PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 3, Fasc. 1 (1982), p. 53-69

$\mathscr{U}(X)$-STABLE MEASURES ON BANACH SPACES

BY

BOHDAN MINCER (WROCŁAW)

Abstract. The probability measures on separable Banach space X which are stable under the group $\mathscr{U}(X)$ of all invertible linear operators on X are investigated. The characterization of such measures on finite-dimensional spaces was obtained by Parthasarathy in [6]. In this paper some generalizations of Parthasarathy's result are given.

1. Parthasarathy proved in [6] (see also [3] and [7]) that the probability measures on the at least two-dimensional Euclidean space $\boldsymbol{R}^n$ which are stable under the group $\mathscr{U}(\boldsymbol{R}^n)$ of all invertible linear transformations of $\boldsymbol{R}^n$ onto itself are precisely the full Gaussian and the degenerate probability measures. Our aim is to discuss $\mathscr{U}(X)$-stable distributions on a real separable Banach space X. The main results of this paper are the following: For a Hilbert space X it is shown that a non-degenerate Gaussian measure is $\mathscr{U}(X)$-stable if and only if it is full and completely stable in the sense of paper [4]. Hence we obtain a characterization of $\mathscr{U}(X)$-stable Gaussian measures on Hilbert spaces in terms of proper values of their covariance operators. Further, it is shown that a non-degenerate $\mathscr{U}(X)$-stable probability measure on X ($\dim X \geqslant 2$) is a full Gaussian measure if and only if there exists a non-trivial finite-dimensional projector in its decomposability semigroup. As an application of this we obtain a Parthasarathy-type theorem for probability distributions on X, which are stable under $\mathscr{U}(X)$ in some "enough regular" way (for so-called strongly $\mathscr{U}(X)$-stable distributions). The question whether in the infinite-dimensional case there are non-Gaussian $\mathscr{U}(X)$-stable probability measures is still open.

Further in this section we shall introduce basic notions and facts.

Let $\|\cdot\|$ be the norm of a real separable Banach space X and let X^* be the topological dual of X.

For a set $A \subset X$ its norm closure is denoted by $\bar{A}$.

By $\mathscr{P}(X)$ we denote the set of all probability measures defined on the class of Borel subsets of X. The set $\mathscr{P}(X)$ with the topology of weak convergence and multiplication defined by the convolution becomes a topological semigroup. We denote the convolution of two measures λ and μ by $\lambda * \mu$, and by μ^{*n} the n-th power in the sense of convolution. If a sequence $\{\mu_n\}$ of probability measures on X converges weakly to a $\mu \in \mathscr{P}(X)$, we write $\mu_n \Rightarrow \mu$. Moreover, by δ_x $(x \in X)$ we denote the probability measure concentrated at the point x.

The characteristic functional of μ is defined on X^* by the formula

$$\hat{\mu}(x^*) = \int_X e^{ix^*(x)} d\mu(x) \qquad (x^* \in X^*).$$

Given $\mu \in \mathscr{P}(X)$, we define $\bar{\mu}$ by

$$\bar{\mu}(E) = \mu(-E), \qquad \text{where } -E = \{-x\colon x \in E\}.$$

The mapping $\mu \to \bar{\mu}$ is a continuous automorphism of $\mathscr{P}(X)$. For any $\mu \in \mathscr{P}(X)$, $\mu^s = \mu * \bar{\mu}$ is called the *symmetrization* of μ.

A probability measure μ is said to be *symmetric* if $\mu = \bar{\mu}$.

A measure μ from $\mathscr{P}(X)$ is said to be *full* if its support is not contained in any proper hyperplane of X. It is clear that μ is full if and only if μ^s is full.

Further, $\mathscr{B}(X)$ will denote the algebra of continuous linear operators on X with the norm topology. By $\mathscr{U}(X)$ we shall denote the group of all invertible operators from X onto X.

The image and the kernel of $A \in \mathscr{B}(X)$ will be denoted by $\operatorname{Im} A$ and $\operatorname{Ker} A$, respectively. An operator A is said to be *n-dimensional* if $\dim \operatorname{Im} A = n$ $(n = 1, 2, \ldots)$. By a *projector* we mean an operator P from $\mathscr{B}(X)$ with the property $P^2 = P$. The zero operator will be denoted by 0, and the unit operator by I.

For any $A \in \mathscr{B}(X)$ and $\mu \in \mathscr{P}(X)$ let $A\mu$ denote the measure defined by the formula $A\mu(E) = \mu(A^{-1}(E))$ for all Borel subsets E of X. For all A in $\mathscr{B}(X)$ and μ, $v \in \mathscr{P}(X)$ we can easily check the equations

$$A(\mu * v) = A\mu * Av, \qquad (A\mu)^\wedge(x^*) = \hat{\mu}(A^* x^*),$$

where A^* denotes the adjoint operator. Moreover, it is clear that the mapping $\langle A, \mu\rangle \to A\mu$ from $\mathscr{B}(X) \times \mathscr{P}(X)$ onto $\mathscr{P}(X)$ is jointly sequentially continuous even if $\mathscr{B}(X)$ is provided with a strong operator topology. Consequently, if a sequence $\{A_n\}$ of linear operators is sequentially strongly compact, then for every $\mu \in \mathscr{P}(X)$ the sequence $\{A_n \mu\}$ is compact in $\mathscr{P}(X)$.

For full measures on finite-dimensional spaces the converse implication is also true. Namely, if the sequence $\{A_n \mu\}$ is compact in $\mathscr{P}(X)$, where μ is full and $A_n \in \mathscr{B}(X)$ $(n = 1, 2, \ldots)$, then the sequence $\{A_n\}$ is compact in $\mathscr{B}(X)$ (see [11], p. 120).

In the study of limit probability distributions Urbanik [11] introduced the concept of decomposability semigroups $\mathscr{D}(\mu)$ of linear operators associated with the probability measure μ. Namely, $\mathscr{D}(\mu)$ consists of all operators A from $\mathscr{B}(X)$ for which $\mu = A\mu * v$ holds for a certain probability measure v. It is clear that $\mathscr{D}(\mu)$ is a semigroup under multiplication of operators and $\mathscr{D}(\mu)$ always contains the operators 0 and I. Moreover, $\mathscr{D}(\mu)$ is closed in $\mathscr{B}(X)$. It has been shown that some probabilistic properties of measures correspond to algebraic and topological properties of their decomposability semigroups (see, e.g., [12]).

The Tortrat representation of infinitely divisible laws on Banach spaces is an important step in our considerations. We recall that for any bounded non-negative Borel measure F on X vanishing at 0 the Poisson measure $e(F)$ associated with F is defined as

$$e(F) = e^{-F(X)} \sum_{n=0}^{\infty} \frac{1}{n!} F^{*n},$$

where $F^{*0} = \delta_0$. Let M be a not necessarily bounded Borel measure vanishing at 0. If there exists a representation

$$M = \sup_n F_n,$$

where F_n are bounded and the sequence $\{e(F_n)\}$ of associated Poisson measures is shift compact, then each cluster point of $\{e(F_n)\}$ will be called a *generalized Poisson measure* and denoted by $\tilde{e}(M)$. The measure $\tilde{e}(M)$ is uniquely defined up to a shift transformation, i.e. for two cluster points, say μ_1 and μ_2, of translates of $\{e(F_n)\}$ there exists an element $x \in X$ such that $\mu_1 = \mu_2 * \delta_x$.

Clearly, $\tilde{e}(M_1) = \tilde{e}(M_2)$ implies $M_1 = M_2$. Moreover, if $\tilde{e}(M)$ is full, then so is M.

For each pair $\tilde{e}(M)$, $\tilde{e}(N)$ of generalized Poisson measures and each $A \in \mathscr{B}(X)$ we can easily check the equations

$$\tilde{e}(M) * \tilde{e}(N) = \tilde{e}(M+N), \qquad A\tilde{e}(M) = \tilde{e}(\widetilde{AM}),$$

where $\widetilde{AM}(S) = AM(S)$ for all Borel subsets S of $X \setminus \{0\}$.

By a *Gaussian measure* on X we mean a measure ϱ such that, for every $x^* \in X^*$, the induced measure $x^*\varrho$ on $\boldsymbol{R}$ is Gaussian.

Tortrat proved in [10], p. 311, the following analogue of the Lévy-Khintchine representation: $\mu \in \mathscr{P}(X)$ is infinitely divisible, i.e. for every positive integer n there exists a probability measure μ_n on X such that $\mu_n^{*n} = \mu$ if and only if μ has a unique representation $\mu = \varrho * \tilde{e}(M)$, where ϱ is a symmetric Gaussian measure on X and $\tilde{e}(M)$ is a generalized Poisson measure.

If ϱ is a Gaussian measure on X, then the characteristic functional of ϱ takes the form

$$\hat{\varrho}(x^*) = \exp\{ix^*(x) - \tfrac{1}{2}x^*(Rx^*)\} \qquad (x^* \in X^*),$$

where x is the mean value of ϱ and R is its covariance operator, i.e. a nuclear operator from X^* into X with the following properties: $x_1^*(Rx_2^*)$

$= x_2^*(Rx_1^*)$ for all $x_1^*, x_2^* \in X^*$ (symmetry) and $x^*(Rx^*) \geqslant 0$ (non-negativity) (see [1]). Clearly, if R is the covariance operator of ϱ and $A \in \mathscr{B}(X)$, then ARA^* is the covariance operator of $A\varrho$. Moreover, for any Gaussian measures ϱ_1 and ϱ_2 with the covariances R_1 and R_2, respectively, the covariance of $\varrho_1 * \varrho_2$ is equal to $R_1 + R_2$.

We note that ϱ is full if and only if its covariance operator R is one-to-one.

If X is a Hilbert space, then every Gaussian covariance is a positive definite, nuclear operator on X.

2. Let μ be a probability measure on a real separable Banach space X and let $\mathscr{A}$ be a subgroup of $\mathscr{U}(X)$. We say that μ is *$\mathscr{A}$-stable* if for any $A, B \in \mathscr{A}$ there exist a $C \in \mathscr{A}$ and a point $x \in X$ such that

$$A\mu * B\mu = C\mu * \delta_x.$$

The notion of stability with respect to arbitrary groups of automorphisms in the case of probability measures on locally compact groups was introduced by Parthasarathy and Schmidt in [7]. Earlier (in [6]) Parthasarathy investigated probability distributions on $\boldsymbol{R}^n$ which are stable under the group of all invertible linear transformations of $\boldsymbol{R}^n$ onto itself. The theory of $\mathscr{A}$-stable measures on Euclidean spaces has been presented in [8]. In particular, [8] contains the Lévy-Khintchine formula for such measures.

It is easy to see that if μ is $\mathscr{A}$-stable, then for any finite set $A_1, \ldots, A_n$ of elements of $\mathscr{A}$ there exist a $C_n \in \mathscr{A}$ and a point $x_n \in X$ for which

$$A_1\mu * \ldots * A_n\mu = C_n\mu * \delta_{x_n}.$$

Consequently, for each positive integer n there are $C_n \in \mathscr{A}$ and $x_n \in X$ such that $\mu^{*n} = C_n\mu * \delta_{x_n}$, e.g. μ is operator-stable in the sense of Sharpe (see [9]). For full measures on $\boldsymbol{R}^n$ the converse is also true. Namely, a full probability measure μ on $\boldsymbol{R}^n$ is operator-stable if and only if μ is stable under a one-parameter subgroup of $\mathscr{U}(\boldsymbol{R}^n)$ (see [9], Theorem 2).

PROPOSITION 2.1. *Let $\mathscr{A}$ be a subgroup of $\mathscr{U}(X)$ and let μ be a probability measure on X which is stable under $\mathscr{A}$. Then μ is infinitely divisible. Let $\mu = \varrho * \tilde{e}(M)$ be the decomposition of μ into its symmetric Gaussian part ϱ and its generalized Poisson part $\tilde{e}(M)$. Then both ϱ and $\tilde{e}(M)$ are stable under $\mathscr{A}$.*

Proof. The proposition follows immediately from the operator stability of μ and from the uniqueness of the Tortrat representation.

We note that a generalized Poisson measure $\tilde{e}(M)$ associated with the measure M is stable under $\mathscr{A}$ if and only if for any pair $A, B \in \mathscr{A}$ there exists a $C \in \mathscr{A}$ such that

$$AM + BM = CM. \tag{2.1}$$

Let ϱ be a Gaussian measure with the covariance operator R. It is clear that ϱ is $\mathscr{A}$-stable if and only if for any $A, B \in \mathscr{A}$ there is a $C \in \mathscr{A}$ for which the equality $ARA^* + BRB^* = CRC^*$ holds.

Let μ be a probability measure on X which is stable under a subgroup $\mathcal{A}$ of $\mathcal{U}(X)$ and assume that λ is a real number from $(-1, 1)$ such that $\lambda I \in \mathcal{A}$. Then the operator λI belongs to $\mathcal{D}(\mu)$. We infer this from the following

PROPOSITION 2.2. *Let μ be an $\mathcal{A}$-stable probability measure on X and assume that A is an operator from $\mathcal{A}$ such that*

(i) *the sequence $\{A^n\}$ converges strongly to* 0;

(ii) *A commutes with every element of $\mathcal{A}$.*

Then A belongs to $\mathcal{D}(\mu)$.

Proof. Since μ is $\mathcal{A}$-stable and $A \in \mathcal{A}$, there exist a sequence $\{C_n\}$ in $\mathcal{A}$ and a sequence $\{x_n\}$ in X such that

$$A\mu * A^2\mu * \ldots * A^n\mu = C_n\mu * \delta_{x_n} \quad \text{for all } n.$$

Hence we get the formula

$$(2.2) \quad C_n^{-1}A\mu * C_n^{-1}A^2\mu * \ldots * C_n^{-1}A^n\mu = \mu * \delta_{C_n^{-1}x_n} \quad (n = 1, 2, \ldots).$$

Since A commutes with C_n^{-1}, it is easy to obtain from (2.2) the following equation:

$$(2.3) \quad \mu * \delta_{C_n^{-1}x_n} * C_n^{-1}A^{n+1}\mu = C_n^{-1}A\mu * A\mu * \delta_{AC_n^{-1}x_n} \quad (n = 1, 2, \ldots).$$

Moreover, from (2.2) and Theorem 5.1, Chapter III in [5], we infer that the sequence $\{C_n^{-1}A\mu\}$ is shift compact and, consequently (by (i) $A^n \to 0$ strongly), there exists a sequence $\{y_n\}$ in X for which

$$C_n^{-1}A^{n+1}\mu * \delta_{y_n} = A^n C_n^{-1}A\mu * \delta_{y_n} \Rightarrow \delta_0.$$

Thus, setting $v_n = C_n^{-1}A^{n+1}\mu * \delta_{y_n}$ and $\mu_n = C_n^{-1}A\mu * \delta_{AC_n^{-1}x_n + y_n - C_n^{-1}x_n}$ $(n = 1, 2, \ldots)$, by (2.3) we have the formula $\mu * v_n = A\mu * \mu_n$, where $v_n \Rightarrow \delta_0$. Since in this case the sequence $\{\mu_n\}$ must be compact (see Theorem 2.1, Chapter III in [5]), it is then clear that there exists a $v \in \mathcal{P}(X)$ for which $\mu = A\mu * v$. This completes the proof.

We say that a probability measure $\mu \in \mathcal{P}(X)$ is *self-decomposable* if the inclusion $\{\lambda I \colon \lambda \in [0, 1]\} \subset \mathcal{D}(\mu)$ holds (see [2]). Thus, if μ is $\mathcal{U}(X)$-stable, then μ is self-decomposable. Moreover, for any $\lambda \in (-1, 0)$ we have also $\lambda I \in \mathcal{D}(\mu)$, and since $\mathcal{D}(\mu)$ is closed, the same is true for $\lambda = -1$. But it is easy to prove that $-I \in \mathcal{D}(\mu)$ if and only if μ is a translation of a symmetric probability measure. Thus, every $\mathcal{U}(X)$-stable measure is in addition a translation of a symmetric one.

Moreover, the following statement is true:

PROPOSITION 2.3. *Let μ be a symmetric $\mathcal{A}$-stable probability measure on X. Then for any finite group $\mathcal{G} \subset \mathcal{A}$ there exists a $T \in \mathcal{A}$ such that $(T^{-1}AT)\mu = \mu$ for all $A \in \mathcal{G}$.*

For the proof see [6], Lemma 3.

Further, we shall need the following lemma:

LEMMA 2.1. *Let N be an infinite-dimensional subspace of X. Then there exists an $A \in \mathscr{U}(X)$ such that the linear manifold $A(N)+N$ is dense in X.*

Proof. Let $\{x_n\}$ be a countable dense subset of X. Since N is infinite dimensional, we can find a pair of sequences $\{y_n\}$ in N and $\{y_n^*\}$ in X^* such that $y_n^*(y_m) = \delta_{n,m}$ $(n, m = 1, 2, \ldots)$. Choose a sequence $\{a_n\}$ of positive numbers such that

$$\Big\| \sum_{n=1}^{\infty} a_n y_n^*(x) x_n \Big\| < \|x\| \quad \text{for all } x \neq 0.$$

If we define the operator S by

$$Sx = \sum_{n=1}^{\infty} a_n y_n^*(x) x_n \quad (x \in X),$$

then $\|S\| < 1$ and $Sa_n^{-1} y_n = x_n$ for all n. Consequently, $S(N) \supset \{x_n\}$. Put $A = I+S$. Obviously, A is invertible and $A(N)+N \supset \{x_n\}$. Thus the lemma is proved.

An application of Lemma 2.1 leads to the following

THEOREM 2.1. *Let μ be a non-degenerate $\mathscr{U}(X)$-stable probability measure on a real separable Banach space X. Then μ is full.*

Proof. Denote by N the smallest closed subspace of X for which there exists an element x_0 such that μ is concentrated on the hyperplane $N+x_0$. Since μ is $\mathscr{U}(X)$-stable, for any $A \in \mathscr{U}(X)$ there exists an operator $B \in \mathscr{U}(X)$ such that the formula

$$\overline{A(N)+N} = \overline{B(N)} = B(N) \tag{2.4}$$

holds. Under our assumption $N \neq \{0\}$. We shall prove that $N = X$. Indeed, if $N \neq X$, we can choose an operator A from $\mathscr{U}(X)$ in such a way that $A(N) \neq N$. Then from (2.4) it follows that N (and X) must be infinite dimensional. But in this case $A(N)+N$ is dense in X for some $A \in \mathscr{U}(X)$ (Lemma 2.1). Hence and from (2.4) we infer that there exists an operator $B \in \mathscr{U}(X)$ such that $B(N) = X$, which contradicts the assumption $N \neq X$. Thus the theorem is proved.

A probability measure μ on X is said to be *completely stable* if for any pair $A, B \in \mathscr{B}(X)$ there exist $C \in \mathscr{B}(X)$ and $x \in X$ such that

$$A\mu * B\mu = C\mu * \delta_x. \tag{2.5}$$

We note that any non-degenerate completely stable distribution μ on the Euclidean space $\boldsymbol{R}^n$ is full. Consequently, if $A, B \in \mathscr{U}(\boldsymbol{R}^n)$, then $C\mu$, where C satisfies (2.5), is also full. Hence $C(\boldsymbol{R}^n) = \boldsymbol{R}^n$, i.e. the operator C is invertible. Thus, every completely stable measure on $\boldsymbol{R}^n$ is $\mathscr{U}(\boldsymbol{R}^n)$-stable. Since any full Gaussian measure on $\boldsymbol{R}^n$ is completely stable, the converse is also true.

Completely stable measures on the infinite-dimensional Hilbert space were investigated in [4]. In particular, in [4] it is shown that in this case there exist even completely stable distributions which are not infinitely divisible. Moreover, a characterization of completely stable Gaussian measures in terms of proper values of their covariance operators is given. Namely, it is shown (see [4], Theorem 3) that a non-degenerate Gaussian measure ϱ on the infinite-dimensional separable Hilbert space H is completely stable if and only if its covariance operator has infinitely many positive eigenvalues $a_1 \geqslant a_2 \geqslant \ldots$ (i.e. ϱ is not concentrated on a finite-dimensional hyperplane of H) and the sequence a_n/a_{2n} $(n = 1, 2, \ldots)$ is bounded.

Further, by H we denote a real separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. We shall prove that the non-degenerate Gaussian measures on H which are $\mathscr{U}(H)$-stable are precisely the full Gaussian completely stable measures. The problem of characterization of $\mathscr{U}(X)$-stable Gaussian measures on an arbitrary Banach space is still open.

The following propositions are true:

PROPOSITION 2.4. *Let $a_1 \geqslant a_2 \geqslant \ldots$ and $b_1 \geqslant b_2 \geqslant \ldots$ be the sequences of eigenvalues of covariance operators S and ASA^* $(A \in \mathscr{B}(H))$, respectively. Then the inequality $b_n \leqslant \|A\|^2 a_n$ $(n = 1, 2, \ldots)$ holds. In particular, if $A \in \mathscr{U}(H)$, we also have $a_n \leqslant \|A^{-1}\|^2 b_n$ for all n.*

For the proof see [4], Lemma 2.

PROPOSITION 2.5. *Let S_1 and S_2 be one-to-one covariance operators with the corresponding sequences of eigenvalues $a_1 \geqslant a_2 \geqslant \ldots$, respectively. Then $S_1 = AS_2A^*$ for some $A \in \mathscr{U}(H)$ if and only if the sequence $\{\max\{a_n/b_n, b_n/a_n\}\}$ is bounded.*

Proof. The necessity follows from Proposition 2.4. To prove the sufficiency we assume that $e_1, e_2, \ldots$ $(f_1, f_2, \ldots)$ is an orthonormal basis of eigenvectors of S_1 (S_2) corresponding to the eigenvalues $a_1, a_2, \ldots$ $(b_1, b_2, \ldots)$, respectively. Further, let U be the unitary operator on H such that $Uf_n = e_n$ $(U^{-1}e_n = f_n)$ for all n. Put

$$H_0 x = \sum_{n=1}^{\infty} \sqrt{a_n/b_n} \langle x, f_n \rangle f_n \quad (x \in H).$$

Since the sequence $\{\max\{a_n/b_n, b_n/a_n\}\}$ is bounded, H_0 is a well-defined linear operator from $\mathscr{U}(H)$. Obviously, H_0 is a Hermitian operator. Consequently, setting $A = UH_0$ we have $A \in \mathscr{U}(H)$ and $A^* = HU^{-1}$. Now it is easy to verify the equation $AS_2A^*e_n = S_1e_n$ $(n = 1, 2, \ldots)$, which shows that $AS_2A^* = S_1$. Thus the proposition is proved.

PROPOSITION 2.6. *Let S be a one-to-one covariance operator on H and $A, B \in \mathscr{U}(H)$. If there exists an operator C from $\mathscr{B}(H)$ such that $ASA^* + BSB^* = CSC^*$, then we can find an invertible operator with the same property.*

Proof. Put $S_1 = ASA^* + BSB^*$. Obviously, the covariance operator S_1 is also one-to-one. By our assumption, $S_1 = CSC^*$. Hence, by Proposition 2.4, we have the formula

$$b_n \leqslant \|C\|^2 a_n \quad (n = 1, 2, \ldots), \tag{2.6}$$

where $a_1 \geqslant a_2 \geqslant \ldots$ and $b_1 \geqslant b_2 \geqslant \ldots$ are the sequences of eigenvalues S and S_1, respectively. We note that $\langle S_1 x, x\rangle \geqslant \langle ASA^* x, x\rangle$ and $\langle S_1 x, x\rangle \geqslant \langle BSB^* x, x\rangle$ for all x. Consequently, if $c_1 \geqslant c_2 \geqslant \ldots$ and $d_1 \geqslant d_2 \geqslant \ldots$ are the sequences of eigenvalues of ASA^* and BSB^*, respectively, then

$$b_n \geqslant \max\{c_n, d_n\} \quad (n = 1, 2, \ldots). \tag{2.7}$$

But by Proposition 2.4 we get

$$c_n \geqslant \|A^{-1}\|^2 a_n, \quad d_n \geqslant \|B^{-1}\|^2 a_n \tag{2.8}$$

for all n. From (2.7) and (2.8) we obtain the inequality

$$b_n \geqslant \max\{\|A^{-1}\|^2 a_n, \|B^{-1}\|^2 a_n\} \quad (n = 1, 2, \ldots)$$

which, together with (2.6) and Proposition 2.5, completes the proof.

THEOREM 2.2. *Let ϱ be a non-degenerate Gaussian measure on a real separable Hilbert space H. Then ϱ is $\mathscr{U}(H)$-stable if and only if it is a full completely stable measure.*

Proof. It follows from Proposition 2.6 that every full Gaussian completely stable probability measure on H is $\mathscr{U}(H)$-stable. Conversely, suppose that ϱ is a non-degenerate Gaussian measure on H which is $\mathscr{U}(H)$-stable. By Theorem 2.1, ϱ is full. Obviously, we may assume that H is infinite dimensional. Let S denote the covariance operator of ϱ and let $a_1 \geqslant a_2 \geqslant \ldots$ be the sequence of eigenvalues of S. Using the same arguments as in the proof of Theorem 1 in [4], we infer from $\mathscr{U}(H)$-stability of ϱ that the sequence $\{a_n/a_{2n}\}$ is bounded. But in this case ϱ is completely stable (Theorem 3 in [4]). Thus the theorem is proved.

Remark 2.1. Let ϱ be a full completely stable Gaussian measure on an infinite-dimensional Hilbert space H. Then the sequence $a_1 \geqslant a_2 \geqslant \ldots$ of eigenvalues of its covariance operator fulfils the condition $\sup_n a_n/a_{2n} < \infty$. Put

$$a = \sup_n a_n/a_{2n}.$$

Let $A, B \in \mathscr{B}(H)$. Then there exists an operator C such that

$$A\varrho * B\varrho = C\varrho * \delta_x$$

for some $x \in X$ and $\|C\| \leqslant a(\|A\| + \|B\|)$. Namely, the operator C constructed in the proof of Theorem 2 in [4] has this property. Moreover, the proof of Proposition 2.6 shows that for $A, B \in \mathscr{U}(H)$ we can find an invertible operator with the same property.

3. Let μ be a probability measure on X and suppose that P is a projector belonging to $\mathscr{D}(\mu)$. Then, by Proposition 1.5 in [11], $I-P$ also belongs to $\mathscr{D}(\mu)$ and the equality

$$\mu = P\mu * (I-P)\mu \tag{3.1}$$

holds.

Let μ be in addition infinitely divisible. Then from (3.1) and the uniqueness of the Tortrat representation of μ one can obtain the following

PROPOSITION 3.1. *Suppose that $\mu = \varrho * \tilde{e}(M)$, where ϱ is a symmetric Gaussian measure with the covariance operator R and $\tilde{e}(M)$ is a generalized Poisson measure. Moreover, let P be a projector on X. Then:*

(i) *$P \in \mathscr{D}(\mu)$ if and only if $P \in \mathscr{D}(\mu) \cap \mathscr{D}(\tilde{e}(M))$;*

(ii) *$P \in \mathscr{D}(\varrho)$ if and only if $R(\operatorname{Im} P^*) \subset \operatorname{Im} P$ and $R(\operatorname{Ker} P^*) \subset \operatorname{Ker} P$;*

(iii) *$P \in \mathscr{D}(\tilde{e}(M))$ if and only if the measure M is concentrated on $\operatorname{Im} P \cup \operatorname{Ker} P$.*

The following lemma is a crucial step in our considerations:

LEMMA 3.1. *Let X be a real separable Banach space of dimension at least two and let $\tilde{e}(M)$ be a non-degenerate generalized Poisson measure on X. If there exists a non-trivial finite-dimensional projector P belonging to $\mathscr{D}(\tilde{e}(M))$, then $\tilde{e}(M)$ is not $\mathscr{U}(X)$-stable.*

Proof. Clearly, by Theorem 2.1 it is sufficient to prove the lemma under the assumption that $\tilde{e}(M)$ is full. Then the measure M is also full.

Suppose that P_1 and P_2 are projectors from $\mathscr{D}(\tilde{e}(M))$. We infer from Proposition 3.1 that M is concentrated on the set

$$(\operatorname{Im} P_1 \cup \operatorname{Ker} P_1) \cap (\operatorname{Im} P_2 \cup \operatorname{Ker} P_2).$$

Consequently, the restrictions $M|\operatorname{Im} P_1$ and $M|\operatorname{Ker} P_1$ are concentrated on the unions

$$(\operatorname{Im} P_1 \cap \operatorname{Im} P_2) \cup (\operatorname{Im} P_1 \cap \operatorname{Ker} P_2), \qquad (\operatorname{Ker} P_1 \cap \operatorname{Im} P_2) \cup (\operatorname{Ker} P_1 \cap \operatorname{Ker} P_2),$$

respectively. Hence, since M is full, $\operatorname{Im} P_1$ is equal to the direct sum of $\operatorname{Im} P_1 \cap \operatorname{Im} P_2$ and $\operatorname{Im} P_1 \cap \operatorname{Ker} P_2$, and $\operatorname{Ker} P_1$ is equal to the direct sum of $\operatorname{Ker} P_1 \cap \operatorname{Im} P_2$ and $\operatorname{Ker} P_1 \cap \operatorname{Ker} P_2$. Consequently, $P_1 P_2 = P_2 P_1$.

Let k be the least positive integer for which there exists a k-dimensional projector belonging to $\mathscr{D}(\tilde{e}(M))$ and let $\mathscr{I}_k$ denote the set of all k-dimensional projectors from $\mathscr{D}(\tilde{e}(M))$. Consider $P_1, P_2 \in \mathscr{I}_k$, $P_1 \neq P_2$. Since P_1 commutes with P_2, the operator $P_1 P_2$ is a projector from $\mathscr{D}(\tilde{e}(M))$. Moreover, the dimension of $P_1 P_2$ is less than k. By the definition of k, this implies $P_1 P_2 = P_2 P_1 = 0$.

Hence, in particular, we infer (note that, for any $P \in \mathscr{I}_k$, $M(\operatorname{Im} P) > 0$ and M is σ-finite) that $\mathscr{I}_k$ is a countable set.

Contrary to the assertion of the lemma, suppose that $\tilde{e}(M)$ is $\mathscr{U}(X)$-stable. Then, by (2.1), given an arbitrary $A \in \mathscr{U}(X)$ we can find $B \in \mathscr{U}(X)$ such that

$$AM + M = BM. \tag{3.2}$$

Now let P be a projector from $\mathscr{D}(\tilde{e}(M))$. Then the projector BPB^{-1} belongs to $\mathscr{D}(B\tilde{e}(M))$. Consequently, since $AM \leqslant BM$ and $M \leqslant BM$ by (3.2), Proposition 3.1 implies $BPB^{-1} \in \mathscr{D}(A\tilde{e}(M)) \cap \mathscr{D}(\tilde{e}(M))$ (note that $\tilde{e}(BM) = B\tilde{e}(M)$ and $\tilde{e}(AM) = A\tilde{e}(M)$). Hence BPB^{-1}, $A^{-1}BPB^{-1}A \in \mathscr{D}(\tilde{e}(M))$. Obviously, if $P \in \mathscr{J}_k$, then also BPB^{-1}, $A^{-1}BPB^{-1}A \in \mathscr{J}_k$. Thus the set $\mathscr{J}_k$ has the following property: for any $A \in \mathscr{U}(X)$ there exists a $P_A \in \mathscr{J}_k$ such that $A^{-1}P_A A \in \mathscr{J}_k$. Fix $P_0 \in \mathscr{J}_k$. Let A be an arbitrary operator from $\mathscr{U}(X)$ and let $A^{-1}P_A A \in \mathscr{J}_k$ for $P_A \in \mathscr{J}_k$. If $P_A \neq P_0$, then $P_A P_0 = 0$ and, consequently, $A^{-1}P_A AA^{-1}P_0 A = 0$. Hence, since $A^{-1}P_A A \in \mathscr{J}_k$, we obtain

$$A^{-1}(\operatorname{Im} P_0) \subset \bigcup_{P \in \mathscr{J}_k} \operatorname{Ker} P. \tag{3.3}$$

If $P_A = P_0$, then $A^{-1}P_0 A \in \mathscr{J}_k$ and, consequently,

$$A^{-1}(\operatorname{Im} P_0) \subset \bigcup_{P \in \mathscr{J}_k} \operatorname{Im} P. \tag{3.4}$$

Since A is arbitrary, (3.3) and (3.4) together imply that

$$\bigcup_{P \in \mathscr{J}_k} (\operatorname{Ker} P \cup \operatorname{Im} P) = X,$$

which contradicts the fact that $\mathscr{J}_k$ is a countable set. The lemma is thus proved.

We note that for any Gaussian measure ϱ on X ($\dim X \geqslant 2$) there are non-trivial finite-dimensional projectors in $\mathscr{D}(\varrho)$. For instance, if ϱ is a full Gaussian measure with the covariance operator R, then every projector of the form $x^*(\cdot)\big(Rx^*/x^*(Rx^*)\big)$ $(x^* \in X^*)$ belongs to $\mathscr{D}(\varrho)$ (this follows, by a simple computation, from Proposition 3.1). Thus, combining Proposition 2.1 and Lemma 3.1 and taking into account Theorem 2.1, we obtain

THEOREM 3.1. *Let X be a real separable Banach space of dimension at least two. Then a non-degenerate $\mathscr{U}(X)$-stable probability measure on X is a full Gaussian measure if and only if there exists a non-trivial finite-dimensional projector in its decomposability semigroup.*

4. Before proceeding to state and prove the main results of this paper we shall establish auxiliary propositions.

PROPOSITION 4.1. *Suppose that, for $n = 1, 2, \ldots$, $\mu_n \in \mathscr{P}(X)$ and $\{A_n\}$, $\{B_n\}$ are two sequences of linear operators on X. If the sequences $\{A_n\mu_n\}$ and $\{B_n\mu_n\}$ are conditionally compact, then so are the sequences $\{(A_n+B_n)\mu_n\}$ and $\{(A_n-B_n)\mu_n\}$. Moreover, if $A_n\mu_n \Rightarrow \nu$ for some $\nu \in \mathscr{P}(X)$ and $B_n\mu_n \Rightarrow \delta_0$, then $(A_n+B_n)\mu_n \Rightarrow \nu$.*

Proof. Since $\{A_n\mu_n\}$ and $\{B_n\mu_n\}$ are conditionally compact, it follows from Theorem 6.7, Chapter II in [5], that given $\varepsilon > 0$ there exist compact sets K_ε^1 and K_ε^2 such that

$$\mu_n\{x\colon A_n x\in K_\varepsilon^1\} > 1-\varepsilon/2, \qquad \mu_n\{x\colon B_n x\in K_\varepsilon^2\} > 1-\varepsilon/2$$

for all n. Then we have

$$\mu_n\{x\colon A_n x\in K_\varepsilon^1 \text{ and } B_n x\in K_\varepsilon^2\} > 1-\varepsilon.$$

We note that

$$\{x\colon A_n x\in K_\varepsilon^1 \text{ and } B_n x\in K_\varepsilon^2\} \subset \{x\colon (A_n+B_n)x\in K_\varepsilon^1+K_\varepsilon^2\}.$$

Consequently,

$$\mu_n\{x\colon (A_n+B_n)x\in K_\varepsilon^1+K_\varepsilon^2\} > 1-\varepsilon \quad \text{for all } n.$$

Since $K_\varepsilon^1+K_\varepsilon^2$ is compact and ε is arbitrary, it follows once more from Theorem 6.7, Chapter II in [5], that $\{(A_n+B_n)\mu_n\}$ is compact.

If A is an operator from $\mathscr{B}(X)$, then $(-A)\mu = \overline{A\mu}$ for any $\mu\in\mathscr{P}(X)$. Hence and from the continuity of the operation $-$ we infer that the compactness of $\{B_n\mu_n\}$ implies the compactness of $\{(-B_n)\mu_n\}$. Consequently, the sequence $\{(A_n-B_n)\mu_n\}$ is also conditionally compact.

Let $A_n\mu_n \Rightarrow v$ and $B_n\mu_n \Rightarrow \delta_0$. By the inequality

$$|((A_n+B_n)\mu_n)^\wedge(x^*)-(A_n\mu_n)^\wedge(x^*)| = |\hat{\mu}_n(A_n^* x^*+B_n^* x^*)-\hat{\mu}_n(A_n^* x^*)|$$

$$\leqslant \int_X |1-e^{iB_n^*x^*(x)}|\,d\mu_n(x) = \int_X |1-e^{ix^*(x)}|\,dB_n\mu_n \qquad (n = 1, 2, \ldots;\ x^*\in X^*)$$

we get

$$\lim_n ((A_n+B_n)\mu_n)^\wedge(x^*) = \lim_n (A_n\mu_n)^\wedge(x^*) = \hat{v}(x^*) \quad \text{for all } x^*\in X^*.$$

Moreover, by the first part of the proposition, the sequence $\{(A_n+B_n)\mu_n\}$ is compact. Hence $(A_n+B_n)\mu_n \Rightarrow v$. The proposition is thus proved.

PROPOSITION 4.2. *Let $\{T_n\}$ be a sequence of one-dimensional operators from $\mathscr{B}(X)$, let $\mu_n\in\mathscr{P}(X)$ $(n = 1, 2, \ldots)$, and assume that*

(i) $T_n\mu_n \Rightarrow \mu\in\mathscr{P}(X)$, $\mu \neq \delta_0$,

(ii) *the sequence of norms $\{\|T_n\|\}$ is bounded.*

Then the sequence $\{T_n\}$ is sequentially compact in the strong operator topology.

Proof. Let T_n be given for each n by the formula $T_n = x_n^*(\cdot)x_n$, where $x_n^*\in X^*$ and x_n is the element of X with $\|x_n\| = 1$. From (ii) it follows that the sequence $\{x_n^*\}$ is bounded and, consequently (X is separable), $\sigma(X^*, X)$-compact. Thus, to prove our statement it is enough to verify that the sequence $\{x_n\}$ is compact.

There exist $a > 0$ and $b > 0$ such that for sufficiently large n we have

(4.1) $$\mu_n\{x\colon \|T_n x\| \geqslant b\} > a.$$

Indeed, if no such a and b exist, then for any $b > 0$ there exists a subsequence of $\{T_n\}$, say $\{T_{n_k}\}$, such that

$$\lim_k \mu_{n_k}\{x\colon \|T_{n_k} x\| \geqslant b\} = \lim_k T_{n_k}\mu_{n_k}\{x\colon \|x\| \geqslant b\} = 0.$$

But, by (i), $T_{n_k}\mu_{n_k} \Rightarrow \mu$. As $\mu \neq \delta_0$, this leads to a contradiction. Moreover, since $\{T_n\mu_n\}$ is conditionally compact, it follows from Theorem 6.7, Chapter II in [5], that given $\varepsilon > 0$ there exists a compact set K_ε such that

(4.2) $$T_n\mu_n(K_\varepsilon) = \mu_n\{x\colon T_n x \in K_\varepsilon\} > 1-\varepsilon$$

for all n. By (4.1) and (4.2) we can find a compact set K and $b > 0$ such that for sufficiently large n, say $n \geqslant N$, we have

$$\mu_n\{x\colon \|T_n x\| \geqslant b \text{ and } T_n x \in K\} > 0.$$

Consequently, since $\|T_n x\| = |x_n^*(x)|$ $(n = 1, 2, \ldots)$, there exists a sequence $\{y_n\}$ in X with $|x_n^*(y_n)| \geqslant b$ and $x_n^*(y_n)x_n \in K$ for $n \geqslant N$. But this implies the compactness of $\{x_n\}$. The proposition is thus proved.

Proposition 4.3. *Let μ be a full measure on X, let P be a one-dimensional projector from $\mathscr{B}(X)$, and $C_n \in \mathscr{B}(X)$ $(n = 1, 2, \ldots)$. If the sequence $\{C_n P\mu\}$ is compact in $\mathscr{P}(X)$, then the sequence $\{C_n P\}$ is compact in $\mathscr{B}(X)$. In particular, if $C_n P\mu \Rightarrow \delta_0$, then $\|C_n P\| \to 0$.*

Proof. First we prove the second part of the proposition. Let $P = x_0^*(\cdot)x_0$, where $x_0^* \in X^*$, $x_0 \in X$, $x_0^*(x_0) = 1$, and $\|x_0\| = 1$. Thus $C_n Px = x_0^*(x)C_n x_0$ $(x \in X;\ n = 1, 2, \ldots)$. Moreover, it is easy to see that if $b < \overline{\lim}_n \|C_n x_0\|$, then the inequality

(4.3) $$\mu\{x\colon |x_0^*(x)|b \geqslant a\} \leqslant \overline{\lim_n}\, \mu\{x\colon |x_0^*(x)|\cdot\|C_n x_0\| \geqslant a\}$$

holds for any $a > 0$. Suppose that $C_n P\mu \Rightarrow \delta_0$. Hence for each $a > 0$ we obtain

$$\lim_n C_n P\mu\{x\colon \|x\| \geqslant a\} = 0.$$

But we have

$$C_n P\mu\{x\colon \|x\| \geqslant a\} = \mu\{x\colon |x_0^*(x)|\cdot\|C_n x_0\| \geqslant a\} \quad (n = 1, 2, \ldots),$$

which together with (4.3) implies that if $b < \overline{\lim}_n \|C_n x_0\|$, then

$$\mu\{x\colon |x_0^*(x)|b \geqslant a\} = 0 \quad \text{for any } a > 0.$$

Since μ is full, the last relation shows that $\|C_n x_0\| \to 0$ and, consequently,

$$\lim_n \|C_n P\| = 0.$$

Now assume that the sequence $\{C_n P\mu\}$ is compact in $\mathscr{P}(X)$. In order to establish that the sequence $\{C_n P\}$ is compact in $\mathscr{B}(X)$ it is enough (by Proposition 4.2 and the second part of Proposition 4.3) to show that the sequence of norms $\{\|C_n P\|\}$ is bounded. But the last assertion follows easily from the fact that if the sequence $\{C_n P\mu\}$ is compact, then for every sequence $\{a_n\}$ chosen so that $a_n > 0$ and $\lim_n a_n = 0$ we have $a_n C_n P\mu \Rightarrow \delta_0$ and, consequently (by the second part of the proposition), $\|a_n C_n P\| \to 0$. Thus the proof is completed.

5. Let $\mathscr{A}$ be a subgroup of $\mathscr{U}(X)$ and let σ be a map from $\mathscr{A} \times \mathscr{A}$ into $\mathscr{A}$. We say that σ is *continuous at* 0 if for any pair of sequences $\{A_n\}$ and $\{B_n\}$ in $\mathscr{A}$ which converge to 0 we have $\sigma(A_n, B_n) \to 0$.

Let μ be a probability measure on X which is stable under $\mathscr{A}$. For each $(A, B) \in \mathscr{A} \times \mathscr{A}$ let $\mathscr{C}(A, B)$ denote the subset of $\mathscr{A}$ consisting of those operators C from $\mathscr{A}$ for which

$$A\mu * B\mu = C\mu * \delta_x \quad \text{for a certain } x \in X.$$

If there exists a selector σ on $\mathscr{A} \times \mathscr{A}$ with $\sigma(A, B) \in \mathscr{C}(A, B)$ for any $A, B \in \mathscr{A}$, which is continuous at 0, then μ is said to be *strongly $\mathscr{A}$-stable.*

Remark 5.1. Let μ be an $\mathscr{A}$-stable measure on X and assume that for any pair of sequences $\{A_n\}$ and $\{B_n\}$ in $\mathscr{A}$ which converge to 0 we can find a sequence $\{C_n\}$ in $\mathscr{A}$ such that

$$A_n \mu * B_n \mu = C_n \mu * \delta_{x_n} \quad \text{for some } x_n \in X \ (n = 1, 2, \ldots)$$

and

$$\lim_n C_n = 0.$$

Then μ is strongly $\mathscr{A}$-stable.

For example, given $A, B \in \mathscr{A}$ it is enough to take as $\sigma(A, B)$ an operator C from $\mathscr{C}(A, B)$ such that

$$\big|\|C\| - r_{A,B}\big| \leqslant \|A\|, \quad \text{where } r_{A,B} = \inf\{\|C\| : C \in \mathscr{C}(A, B)\}.$$

Remark 5.2. Let μ be a full $\mathscr{A}$-stable measure on a finite-dimensional space. Then μ is strongly $\mathscr{A}$-stable.

To see this we note that for full measures μ on the finite-dimensional space X the convergence $A_n \mu \Rightarrow \delta_0$ for $A_n \in \mathscr{B}(X)$ implies

$$\lim_n A_n = 0$$

(it is a simple consequence of statement (ii), p. 120, in [11]).

Moreover, it is clear that any μ which is stable must also be strongly $\{aI\colon a > 0\}$-stable.

PROPOSITION 5.1. *Let μ be a probability measure on X which is strongly $\mathcal{U}(X)$-stable and let $A_n, B_n \in \mathcal{U}(X)$ $(n = 1, 2, \ldots)$. If the sequences of norms $\{\|A_n\|\}$ and $\{\|B_n\|\}$ are bounded, then we can find a sequence $\{C_n\}$ in $\mathcal{U}(X)$ such that*

$$A_n\mu * B_n\mu = C_n\mu * \delta_{x_n} \quad \textit{for some } x_n \in X \ (n = 1, 2, \ldots)$$

and $\{\|C_n\|\}$ is bounded.

Proof. Let μ be a strongly $\mathcal{U}(X)$-stable measure. Given $A, B \in \mathcal{U}(X)$ we put $r_{A,B} = \inf\{\|C\|\colon C \in \mathcal{C}(A, B)\}$. Then for any pair of sequences $\{A_n'\}$ and $\{B_n'\}$ in $\mathcal{U}(X)$ which converge to 0 we have

$$\lim_n r_{A_n', B_n'} = 0.$$

Let $\{A_n\}$, $\{B_n\}$ be two sequences of operators from $\mathcal{U}(X)$ with

$$\sup_n \{\|A_n\| + \|B_n\|\} < \infty.$$

Then, for any sequence of positive real numbers $\{a_n\}$ which converge to 0, we have

$$\lim_n r_{a_n A_n, a_n B_n} = 0.$$

We note that for each $a > 0$ and $A, B \in \mathcal{U}(X)$ we have $r_{aA,aB} = a r_{A,B}$. Consequently, $a_n r_{A_n,B_n} \to 0$ as $n \to \infty$ for every sequence $\{a_n\}$ chosen so that $a_n > 0$ and $\lim_n a_n = 0$. Hence

$$\sup_n r_{A_n,B_n} < \infty.$$

For each n we choose an operator C_n from $\mathcal{C}(A_n, B_n)$ such that $\big|\|C_n\| - r_{A_n,B_n}\big| \leqslant 1$. It is clear that the sequence $\{C_n\}$ has the required properties. The proposition is thus proved.

Now, we are ready to prove the main result of this paper.

THEOREM 5.1. *Let X be a real separable Banach space of dimension at least two and let μ be a non-degenerate probability measure on X which is strongly $\mathcal{U}(X)$-stable. Then μ is full Gaussian.*

Proof. Let μ be a non-degenerate and strongly $\mathcal{U}(X)$-stable measure on X. It is clear that the symmetrized distribution μ^s is also strongly $\mathcal{U}(X)$-stable. In this case for any $A, B \in \mathcal{U}(X)$ there exists a $C \in \mathcal{U}(X)$ such that

$$A\mu^s * B\mu^s = C\mu^s. \tag{5.1}$$

If we prove that μ^s is Gaussian, then it will follow from Cramer's theorem that so is μ. By Theorem 2.1, μ and, consequently, μ^s are full. For simplicity of the notation we put $v = \mu^s$. According to Theorem 3.1 it will be sufficient to show that there is a one-dimensional projector P in $\mathscr{D}(v)$.

Given a projector P, we put $P^\perp = I - P$. It is obvious that $\{I, P - P^\perp\}$ is a finite subgroup of $\mathscr{U}(X)$. Consequently, by Proposition 2.3 there exists a one-dimensional projector P_0 such that $(P_0 - P_0^\perp)v = v$. Put

$$A_n = P_0 + \frac{1}{n}P_0^\perp, \quad B_n = P_0^\perp + \frac{1}{n}P_0 \quad (n = 1, 2, \ldots).$$

Obviously, the operators A_n and B_n belong to $\mathscr{U}(X)$. By (5.1) for every integer n we can find an operator C_n in $\mathscr{U}(X)$ such that

$$A_n v * B_n v = C_n v. \tag{5.2}$$

We note that the sequences of norms $\{\|A_n\|\}$ and $\{\|B_n\|\}$ are bounded. Consequently, by Proposition 5.1 we may assume that also the sequence $\{\|C_n\|\}$ is bounded. From (5.2) we obtain

$$C_n^{-1} A_n v * C_n^{-1} B_n v = v \quad (n = 1, 2, \ldots). \tag{5.3}$$

Hence, by Theorem 2.2, Chapter II in [5], the sequences $\{C_n^{-1} A_n v\}$ and $\{C_n^{-1} B_n v\}$ are shift compact. Since $\hat{v}(x^*) \geqslant 0$ for all $x^* \in X^*$, we infer that $\{C_n^{-1} A_n v\}$ and $\{C_n^{-1} B_n v\}$ are compact. Applying the formula $(P_0 - P_0^\perp)v = v$ and Proposition 4.1 it is easy to prove that for any sequence $\{T_n\}$ in $\mathscr{B}(X)$ the compactness of $\{T_n v\}$ implies the compactness of $\{T_n P_0 v\}$ and $\{T_n P_0^\perp v\}$. Hence, in particular, the sequences $\{C_n^{-1} A_n P_0 v\}$ and $\{C_n^{-1} B_n P_0^\perp v\}$ are compact. Consequently,

$$\frac{1}{n} C_n^{-1} A_n P_0 v \Rightarrow \delta_0 \quad \text{and} \quad \frac{1}{n} C_n^{-1} B_n P_0^\perp v \Rightarrow \delta_0.$$

But for $n = 1, 2, \ldots$ we have

$$C_n^{-1} A_n = C_n^{-1} P_0 + \frac{1}{n} C_n^{-1} P_0^\perp, \quad C_n^{-1} B_n = C_n^{-1} P_0^\perp + \frac{1}{n} C_n^{-1} P_0,$$

$$C_n^{-1} A_n P_0 = C_n^{-1} P_0, \quad C_n^{-1} B_n P_0^\perp = C_n^{-1} P_0^\perp.$$

Thus (5.3) can be rewritten in the form

$$\left(C_n^{-1} P_0 + \frac{1}{n} C_n^{-1} P_0^\perp\right) v * \left(C_n^{-1} P_0^\perp + \frac{1}{n} C_n^{-1} P_0\right) v = v \quad (n = 1, 2, \ldots), \tag{5.4}$$

where the sequences $\{C_n^{-1} P_0 v\}$ and $\{C_n^{-1} P_0^\perp v\}$ are compact. Passing, if necessary, to subsequences we may assume without loss of generality that these sequences are convergent. Moreover, we have

$$\lim_n \frac{1}{n} C_n^{-1} P_0^\perp v = \lim_n \frac{1}{n} C_n^{-1} P_0 v = \delta_0.$$

Hence and from Proposition 4.1 we get

$$\text{(5.5)}\qquad \begin{aligned} \lim_n C_n^{-1} P_0 v &= \lim_n \Big(C_n^{-1} P_0 + \frac{1}{n} C_n^{-1} P_0^{\perp} \Big) v, \\ \lim_n C_n^{-1} P_0^{\perp} v &= \lim_n \Big(C_n^{-1} P_0^{\perp} + \frac{1}{n} C_n^{-1} P_0 \Big) v. \end{aligned}$$

On the other hand, from (5.4) by a simple computation we obtain

$$C_n^{-1} P_0 C_n v = C_n^{-1} P_0 v * \frac{1}{n} C_n^{-1} P_0 v, \qquad C_n^{-1} P_0^{\perp} C_n v = C_n^{-1} P_0^{\perp} v * \frac{1}{n} C_n^{-1} P_0^{\perp} v$$

$$(n = 1, 2, \ldots)$$

and, consequently,

$$\text{(5.6)}\quad \lim_n C_n^{-1} P_0 C_n v = \lim_n C_n^{-1} P_0 v, \qquad \lim_n C_n^{-1} P_0^{\perp} C_n v = \lim_n C_n^{-1} P_0^{\perp} v.$$

Thus (5.4)-(5.6) together imply that

$$\text{(5.7)}\qquad \lim_n C_n^{-1} P_0 C_n v * \lim_n C_n^{-1} P_0^{\perp} C_n v = v.$$

Since $\{C_n^{-1} P_0 v\}$ is convergent, we infer from Proposition 4.3 that $\{C_n^{-1} P_0\}$ is compact. Consequently, by the assumption on the sequence $\{\|C_n\|\}$ the norms $\|C_n^{-1} P_0 C_n\|$ $(n = 1, 2, \ldots)$ are bounded in common. This together with the convergence of $\{C_n^{-1} P_0 C_n v\}$ proves (by Proposition 4.2) that the sequence $\{C_n^{-1} P_0 C_n\}$ has a strongly convergent subsequence. Denoting by P its limit, by (5.7) we have

$$Pv * \lim_n C_n^{-1} P_0^{\perp} C_n v = v,$$

which shows that $P \in \mathscr{D}(v)$. Obviously, P is a projector from $\mathscr{B}(X)$ of dimension at most one. Moreover, $P \neq 0$. Indeed, if $P = 0$, then there exists a subsequence of indices $n_1 < n_2 < \ldots$ for which $\{C_{n_k}^{-1} P_0 C_{n_k}\}$ converges strongly to 0 and, consequently,

$$\lim_k C_{n_k}^{-1} P_0 v = \lim_k C_{n_k}^{-1} P_0 C_{n_k} v = \delta_0.$$

But Proposition 4.2 then implies that $\|C_{n_k}^{-1} P_0\| \to 0$, which contradicts the fact that the sequence $\{\|C_n\|\}$ is bounded. The theorem is thus proved.

Remark 5.3. In the statement of Theorem 5.1 we have assumed that μ is strongly $\mathscr{U}(X)$-stable. However, it is enough to assume that μ is $\mathscr{U}(X)$-stable and has the property proved in Proposition 5.1.

From Remark 2.1 it follows that if H is a Hilbert space, then every $\mathscr{U}(H)$-stable Gaussian measure ϱ is also strongly $\mathscr{U}(H)$-stable. Thus, combining Theorems 2.2 and 5.1 and taking into account Theorem 3 in [4], we get a characterization of strongly $\mathscr{U}(H)$-stable measures on the infinite-dimensional Hilbert space.

THEOREM 5.2. *Suppose that the Hilbert space H is infinite dimensional. Then a non-degenerate probability measure μ on H is strongly $\mathscr{U}(H)$-stable if and only if μ is full Gaussian and the sequence $a_1 \geqslant a_2 \geqslant \ldots$ of eigenvalues of its covariance operator fulfils the condition* $\sup_n a_n/a_{2n} < \infty$.

REFERENCES

[1] S. Chevet, S. A. Chobanjan, W. Linde et V. T. Tarieladze, *Caractérisation de certaines classes d'espaces de Banach par les mesures gaussiennes*, C. R. Acad. Sci. Paris, Sér. A, 285 (1977), p. 793-796.

[2] A. Kumar and B. M. Schreiber, *Self-decomposable probability measures on Banach spaces*, Studia Math. 53 (1975), p. 55-71.

[3] B. Mincer, *On* GL(n, R)*-stable measures*, Probability and Mathematical Statistics 2 (1982), p. 193-195.

[4] — and K. Urbanik, *Completely stable measures on Hilbert spaces*, Colloq. Math. 42 (1979), p. 301-307.

[5] K. R. Parthasarathy, *Probability measures on metric spaces*, New York—London 1967.

[6] — *Every completely stable distribution is normal*, Sankhyā, Series A, 35 (1973), p. 35-38.

[7] — and K. Schmidt, *Stable positive functions*, Trans. Amer. Math. Soc. 203 (1975), p. 161-174.

[8] K. Schmidt, *Stable probability measures on* R^v, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 33 (1975), p. 19-31.

[9] M. Sharpe, *Operator-stable probability distributions on vector groups*, Trans. Amer. Math. Soc. 136 (1969), p. 51-65.

[10] A. Tortrat, *Structure des lois indéfiniment divisibles dans un espace vectoriel topologique (séparé)* X, Symposium on Probability Methods in Analysis; p. 299-328 in: Lecture Notes in Math. 31, Berlin—Heidelberg—New York 1967.

[11] K. Urbanik, *Lévy's probability measures on Euclidean spaces*, Studia Math. 44 (1972), p. 119-148.

[12] — *Operator semigroups associated with probability measures*, Bull. Acad. Polon. Sci., Sér. Sci. Math., Astronom. Phys., 23 (1975), p. 75-76.

Institute of Mathematics, Wrocław University
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

Received on 3. 4. 1980