

**Proof.** First we prove the second part of the proposition. Let \( P = x^n_0 (\cdot) x_0 \), where \( x^n_0 \in X^* \), \( x_0 \in X \), \( x^n_0 (x_0) = 1 \), and \( \| x_0 \| = 1 \). Thus \( C_n P x = x^n_0 (x) C_n x_0 \) \((x \in X; n = 1, 2, \ldots) \). Moreover, it is easy to see that if \( b < \lim\sup \| C_n x_0 \| \), then the inequality
\[
(4.3) \quad \mu \{ x : \| x^n_0 (x) \| \geq a \} \leq \lim\sup \mu \{ x : \| x^n_0 (x) \| \cdot \| C_n x_0 \| \geq a \}
\]
holds for any \( a > 0 \). Suppose that \( C_n P \mu \Rightarrow \delta_0 \). Hence for each \( a > 0 \) we obtain
\[
\lim_{n} C_n P \mu \{ x : \| x \| \geq a \} = 0.
\]
But we have
\[
C_n P \mu \{ x : \| x \| \geq a \} = \mu \{ x : \| x^n_0 (x) \| \cdot \| C_n x_0 \| \geq a \} (n = 1, 2, \ldots),
\]
which together with (4.3) implies that if \( b < \lim\sup \| C_n x_0 \| \), then
\[
\mu \{ x : \| x^n_0 (x) \| b \geq a \} = 0 \quad \text{for any } a > 0.
\]
Since \( \mu \) is full, the last relation shows that \( \|C_n x_0\| \to 0 \) and, consequently,

\[
\lim_n \|C_n P\| = 0.
\]

Now assume that the sequence \( \{C_n P \mu\} \) is compact in \( \mathcal{P}(X) \). In order to establish that the sequence \( \{C_n P\} \) is compact in \( \mathcal{B}(X) \) it is enough (by Proposition 4.2 and the second part of Proposition 4.3) to show that the sequence of norms \( \{\|C_n P\|\} \) is bounded. But the last assertion follows easily from the fact that if the sequence \( \{C_n P \mu\} \) is compact, then for every sequence \( \{a_n\} \) chosen so that \( a_n \geq 0 \) and \( \lim a_n = 0 \) we have \( a_n C_n P \mu \Rightarrow \delta_0 \) and, consequently (by the second part of the proposition), \( \|a_n C_n P\| \to 0 \). Thus the proof is completed.

5. Let \( \mathcal{A} \) be a subgroup of \( \mathcal{U}(X) \) and let \( \sigma \) be a map from \( \mathcal{A} \times \mathcal{A} \) into \( \mathcal{A} \). We say that \( \sigma \) is continuous at 0 if for any pair of sequences \( \{A_n\} \) and \( \{B_n\} \) in \( \mathcal{A} \) which converge to 0 we have \( \sigma(A_n, B_n) \to 0 \).

Let \( \mu \) be a probability measure on \( X \) which is stable under \( \mathcal{A} \). For each \( (A, B) \in \mathcal{A} \times \mathcal{A} \) let \( \mathcal{C}(A, B) \) denote the subset of \( \mathcal{A} \) consisting of those operators \( C \) from \( \mathcal{A} \) for which

\[
A \mu \ast B \mu = C \mu \ast \delta_x
\]

for a certain \( x \in X \).

If there exists a selector \( \sigma \) on \( \mathcal{A} \times \mathcal{A} \) with \( \sigma(A, B) \in \mathcal{C}(A, B) \) for any \( A, B \in \mathcal{A} \), which is continuous at 0, then \( \mu \) is said to be strongly \( \mathcal{A} \)-stable.

Remark 5.1. Let \( \mu \) be an \( \mathcal{A} \)-stable measure on \( X \) and assume that for any pair of sequences \( \{A_n\} \) and \( \{B_n\} \) in \( \mathcal{A} \) which converge to 0 we can find a sequence \( \{C_n\} \) in \( \mathcal{A} \) such that

\[
A_n \mu \ast B_n \mu = C_n \mu \ast \delta_{x_n}
\]

for some \( x_n \in X \) \( (n = 1, 2, \ldots) \)

and

\[
\lim_n C_n = 0.
\]

Then \( \mu \) is strongly \( \mathcal{A} \)-stable.

For example, given \( A, B \in \mathcal{A} \) it is enough to take as \( \sigma(A, B) \) an operator \( C \) from \( \mathcal{C}(A, B) \) such that

\[
||C|| - r_{A,B} \leq ||A||,
\]

where \( r_{A,B} = \inf\{||C|| : C \in \mathcal{C}(A, B)\} \).

Remark 5.2. Let \( \mu \) be a full \( \mathcal{A} \)-stable measure on a finite-dimensional space. Then \( \mu \) is strongly \( \mathcal{A} \)-stable.

To see this we note that for full measures \( \mu \) on the finite-dimensional space \( X \) the convergence \( A_n \mu \Rightarrow \delta_0 \) for \( A_n \in \mathcal{B}(X) \) implies

\[
\lim_n A_n = 0
\]

(it is a simple consequence of statement (ii), p. 120, in [11]).
Moreover, it is clear that any $\mu$ which is stable must also be strongly 
\{aI: a > 0\}-stable.

**Proposition 5.1.** Let $\mu$ be a probability measure on $X$ which is strongly 
$\mathcal{U}(X)$-stable and let $A_n, B_n \in \mathcal{U}(X)$ \( (n = 1, 2, \ldots) \). If the sequences of norms 
\{||A_n||\} and \{||B_n||\} are bounded, then we can find a sequence \{\(C_n\)\} in \(\mathcal{U}(X)\) such that

\[ A_n \mu \ast B_n \mu = C_n \mu \ast \delta_{x_n} \quad \text{for some } x_n \in X \ (n = 1, 2, \ldots) \]

and \{||C_n||\} is bounded.

**Proof.** Let $\mu$ be a strongly $\mathcal{U}(X)$-stable measure. Given $A, B \in \mathcal{U}(X)$ we put $r_{A,B} = \inf \{||C||: C \in \mathcal{F}(A, B)\}$. Then for any pair of sequences \{\(A'_n\)\} and \{\(B'_n\)\} in \(\mathcal{U}(X)\) which converge to 0 we have

\[ \lim_{n} r_{A'_n,B'_n} = 0. \]

Let \{\(A_n\), \(B_n\)\} be two sequences of operators from $\mathcal{U}(X)$ with

\[ \sup_{n} \{||A_n|| + ||B_n||\} < \infty. \]

Then, for any sequence of positive real numbers \{\(a_n\)\} which converge to 0, we have

\[ \lim_{n} r_{a_n A_n, a_n B_n} = 0. \]

We note that for each $a > 0$ and $A, B \in \mathcal{U}(X)$ we have $r_{aA,aB} = ar_{A,B}$. Consequently, $a_n r_{A'_n,B'_n} \to 0$ as $n \to \infty$ for every sequence \{\(a_n\)\} chosen so that $a_n > 0$ and $\lim a_n = 0$. Hence

\[ \sup_{n} r_{A'_n,B'_n} < \infty. \]

For each $n$ we choose an operator $C_n$ from $\mathcal{F}(A_n, B_n)$ such that

\[ ||C_n|| - r_{a_n A_n, a_n B_n} \leq 1. \]

It is clear that the sequence \{\(C_n\)\} has the required properties. The proposition is thus proved.

Now, we are ready to prove the main result of this paper.

**Theorem 5.1.** Let $X$ be a real separable Banach space of dimension at least two and let $\mu$ be a non-degenerate probability measure on $X$ which is strongly $\mathcal{U}(X)$-stable. Then $\mu$ is full Gaussian.

**Proof.** Let $\mu$ be a non-degenerate and strongly $\mathcal{U}(X)$-stable measure on $X$. It is clear that the symmetrized distribution $\mu^s$ is also strongly $\mathcal{U}(X)$-stable. In this case for any $A, B \in \mathcal{U}(X)$ there exists a $C \in \mathcal{U}(X)$ such that

\[ A \mu^s \ast B \mu^s = C \mu^s. \]

If we prove that $\mu^s$ is Gaussian, then it will follow from Cramer's theorem that so is $\mu$. By Theorem 2.1, $\mu$ and, consequently, $\mu^s$ are full. For simplicity of
the notation we put \( v = \mu^* \). According to Theorem 3.1 it will be sufficient to show that there is a one-dimensional projector \( P \) in \( \mathcal{D}(v) \).

Given a projector \( P \), we put \( P^\perp = I - P \). It is obvious that \( \{ I, P - P^\perp \} \) is a finite subgroup of \( \mathcal{U}(X) \). Consequently, by Proposition 2.3 there exists a one-dimensional projector \( P_0 \) such that \( (P_0 - P^\perp) v = v \). Put

\[
A_n = P_0 + \frac{1}{n} P^\perp_0, \quad B_n = P^\perp_0 + \frac{1}{n} P_0 \quad (n = 1, 2, \ldots).
\]

Obviously, the operators \( A_n \) and \( B_n \) belong to \( \mathcal{U}(X) \). By (5.1) for every integer \( n \) we can find an operator \( C_n \) in \( \mathcal{U}(X) \) such that

\[
(5.2) \quad A_n v \ast B_n v = C_n v.
\]

We note that the sequences of norms \( \{\|A_n\|\} \) and \( \{\|B_n\|\} \) are bounded. Consequently, by Proposition 5.1 we may assume that also the sequence \( \{\|C_n\|\} \) is bounded. From (5.2) we obtain

\[
(5.3) \quad C_n^{-1} A_n v \ast C_n^{-1} B_n v = v \quad (n = 1, 2, \ldots).
\]

Hence, by Theorem 2.2, Chapter II in [5], the sequences \( \{C_n^{-1} A_n v\} \) and \( \{C_n^{-1} B_n v\} \) are shift compact. Since \( \delta(x^*) \geq 0 \) for all \( x^* \in X^* \), we infer that \( \{C_n^{-1} A_n v\} \) and \( \{C_n^{-1} B_n v\} \) are compact. Applying the formula \( (P_0 - P^\perp_0) v = v \) and Proposition 4.1 it is easy to prove that for any sequence \( \{T_n\} \) in \( \mathcal{B}(X) \) the compactness of \( \{T_n v\} \) implies the compactness of \( \{T_n P_0 v\} \) and \( \{T_n P^\perp_0 v\} \).

Hence, in particular, the sequences \( \{C_n^{-1} A_n P_0 v\} \) and \( \{C_n^{-1} B_n P^\perp_0 v\} \) are compact. Consequently,

\[
\frac{1}{n} C_n^{-1} A_n P_0 v \Rightarrow \delta_0 \quad \text{and} \quad \frac{1}{n} C_n^{-1} B_n P^\perp_0 v \Rightarrow \delta_0.
\]

But for \( n = 1, 2, \ldots \) we have

\[
C_n^{-1} A_n = C_n^{-1} P_0 + \frac{1}{n} C_n^{-1} P^\perp_0, \quad C_n^{-1} B_n = C_n^{-1} P^\perp_0 + \frac{1}{n} C_n^{-1} P_0,
\]

\[
C_n^{-1} A_n P_0 = C_n^{-1} P_0, \quad C_n^{-1} B_n P^\perp_0 = C_n^{-1} P^\perp_0.
\]

Thus (5.3) can be rewritten in the form

\[
(5.4) \quad \left( C_n^{-1} P_0 + \frac{1}{n} C_n^{-1} P^\perp_0 \right) v \ast \left( C_n^{-1} P^\perp_0 + \frac{1}{n} C_n^{-1} P_0 \right) v = v \quad (n = 1, 2, \ldots),
\]

where the sequences \( \{C_n^{-1} P_0 v\} \) and \( \{C_n^{-1} P^\perp_0 v\} \) are compact. Passing, if necessary, to subsequences we may assume without loss of generality that these sequences are convergent. Moreover, we have

\[
\lim_{n} \frac{1}{n} C_n^{-1} P_0 v = \lim_{n} \frac{1}{n} C_n^{-1} P_0 v = \delta_0.
\]
Hence and from Proposition 4.1 we get

\[ \lim_{n} C_n^{-1} P_0 v = \lim_{n} \left( C_n^{-1} P_0 + \frac{1}{n} C_n^{-1} P_0^+ \right) v, \]

(5.5)

\[ \lim_{n} C_n^{-1} P_0^+ v = \lim_{n} \left( C_n^{-1} P_0^+ + \frac{1}{n} C_n^{-1} P_0 \right) v. \]

On the other hand, from (5.4) by a simple computation we obtain

\[ C_n^{-1} P_0 C_n v = C_n^{-1} P_0 v + \frac{1}{n} C_n^{-1} P_0 v, \quad C_n^{-1} P_0^+ C_n v = C_n^{-1} P_0^+ v + \frac{1}{n} C_n^{-1} P_0^+ v \]

\[ (n = 1, 2, \ldots) \]

and, consequently,

\[ \lim_{n} C_n^{-1} P_0 C_n v = \lim_{n} C_n^{-1} P_0 v, \quad \lim_{n} C_n^{-1} P_0^+ C_n v = \lim_{n} C_n^{-1} P_0^+ v. \]

(5.6)

Thus (5.4)-(5.6) together imply that

\[ \lim_{n} C_n^{-1} P_0 C_n v \ast \lim_{n} C_n^{-1} P_0^+ C_n v = v. \]

(5.7)

Since \( \{C_n^{-1} P_0 v\} \) is convergent, we infer from Proposition 4.3 that \( \{C_n^{-1} P_0\} \) is compact. Consequently, by the assumption on the sequence \( \{\|C_n\|\} \) the norms \( \|C_n^{-1} P_0 C_n\| \) \( (n = 1, 2, \ldots) \) are bounded in common. This together with the convergence of \( \{C_n^{-1} P_0 C_n v\} \) proves (by Proposition 4.2) that the sequence \( \{C_n^{-1} P_0 C_n\} \) has a strongly convergent subsequence. Denoting by \( P \) its limit, by (5.7) we have

\[ P v \ast \lim_{n} C_n^{-1} P_0^+ C_n v = v, \]

which shows that \( P \in \mathcal{D}(v) \). Obviously, \( P \) is a projector from \( \mathcal{B}(X) \) of dimension at most one. Moreover, \( P \neq 0 \). Indeed, if \( P = 0 \), then there exists a subsequence of indices \( n_1 < n_2 < \ldots \) for which \( \{C_n^{-1} P_0 C_n\} \) converges strongly to \( 0 \) and, consequently,

\[ \lim_{k} C_{n_k}^{-1} P_0 v = \lim_{k} C_{n_k}^{-1} P_0 C_{n_k} v = \delta_0. \]

But Proposition 4.2 then implies that \( \|C_{n_k}^{-1} P_0\| \to 0 \), which contradicts the fact that the sequence \( \{\|C_n\|\} \) is bounded. The theorem is thus proved.

Remark 5.3. In the statement of Theorem 5.1 we have assumed that \( \mu \) is strongly \( \mathcal{D}(X) \)-stable. However, it is enough to assume that \( \mu \) is \( \mathcal{U}(X) \)-stable and has the property proved in Proposition 5.1.

From Remark 2.1 it follows that if \( H \) is a Hilbert space, then every \( \mathcal{U}(H) \)-stable Gaussian measure \( \varrho \) is also strongly \( \mathcal{U}(H) \)-stable. Thus, combining Theorems 2.2 and 5.1 and taking into account Theorem 3 in [4], we get a
characterization of strongly $\mathcal{U}(H)$-stable measures on the infinite-dimensional Hilbert space.

**Theorem 5.2.** Suppose that the Hilbert space $H$ is infinite dimensional. Then a non-degenerate probability measure $\mu$ on $H$ is strongly $\mathcal{U}(H)$-stable if and only if $\mu$ is full Gaussian and the sequence $a_1 \geq a_2 \geq \ldots$ of eigenvalues of its covariance operator fulfills the condition $\sup_n a_n/a_{2n} < \infty$.

**REFERENCES**


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