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UNIVERSAL MULTIPLY SELF-DECOMPOSABLE PROBABILITY MEASURES ON BANACH SPACES

BY

NGUYEN VAN THU (HANOI)

Abstract. The aim of the present paper* is to prove the existence of universal Doéblin probability measures for classes of multiply self-decomposable probability measures on Banach spaces.

1. Introduction and notation. Throughout the paper we shall denote by X a real separable Banach space with the norm $\|\cdot\|$ and the topological dual space X^*. Given $r, s > 0$, let

$$B_{r,s} = \{x \in X: r < \|x\| \leqslant s\}, \quad B_s = \{x \in X: \|x\| < s\},$$

and let B_s' be the complement of B_s. We shall consider only σ-additive measures defined on Borel subsets of X. Given a bounded linear operator A and a measure μ on X let $A\mu$ denote a measure defined by

$$(A\mu)(E) = \mu(A^{-1}E) \quad (E \subset X).$$

In particular, if $Ax = cx$ $(x \in X)$ for some $c \in R^1$, then $A\mu$ will be denoted by the usual symbol $T_c \mu$.

Let $L_0(X)$ denote the class of all infinite divisible (i.d.) probability measures (p.m.'s) on X endowed with the weak convergence $\Rightarrow$. It is well known [3] that for every measure $\mu \in L_0(X)$ its characteristic functional (ch.f.) $\hat{\mu}$ has a unique representation

$$(1.1) \quad \hat{\mu}(y) = \exp\{i\langle x_0, y\rangle - \tfrac{1}{2}\langle Ry, y\rangle + \int_X k(x, y) M(dx)\} \quad (y \in X^*),$$

where x_0 is a vector in X, R a covariance operator corresponding to the

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symmetric Gaussian component of μ, and M a Lévy measure on X. The kernel k is given by the formula

$$(1.2)\qquad k(x, y) = e^{i\langle x,y\rangle}-1-i\langle x, y\rangle 1_{B_1}(X) \quad (x\in X, y\in X^*),$$

where 1_E denotes the indicator of a subset E of X.

In the sequel we shall identify μ with the triple $[x_0, R, M]$ in (1.1). In particular, if $x_0 = 0$ and $R = 0$, then μ will be denoted simply by $[M]$ which, for a finite measure M, is of the form

$$[M](E) = e(M)(E) = e^{-M(X)} \sum_{k=0}^{\infty} M^{*k}(E)/k! \quad (E \subset X),$$

where the asterisk $*$ denotes the convolution operation. Further, if $t > 0$ and $\mu = [x_0, R, M]$, then we denote by μ^t the p.m. $[tx_0, tR, tM]$.

A p.m. μ on X is called *self-decomposable* if for every $c\in(0, 1)$ there exists a p.m. μ_c such that

$$(1.3)\qquad \mu = T_c\mu * \mu_c.$$

Note [7] that if μ is self-decomposable, then μ and μ_c are both i.d.

Multiply self-decomposable p.m.'s were studied by Urbanik [12] on R^1 and by the author [9] on general Banach spaces. Recall [9] that a p.m. μ on X is said to be *n times self-decomposable* if for every $c\in(0, 1)$ the decomposition (1.3) holds, where the measure μ_c is $n-1$ times self-decomposable. Let $L_n(X)$ denote the class of all n times ($n = 1, 2, \ldots$) self-decomposable p.m.'s on X. The class $L_\infty(X)$ of *completely self-decomposable* p.m.'s on X is defined as the intersection of all $L_n(X)$, $n = 1, 2, \ldots$

In the sequel we shall extend the definition of classes $L_n(X)$ to the fractional case $L_\alpha(X)$ ($\alpha > 0$) by introducing operators J^α ($\alpha > 0$) on some σ-finite measures on X. Such operators stand for some analogues of ordinary fractional integration on functions.

In [4] Doéblin proved that there exists an i.d.p.m. belonging to the domain of partial attraction of every one-dimensional i.d.p.m. A natural generalization of this theorem in a Hilbert space was done by Barańska [1], and in a Banach space by Ho Dang Phuc [6]. A new version of the theorem was obtained by the author. Namely, in [10] we presented an operator approach to Doéblin's theorem.

Let A be a bounded linear operator on X and let K be a subclass of $L_0(X)$. A p.m. μ on X is said to be *A-universal* for K if $\mu\in K$ and for every p.m. $\nu\in K$ there exist subsequences $\{n_k\}$ and $\{m_k\}$ of natural numbers such that the sequence $\{A^{n_k}\mu^{m_k}\}$ is shift convergent to ν. The case $K = L_0(X)$ was treated in [10]. In the sequel we shall prove the existence of A-universal p.m.'s for $L_\alpha(X)$ ($\alpha > 0$). Our results are new even in the one-dimensional case.

2. Fractional calculus on semi-finite measures on X. The starting point to this study is a known formula for Lévy measures corresponding to n times ($n = 1, 2, \ldots$) self-decomposable p.m.'s on X. Namely, in [9], formula (5.2), we proved that $\mu \in L_n(X)$ ($n = 1, 2, \ldots$) if and only if its Lévy measure M is of the form

$$M(E) = \int_X c_n(x) \int_0^\infty 1_E(e^{-t}x) t^{n-1} dt m(dx) \quad (E \subset X), \tag{2.1}$$

where m is a finite measure on X vanishing at 0 and

$$c_n^{-1}(x) = \int_0^\infty \Phi(e^{-t}x) t^{n-1} dt \quad (x \in X), \tag{2.2}$$

Φ being a weight function on X in Urbanik's sense [13].

Putting $G(dx) = (n-1)!\, c_n(x) m(dx)$ and taking into account (2.1) we get a measure G which is finite on every B_r' ($r > 0$), $G(\{0\}) = 0$, and

$$M(E) = \frac{1}{(n-1)!} \int_X \int_0^\infty 1_E(e^{-t}x) t^{n-1} dt G(dx) \quad (E \subset X), \tag{2.3}$$

where the constant $(n-1)!$ is introduced for further convenience.

Let $M(X)$ denote the class of all σ-finite measures M on X such that $M(\{0\}) = 0$ and $M(B_r') < \infty$ for every $r > 0$. A sequence $\{M_n\} \subset M(X)$ is said to be *convergent* to M if $M_n | B_r'$ converges weakly to $M | B_r'$ for every $r > 0$.

Formula (2.3) suggests a more general setting. Namely, for $\alpha > 0$ and $G \in M(X)$ we put

$$J^\alpha G(E) = \frac{1}{\Gamma(\alpha)} \int_X \int_0^\infty 1_E(e^{-t}x) t^{\alpha-1} dt G(dx) \tag{2.4}$$

for all Borel subsets E of X.

It is evident that for any $G_1, G_2 \in M(X)$, $a > 0$, and for every linear bounded operator A on X we have

$$J^\alpha(aAG_1 + G_2) = aAJ^\alpha G_1 + J^\alpha G_2. \tag{2.5}$$

Further, we have the following

2.1. Proposition. *For any $\alpha > 0$ and $G \in M(X)$, $J^\alpha G \in M(X)$ if and only if*

$$\int_{B_1'} \log^\alpha \|x\|\, G(dx) < \infty. \tag{2.6}$$

Moreover, if $J^\alpha G \in M(X)$, then for every $p > 0$ we have

$$\int_{B_1} \|x\|^p G(dx) < \infty$$

if and only if

$$\int_{B_1} \|x\|^p J^\alpha G(dx) < \infty.$$

Proof. Given $r > 0$, by (2.4) and by a simple computation we have

$$J^{\alpha} G(B_r') = \frac{1}{\Gamma(\alpha+1)} \int_{B_r'} \log^{\alpha} \|x\| r^{-1} G(dx), \tag{2.7}$$

which implies the first part of the proposition. Further, for any $p > 0$ and $\delta > 0$ we obtain

$$\begin{aligned}(2.8)\quad & \int_{B_\delta} \|x\|^p J^{\alpha} G(dx) \\ & = p^{-\alpha} \int_{B_\delta} \|x\|^p G(dx) + \frac{\delta^p}{\Gamma(\alpha)} \int_{B_\delta'} \int_0^{\infty} e^{-tp} (t + \log \|x\| \delta^{-1})^{\alpha-1} dt G(dx),\end{aligned}$$

which, by a simple reasoning, implies the second part of the proposition. Thus the proof is completed.

2.2. Theorem. *For any $\alpha, \beta > 0$ and $G \in M(X)$, $J^{\alpha} G$, $J^{\beta} J^{\alpha} G \in M(X)$ if and only if $J^{\alpha+\beta} G \in M(X)$. In any case we have*

$$J^{\alpha+\beta} G = J^{\beta} J^{\alpha} G. \tag{2.9}$$

Proof. For any $\alpha, \beta > 0$ and $G \in M(X)$ we have

$$\begin{aligned}\int_{B_1'} \log^{\beta} \|x\| J^{\alpha} G(dx) &= \frac{1}{\Gamma(\alpha)} \int_X \int_0^{\infty} 1_{B_1'}(e^{-t} x) \log^{\beta} \|e^{-t} x\| t^{\alpha-1} dt G(dx) \\ &= \frac{1}{\Gamma(\alpha)} \int_{B_1'} \int_0^{\log\|x\|} (\log \|x\| - t)^{\beta} t^{\alpha-1} dt G(dx) \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \int_{B_1'} \log^{\alpha+\beta} \|x\| G(dx).\end{aligned}$$

Thus

$$\int_{B_1'} \log^{\beta} \|x\| J^{\alpha} G(dx) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \int_{B_1'} \log^{\alpha+\beta} \|x\| G(dx), \tag{2.10}$$

which, by Proposition 2.1, implies that $J^{\alpha+\beta} G \in M(X)$ iff $J^{\alpha} G$, $J^{\beta} J^{\alpha} G \in M(X)$.

On the other hand, by (2.4), for every Borel subset E of X we get

$$\begin{aligned}J^{\beta} J^{\alpha} G(E) &= \frac{1}{\Gamma(\beta)} \int_X \int_0^{\infty} 1_E(e^{-t} x) t^{\beta-1} dt J^{\alpha} G(dx) \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_X \int_0^{\infty} \int_0^{\infty} 1_E(e^{-t-s} x) t^{\beta-1} s^{\alpha-1} dt ds G(dx) \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_X \int_0^{\infty} \int_0^{u} 1_E(e^{-u} x)(u-s)^{\beta-1} s^{\alpha-1} ds du G(dx) \\ &= \frac{1}{\Gamma(\alpha+\beta)} \int_X \int_0^{\infty} 1_E(e^{-t} x) t^{\alpha+\beta-1} dt G(dx) = J^{\alpha+\beta} G(E),\end{aligned}$$

which proves (2.9). Thus the proof is completed.

The following theorem can be considered as an analogue of the Dominated Convergence Theorem for ordinary integrals.

2.3. THEOREM. *Suppose that for* $\alpha > 0$ *and* $G_n, G, H \in M(X)$ *we have* $G_n \Rightarrow G$, $G_n \leqslant H$ *for every* $n = 1, 2, \ldots$, *and* $J^\alpha H \in M(X)$. *Then*

$$J^\alpha G_n \Rightarrow J^\alpha G.$$

Proof. From the assumption that $J^\alpha H \in M(X)$ it follows that for every $r > 0$

$$J^\alpha H(B_r') = \frac{1}{\Gamma(\alpha)} \int_0^\infty H(B'_{re^t}) t^{\alpha-1} dt < \infty. \tag{2.11}$$

Let f be a bounded continuous function on B_r' with

$$C = \sup\{|f(x)|: x \in B_r'\}.$$

Then for every $t > 0$ we have

$$\Big|\int_{B'_{re^t}} f(e^{-t}x) G_n(dx)\Big| \leqslant CG_n(B'_{re^t}) \leqslant CH(B'_{re^t}) \quad (n = 1, 2, \ldots). \tag{2.12}$$

Further, since $G_n \Rightarrow G$, we have

$$\lim_{n\to\infty} \int_{B'_{re^t}} f(e^{-t}x) G_n(dx) = \int_{B'_{re^t}} f(e^{-t}x) G(dx),$$

which together with (2.11) and (2.12), and the Dominated Convergence Theorem implies

$$\lim_{n\to\infty} \int_{B_r'} f(x) J^\alpha G_n(dx) = \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_{B'_{re^t}} f(e^{-t}x) G_n(dx) t^{\alpha-1} dt$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_{B'_{re^t}} f(e^{-t}x) G(dx) t^{\alpha-1} dt = \int_{B_r'} f(x) J^\alpha G(dx).$$

Consequently, $J^\alpha G_n \Rightarrow J^\alpha G$. Thus the theorem is proved.

2.4. COROLLARY. *Suppose that* $G_n, G \in M(X)$, $G_n \Rightarrow G$ *and, for some* $s > 0$, G_n $(n = 1, 2, \ldots)$ *are concentrated on* B_s. *Then for every* $\alpha > 0$ *we have* $J^\alpha G_n \Rightarrow J^\alpha G$.

Proof. Write $H = \sup G_n$. Then $H \in M(X)$ and H is concentrated on B_s. Now, by Theorem 2.3 we get the Corollary.

2.5. THEOREM. *Suppose that for* $\alpha > 0$ *and* $G_n, G, M_n, M \in M(X)$ *we have* $M_n = J^\alpha G_n$ $(n = 1, 2, \ldots)$, $G_n \Rightarrow G$, *and* $M_n \Rightarrow M$. *Then* $M = J^\alpha G$.

Proof. Choose s, r $(0 < s < r)$ such that $B_{s,r}$ is a continuous set for G.

Then, for every bounded continuous function f on $B_{s,r}$, we have

$$\lim_{n\to\infty} \int_X f(x) M_n | B_{s,r}(dx)$$

$$= \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_{B_{s,r}} \int_0^{\log\|x\|s^{-1}} f(e^{-t}x) t^{\alpha-1} dt G_n(dx) +$$

$$+ \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_{B_r'} \int_{\log\|x\|r^{-1}}^{\log\|x\|s^{-1}} f(e^{-t}x) t^{\alpha-1} dt G_n(dx)$$

$$= \frac{1}{\Gamma(\alpha)} \int_{B_{s,r}} \int_0^{\log\|x\|s^{-1}} f(e^{-t}x) t^{\alpha-1} dt G(dx) + \frac{1}{\Gamma(\alpha)} \int_{B_r'} \int_{\log\|x\|r^{-1}}^{\log\|x\|s^{-1}} f(e^{-t}x) t^{\alpha-1} dt G(dx)$$

$$= \int_X f(x) J^\alpha G | B_{s,r}(dx),$$

which shows that $M | B_{s,r} = J^\alpha G | B_{s,r}$ and, consequently, $M = J^\alpha G$. Thus the theorem is proved.

Given $\alpha > 0$, $c = e^{-t}$, $t > 0$, and $M \in M(X)$ we put

$$\Delta_t^\alpha M(E) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} T_{c^k} M(E) \tag{2.13}$$

for all Borel subsets E of X such that $0 \notin \bar{E}$, where

$$\binom{\alpha}{0} = 1 \quad \text{and} \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} \quad (k = 1, 2, \dots).$$

Since

$$\binom{\alpha}{k} = O(k^{-\alpha-1}), \quad k \to \infty,$$

it follows that if $0 \notin \bar{E}$, then the series (2.13) is absolutely convergent, and hence it defines a signed measure on the field of Borel subsets E of X such that $0 \notin \bar{E}$. It is clear that $\Delta_t^\alpha M$ is σ-additive on every B_r' $(r > 0)$.

In the sequel we shall need the following function on $(0, \infty)$:

$$p_\alpha(x) = \frac{1}{\Gamma(\alpha)} \sum_{0 \leqslant k < x} (-1)^k \binom{\alpha}{k} (x-k)^{\alpha-1} \quad (x \geqslant 0). \tag{2.14}$$

Such a function plays an important role in the study of ordinary fractional integrals. Recall [14] that $p_\alpha \in L^1(0, \infty)$ $(\alpha > 0)$ and

$$\int_0^\infty p_\alpha(x) dx = 1. \tag{2.15}$$

Further, for any $\alpha, t > 0$ we define a signed measure on $(0, \infty)$ by

$$m_t^\alpha(dx) = t^{-1} p_\alpha(x/t)\, dx. \tag{2.16}$$

Then we get the following lemma:

2.6. LEMMA. *For every $\alpha > 0$ the signed measures m_t^α $(t > 0)$ have a common finite variation on $(0, \infty)$ and $m_t^\alpha \Rightarrow \delta_0$ as $t \downarrow 0$.*

Proof. By (2.14), (2.16), and by the fact that $p_\alpha \in L^1(0, \infty)$ the measures m_t^α $(t > 0)$ have a common finite variation on $(0, \infty)$. For $v \geqslant 0$ we put

$$f_{\alpha,t}(v) = \int_0^v m_t^\alpha(du). \tag{2.17}$$

Then, by (2.16) we get the formula

$$f_{\alpha,t}(v) = \frac{t^{-\alpha}}{\Gamma(\alpha+1)} \sum_{0 \leqslant k < v/t} (-1)^k \binom{\alpha}{k} (v-kt)^\alpha. \tag{2.18}$$

Therefore

$$\lim_{t \downarrow 0} f_{\alpha,t}(v) = \begin{cases} 0 & \text{if } v = 0, \\ 1 & \text{if } v > 0, \end{cases} \tag{2.19}$$

which implies that $m_t^\alpha \Rightarrow \delta_0$ as $t \downarrow 0$. Thus the lemma is proved.

2.7. LEMMA. *Suppose that for $\alpha > 0$ and $M, G \in M(X)$ we have $M = J^\alpha G$. Then $t^{-\alpha} \Delta_t^\alpha M \Rightarrow G$ on every B_r' $(r > 0)$ as $t \downarrow 0$.*

Proof. Given $r > 0$ and a bounded continuous function f on B_r' we have, by (2.13), the formulas

$$\begin{aligned} & t^{-\alpha} \int_{B_r'} f(x) \Delta_t^\alpha M(dx) \\ &= \frac{t^{-\alpha}}{\Gamma(\alpha)} \int_X \sum_{k=0}^\infty (-1)^k \binom{\alpha}{k} \int_0^\infty 1_{B_r'}(e^{-s-kt}x) f(e^{-s-kt}x) s^{\alpha-1}\, ds G(dx) \\ &= \frac{t^{-\alpha}}{\Gamma(\alpha)} \int_{B_r'} \int_0^{\log\|x\|/r} f(e^{-u}x) \sum_{0 \leqslant k < u/t} (-1)^k \binom{\alpha}{k} (u-kt)^{\alpha-1}\, du G(dx). \end{aligned}$$

Hence and by (2.14) and (2.16) we get

$$t^{-\alpha} \int_{B_r'} f(x) \Delta_t^\alpha M(dx) = \int_{B_r'} \int_0^{\log\|x\|/r} f(e^{-u}x) m_t^\alpha(du) G(dx),$$

which, by Lemma 2.6, implies that

$$\lim_{t \downarrow 0} t^{-\alpha} \int_{B_r'} f(x) \Delta_t^\alpha M(dx) = \int_{B_r'} f(x) G(dx).$$

Consequently, $t^{-\alpha} \Delta_t^\alpha M \Rightarrow G$ as $t \downarrow 0$. Thus the lemma is proved.

2.8. LEMMA. *Suppose that for $\alpha > 0$ and $M, G \in M(R^1)$ we have*

$$\int_{|x|>1} \log^{\alpha}|x|\, G(dx) < \infty \tag{2.20}$$

and assume that $t^{-\alpha}\Delta_t^{\alpha} M \Rightarrow G$ on every set $\{x \in R^1 \colon |x| > r\}$ $(t > 0)$ as $t \downarrow 0$. Then $M = J^{\alpha} G$.

Proof. We may assume that M and G are both concentrated on $(0, \infty)$. For $u \in R^1$ and $t > 0$ we put

$$q(u) = \int_{e^{-u}}^{\infty} M(dx), \tag{2.21}$$

$$g(u) = \int_{e^{-u}}^{\infty} G(dx), \tag{2.22}$$

$$\Delta_t^{\alpha} q(u) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} q(u-kt). \tag{2.23}$$

By (2.13), (2.21), and (2.23) we get

$$t^{-\alpha}\Delta_t^{\alpha} q(u) = \int_{e^{-u}}^{\infty} t^{-\alpha}\Delta_t^{\alpha} M(dx) \quad (u \in R^1), \tag{2.24}$$

which, by the assumption of the lemma, implies

$$\lim_{t\downarrow 0} t^{-\alpha}\Delta_t^{\alpha} q(u) = \int_{e^{-u}}^{\infty} G(dx) = g(u) \tag{2.25}$$

for every point u of continuity of g. Further, from (2.22) it follows that for every $a \in R^1$

$$\int_{-\infty}^{a} (a-u)^{\alpha-1} g(u)\,du = \alpha^{-1} \int_{e^{-a}}^{\infty} (a+\log x)^{\alpha} G(dx),$$

which, by (2.20), implies that for every $a \in R^1$

$$\int_{-\infty}^{a} (a-u)^{\alpha-1} g(u)\,du < \infty. \tag{2.26}$$

Finally, formulas (2.25) and (2.26) together give an integral representation of q (cf. [8]). Namely,

$$q(u) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{u} (u-t)^{\alpha-1} g(t)\,dt \quad (u \in R^1),$$

which, by (2.21) and (2.22), implies $M = J^{\alpha} G$. Thus the lemma is proved.

2.9. THEOREM. *For any $\alpha > 0$ and $M, G \in M(X)$ the relation $M = J^{\alpha} G$ holds if and only if condition (2.6) is satisfied and $t^{-\alpha}\Delta_t^{\alpha} M \Rightarrow G$ on every B_r' $(r > 0)$ as $t \downarrow 0$. Consequently, the operator J^{α} $(\alpha > 0)$ is one-to-one.*

Proof. The necessity follows from Proposition 2.1 and Lemma 2.7. We prove the sufficiency.

Suppose that for $\alpha > 0$ we have $t^{-\alpha}\Delta_t^\alpha M \Rightarrow G$ on every B'_r $(r > 0)$ as $t \downarrow 0$ and G satisfies (2.6). Given a functional $y \in X^*$ with $\|y\| = 1$ let yM and yG denote projections of M and G on R^1, respectively. By (2.6) it is evident that

$$\int_{|x|>1} \log^\alpha |x| \, yG(dx) < \infty$$

and, moreover, $t^{-\alpha}\Delta_t^\alpha yM \Rightarrow yG$ on every set $\{x \in R^1 \colon |x| > r\}$ $(r > 0)$ as $t \downarrow 0$. Hence and from Lemma 2.8 it follows that $yM = J^\alpha yG$. Consequently, $yM = yJ^\alpha G$, and since y is arbitrarily chosen, we get $M = J^\alpha G$, which completes the proof.

Recall that a Banach space X is of *type* p $(0 < p \leqslant 2)$ if for every sequence $\{x_n\} \subset X$ with $\sum_n \|x_n\|^p < \infty$ the series $\sum_n x_n \varepsilon_n$ converges with probability 1, where $\{\varepsilon_n\}$ is the Rademacher sequence. Every Banach space X is of type 1 and every Hilbert space is of type 2. Further, X is of type p $(0 < p \leqslant 2)$ if and only if every $M \in M(X)$ with $\int_{B_1} \|x\|^p M(dx) < \infty$ is a Lévy measure. Hence and by Proposition 2.1 we get the following

2.10. Proposition. *Suppose that X is of type p $(0 < p \leqslant 2)$, $\alpha > 0$, and $G \in M(X)$ with $\int_{B_1} \|x\|^p G(dx) < \infty$. Then $J^\alpha G$ is a Lévy measure if and only if*

$$\int_{B'_1} \log^\alpha \|x\| \, G(dx) < \infty. \tag{2.27}$$

This proposition implies the following

2.11. Corollary. *For every Lévy measure on a Hilbert space H and for every $\alpha > 0$, $J^\alpha G$ is a Lévy measure if and only if condition* (2.27) *is satisfied.*

3. Universal multiply self-decomposable p.m.'s. Operators J^α $(\alpha > 0)$ defined by (2.4) allow us to subclassify i.d.p.m.'s on X into decreasing subclasses $L_\alpha(X)$ $(\alpha > 0)$ which, for $\alpha = n$ $(n = 1, 2, \ldots)$, coincide with classes of n times self-decomposable p.m.'s on X. Namely, given $\alpha > 0$, we put

$$L_\alpha(X) = \{\mu = [x, R, M] \in L_0(X) \colon M = J^\alpha G \text{ for some } G \in M(X)\}.$$

By (2.3) and Theorem 2.9 we get the following characterization of multiply self-decomposable p.m.'s on X:

3.1. THEOREM. *A p.m.* $\mu = [x, R, M] \in L_0(X)$ *is n times self-decomposable if and only if there exists a* $G \in M(X)$ *such that*

$$\int_{B_1'} \log^n \|x\| \, G(dx) < \infty$$

and $t^{-n} \Delta_t^n M \Rightarrow G$ *as* $t \downarrow 0$ *on every* B_r' $(r > 0)$.

3.2. THEOREM. *For any* $0 < \alpha < \beta$ *we have*

$$L_\beta(X) \subset L_\alpha(X). \tag{3.1}$$

Proof. Suppose that $\mu \in L_\beta(X)$. We prove that $\mu \in L_\alpha(X)$. Clearly, one may assume that $\mu = [M]$, where $M = J^\beta G$ for some $G \in M(X)$. By Theorem 2.2 we have $J^{\beta-\alpha} G \in M(X)$ and $M = J^\alpha J^{\beta-\alpha} G$, which shows that $[M] \in L_\alpha(X)$ and (3.1) is proved.

In [10] we proved that if A is a bounded invertible linear operator on X such that

$$\|A^k\| \to 0 \quad \text{as } k \to \infty, \tag{3.2}$$

then there exists a $\mu \in L_0(X)$ such that μ is A-universal for $L_0(X)$. Moreover, if X is a finite-dimensional space, then from the existence of A-universal p.m.'s for $L_0(X)$ it follows that A is invertible and condition (3.2) is satisfied. The same is true for $L_\alpha(X)$. Namely, we get the following theorems:

3.3. THEOREM. *Let A be a linear operator on* R^d $(d = 1, 2, \ldots)$ *such that for some* $\alpha > 0$ *there exists an A-universal p.m. for* $L_\alpha(R^d)$. *Then A is invertible and condition* (3.2) *is satisfied.*

Proof is the same as the proof of Lemma 1 in [10] and will be omitted.

3.4. THEOREM. *For every invertible bounded linear operator A on X satisfying condition* (3.2) *and for every* $\alpha > 0$ *there exists an A-universal p.m. for* $L_\alpha(X)$.

Proof. Suppose that A is an invertible bounded linear operator on X such that condition (3.2) is satisfied. By (3.2), there exist constants $c > 0$ and $a > 1$ such that

$$\|A^k\| \leqslant c a^{-k} \quad (k = 1, 2, \ldots). \tag{3.3}$$

Given $\alpha > 0$ we infer from the definition of $L_\alpha(X)$ and Lemma 2.4 in [11] that there exists a countable dense subset $\{p_k\}$ of $L_\alpha(X)$ such that $p_k = [M_k] * \delta_{x_k}$, $M_k = J^\alpha G_k$, where G_k is a measure concentrated on B_k, $G_k(\{0\}) = 0$ and $G_k(X) \leqslant k$ $(k = 1, 2, \ldots)$.

Put

$$G = \sum_{k=1}^{\infty} [a^{k^2}]^{-1} A^{-k^3} G_k, \tag{3.4}$$

where for a real number b its integer part is denoted by $[b]$ and the constant a is determined by (3.3). Since $G_k(X) \leqslant k$ $(k = 1, 2, \ldots)$, G is a finite measure on X and, moreover, $G(\{0\}) = 0$.

Let $\beta = \max(e, \|A^{-1}\|)$. Then, for every $k = 1, 2, \ldots$,

$$\int_{B'_1} \log^\alpha \|x\| A^{-k^3} G_k(dx) = \int_X \log^\alpha \max(1, \|A^{-k^3} x\|) G_k(dx)$$

$$\leqslant \int_{B_k} \log^\alpha \max(1, \|A^{-1}\|^k \|x\|) G_k(dx) \leqslant k \log^\alpha k\beta^{k^3}$$

$$\leqslant 2^\alpha k^{3\alpha+1} \log^\alpha \beta.$$

Hence and by (3.4) we get

$$\int_{B'_1} \log^\alpha \|x\| G(dx) \leqslant 2^\alpha \log^\alpha \beta \sum_{k=1}^{\infty} [a^{k^2}]^{-1} k^{3\alpha+1} < \infty, \tag{3.5}$$

which, by the fact that G is finite and by Proposition 2.10, implies that $J^\alpha G$ is a Lévy measure. Put $M = J^\alpha G$ and $p = [M]$. It is evident that $p \in L_\alpha(X)$. Our further aim is to prove that p is A-universal for $L_\alpha(X)$.

Accordingly, let q be an arbitrary p.m. in $L_\alpha(X)$. Then there is a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ converging to q. Let us put $t_k = [a^{n_k^2}]$ $(k = 1, 2, \ldots)$. We shall prove that the sequence $v_k := A^{n_k^3} p^{t_k}$ $(k = 1, 2, \ldots)$ is shift convergent to q.

For $k = 1, 2, \ldots$ we write

$$N_k^1 = \sum_{n > n_k} t_k [a^{n^2}]^{-1} A^{n_k^3 - n^3} G_n, \tag{3.6}$$

$$N_k^2 = \sum_{n < n_k} t_k [a^{n^2}]^{-1} A^{n_k^3 - n^3} G_n, \tag{3.7}$$

$$H_k^i = J^\alpha N_k^i \quad (i = 1, 2). \tag{3.8}$$

It is clear that N_k^i and H_k^i are Lévy measures and

$$v_k = p_{n_k} * [H_k^1] * [H_k^2] * \delta_{-x_{n_k}} \quad (k = 1, 2, \ldots). \tag{3.9}$$

Further, for every $k = 1, 2, \ldots$ we have

$$N_k^1(X) \leqslant \sum_{n > n_k} n t_k [a^{n^2}]^{-1} \leqslant \sum_{n=1}^{\infty} (n_k + n) [a^{n_k^2}] [a^{(n_k+n)^2}]^{-1}$$

$$\leqslant a(a-1)^{-1} \sum_{n=1}^{\infty} (n_k + n) a^{-(2n_k+n)n},$$

which implies

$$\lim_{k \to \infty} N_k^1(X) = 0. \tag{3.10}$$

Let s be an arbitrary positive number. By (3.6) we get

$$\int_{B'_1} \log^s \|x\| N_k^1(dx) = \sum_{n>n_k} t_k [a^{n^2}]^{-1} \int_X \log^s \max(1, \|x\|) A^{n_k^3 - n^3} G_n(dx)$$

$$\leqslant \sum_{n>n_k} t_k [a^{n^2}]^{-1} \int_X \log^s \max(1, \|A^{-1}\|^{n^3 - n_k^3} \|x\|) G_n(dx)$$

$$\leqslant \sum_{n>n_k} n t_k [a^{n^2}]^{-1} \log^s n\beta^{n^3 - n_k^3}$$

$$\leqslant a(a-1)^{-1} \log^s \beta \sum_{m=1}^{\infty} (n_k+m)(m^3 + 3n_k^3 m + 3n_k m^2 + n_k + m)^s a^{-(2n_k+m)m},$$

where $\beta = \max(e, \|A^{-1}\|)$. Consequently,

$$\lim_{k\to\infty} \int_{B'_1} \log^s \|x\| N_k^1(dx) = 0. \tag{3.11}$$

Hence and from (3.10) it follows that for any $s, \delta > 0$

$$\lim_{k\to\infty} \int_{B'_\delta} \log^s(\|x\| \delta^{-1}) N_k^1(dx) = 0. \tag{3.12}$$

Next, by (3.3) and (3.7) we get

$$\int_X \|x\| N_k^2(dx) = \sum_{n<n_k} \int_X \|A^{n_k^3 - n^3} x\| t_k [a^{n^2}]^{-1} G_n(dx)$$

$$\leqslant c \sum_{n<n_k} n^2 a^{n^3 - n_k^3} [a^{n_k^2}] [a^{n^2}]^{-1}$$

$$\leqslant ca(a-1)^{-1} \sum_{n<n_k} n^2 a^{-n^2} a^{n^3 - n_k^3 + n_k^2}$$

$$\leqslant ca(a-1)^{-1} \sum_{n<n_k} n^2 a^{-n^2} a^{(n_k-1)^3 - n_k^3 + n_k^2}$$

$$\leqslant ca(a-1)^{-1} a^{-2n_k^2 + 3n_k - 1} \sum_{n=1} n^2 a^{-n^2},$$

where the constants a and c are determined by (3.3). Consequently,

$$\lim_{k\to\infty} \int_X \|x\| N_k^2(dx) = 0. \tag{3.13}$$

Hence it follows that for any $s, \delta > 0$

$$\lim_{k\to\infty} \int_{B'_\delta} \log^s \|x\| \delta^{-1} N_k^2(dx) = 0. \tag{3.14}$$

Further, formulas (2.7) and (3.8) imply that, for any $r > 0$ and $i = 1, 2$,

$$H_k^i(B_r') = J^\alpha N_k^i(B_r') = \frac{1}{\Gamma(\alpha+1)} \int_{B_r'} \log^\alpha \|x\| r^{-1} N_k^i(dx).$$

Consequently, by (3.12) and (3.14), for $r > 0$ and $i = 1, 2$, we get

$$\lim_{k\to\infty} H_k^i(B_r') = 0. \tag{3.15}$$

On the other hand, from (2.8) it follows that for any $\delta > 0$ and $i = 1, 2$

$$\int_{B_\delta} \|x\| H_k^i(dx) = \int_{B_\delta} \|x\| J^\alpha N_k^i(dx)$$
$$= \int_{B_\delta} \|x\| N_k^i(dx) + \frac{\delta}{\Gamma(\alpha)} \int_0^\infty e^{-t}(t+\log\|x\|\delta^{-1})^{\alpha-1} dt N_k^i(dx).$$

Therefore, for $i = 1, 2$ the following inequality holds:

$$\int_{B_\delta} \|x\| H_k^i(dx) \leqslant \begin{cases} \int_{B_\delta} \|x\| N_k^i(dx) + \delta N_k^i(B_\delta') & \text{if } 0 < \alpha < 1, \\ \int_{B_\delta} \|x\| N_k^i(dx) + 2^{\alpha-1}\delta N_k^i(B_\delta') + \\ + 2^{\alpha-1}\frac{\delta}{\Gamma(\alpha)} \int_{B_\delta'} \log^{\alpha-1}\|x\|\delta^{-1} N_k^i(dx) & \text{if } \alpha \geqslant 1. \end{cases} \tag{3.16}$$

Consequently, by (3.10), (3.12)-(3.14), and (3.16), we have

$$\lim_{k\to\infty} \int_B \|x\| H_k^i(dx) = 0 \quad (\delta > 0,\ i = 1, 2). \tag{3.17}$$

Noting that every Banach space is of type 1 we infer from formulas (3.15), (3.17), and Corollary 2.8 in [5] that, for $i = 1, 2$,

$$[H_k^i] \Rightarrow \delta_0 \quad \text{as } k \to \infty. \tag{3.18}$$

Finally, since $p_{n_k} \Rightarrow q$, formulas (3.9) and (3.18) imply that

$$\lim_{k\to\infty} v_k * \delta_{x_{n_k}} = q,$$

which shows that the sequence $\{v_k\}$ is shift covergent to q. Thus the theorem is proved.

REFERENCES

[1] J. Barańska, *Universal Doëblin distributions in a Hilbert space*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys., 26 (1978), p. 343-346.

[2] P. L. Butzer and U. Westphal, *An access to fractional differentiation via fractional difference quotients*, Lecture Notes in Math. 457 (1975), p. 116-145.

[3] E. Dettweiler, *Grenzwertsätze für Wahrscheinlichkeitsmasse auf Badrikianschen Räumen*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 34 (1976), p. 285-311.

[4] W. Doéblin, *Sur l'ensemble de puissance d'une loi de probabilité*, Bull. Sci. Math. 6 (1939), p. 71-96.

[5] E. Giné, *A survey on the central limit problem in Banach spaces*, Séminaire Maurey-Schwartz 1977/78, Exposé 25.

[6] Ho Dang Phuc, *Universal distribution for infinitely divisible distributions in a Banach space* (preprint), Graduate work at the Wrocław University, 1978.

[7] A. Kumar and B. M. Schreiber, *Self-decomposable probability measures on Banach spaces*, Studia Math. 53 (1975), p. 55-71.

[8] A. Marchaud, *Sur les dérivées et sur les différences des fonctions de variables réelles*, J. de Math. 6 (1927), p. 337-425.

[9] N. V. Thu, *Multiply self-decomposable probability measures on Banach spaces*, Studia Math. 66 (1979), p. 161-175.

[10] — *A new version of Doéblin's theorem*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys., 29 (1981), p. 643-648.

[11] — *Fractional calculus in probability*, Probability and Mathematical Statistics 4 (1984) (to appear).

[12] K. Urbanik, *Limit laws for sequences of normed sums satisfying some stability conditions*, p. 225-237 in: *Multivariate analysis. III* (P. R. Krishnaiah, ed.), Academic Press, New York 1973.

[13] — *Lévy's measures on Banach spaces*, Studia Math. 63 (1978), p. 284-308.

[14] U. Westphal, *An approach to fractional powers of operators via fractional differences*, Proc. London Math. Soc. 29 (1974), p. 557-576.

Hanoi Institute of Mathematics
Box 631, Bo-ho
Hanoi, Vietnam

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