WEAK-TYPE INEQUALITY FOR THE MARTINGALE SQUARE FUNCTION AND A RELATED CHARACTERIZATION OF HILBERT SPACES∗

BY

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Abstract. Let $f$ be a martingale taking values in a Banach space $B$ and let $S(f)$ be its square function. We show that if $B$ is a Hilbert space, then

$$P(S(f) \geq 1) \leq \sqrt{e}\|f\|_1,$$

and the constant $\sqrt{e}$ is the best possible. This extends the result of Cox, who established this bound in the real case. Next, we show that this inequality characterizes Hilbert spaces in the following sense: if $B$ is not a Hilbert space, then there is a martingale $f$ for which the above weak-type estimate does not hold.

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1. INTRODUCTION

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, filtered by $(\mathcal{F}_n)_{n \geq 0}$, a non-decreasing sequence of sub-σ-fields of $\mathcal{F}$. Let $f = (f_n)_{n \geq 0}$ and $g = (g_n)_{n \geq 0}$ be adapted martingales taking values in a certain separable Banach space $(B, \| \cdot \|)$. The difference sequences $df = (df_n)_{n \geq 0}$ and $dg = (dg_n)_{n \geq 0}$ of the martingales $f$ and $g$ are defined by $df_0 = f_0$ and $df_n = f_n - f_{n-1}$ for $n \geq 1$, and similarly for $dg_n$. We say that $g$ is a $\pm 1$-transform of $f$ if there is a deterministic sequence $\varepsilon = (\varepsilon_n)_{n \geq 0}$ of signs such that $dg_n = \varepsilon_n df_n$ for each $n$.

It is well-known that martingale inequalities reflect the geometry of Banach spaces in which the martingales take values: see e.g. [1]–[4] and [7]. We shall mention here only one fact, closely related to the result studied in the present paper. As proved by Burkholder in [2], if $f$ takes values in a separable Hilbert space and
$g$ is its $\pm 1$-transform, then

\begin{equation}
\mathbb{P}(\sup_n \|g_n\| \geq 1) \leq 2\|f\|_1
\end{equation}

and the constant 2 is the best possible (here, as usual, $\|f\|_1 = \sup_n \|f_n\|_1$). In fact, the implication can be reversed: if $B$ is a separable Banach space with the property that (1.1) holds for any $B$-valued martingale $f$ and its $\pm 1$-transform $g$, then $B$ is a Hilbert space. For details, see Burkholder [2] and Lee [6].

In this paper we shall study a related problem and characterize the class of Hilbert spaces by another weak-type estimate. Let us introduce the square function of $f$ by the formula

$$S(f) = \left( \sum_{k=0}^{\infty} \|df_k\|^2 \right)^{1/2}.$$ 

We shall also use the notation

$$S_n(f) = \left( \sum_{k=0}^{n} \|df_k\|^2 \right)^{1/2}$$

for the truncated square function, $n = 0, 1, 2, \ldots$. Suppose that $B$ is a given and fixed separable Banach space and let $\beta(B)$ denote the least extended real number $\beta$ such that, for any martingale $f$ taking values in $B$,

$$\mathbb{P}(S(f) \geq 1) \leq \beta(B)\|f\|_1.$$ 

Using the method of moments, Cox [5] showed that $\beta(\mathbb{R}) = \sqrt{e}$: consequently, $\beta(B) \geq \sqrt{e}$ for any non-degenerate $B$. We will extend this result to the following.

**Theorem 1.1.** We have $\beta(B) = \sqrt{e}$ if and only if $B$ is a Hilbert space.

Let us sketch the proof. To show that for any martingale $f$ taking values in a Hilbert space $(\mathcal{H}, \| \cdot \|)$ we have

\begin{equation}
\mathbb{P}(S(f) \geq 1) \leq \sqrt{e}\|f\|_1,
\end{equation}

we may restrict ourselves to the class of simple martingales. Recall that $f$ is simple if for any $n$ the random variable $f_n$ takes only a finite number of values and there is a deterministic $N$ such that $f_N = f_{N+1} = f_{N+2} = \ldots$ We must prove that

$$\text{EV}(f_n, S_n(f)) \leq 0, \quad n = 0, 1, 2, \ldots,$$

where $V(x, y) = 1_{\{y \geq 1\}} - \sqrt{e}|y|$ for $x \in \mathcal{H}$ and $y \in [0, \infty)$.

To do this, we will use Burkholder’s method and construct a function $U: \mathcal{H} \times [0, \infty) \to \mathbb{R}$, which satisfies the following three conditions:

1° We have the majorization $U \geq V$. 

2° For any \( x \in \mathcal{H}, y \geq 0 \) and any simple mean-zero random variable \( T \) taking values in \( \mathcal{H} \) we have \( \mathbb{E}U(x + T, \sqrt{y^2 + |T|^2}) \leq U(x, y) \).

3° For any \( x \in \mathcal{H} \) we have \( \mathbb{E}(x, |x|) \leq 0 \).

Then (1.2) follows.

To see this, apply 2° conditionally on \( \mathcal{F}_n \), with \( x = f_n, y = S_n(f) \) and \( T = df_{n+1}. \) As the result, we obtain the inequality

\[
\mathbb{E}[U(f_{n+1}, S_{n+1}(f)) | \mathcal{F}_n] \leq U(f_n, S_n(f)),
\]

so, in other words, the process \( \{U(f_n, S_n(f))\}_{n \geq 0} \) is a supermartingale. Hence, by 1° and 3°,

\[
\mathbb{E}V(f_n, S_n(f)) \leq \mathbb{E}U(f_n, S_n(f)) \leq \mathbb{E}U(f_0, |f_0|) \leq 0
\]

and we are done.

The special function \( U \) is constructed and studied in the next section. In Section 3 we prove the remaining part of Theorem 1.1: we shall show that the validity of (1.2) for all \( B \)-valued martingales implies the parallelogram identity.

### 2. A SPECIAL FUNCTION

Let \( \mathcal{H} \) be a separable Hilbert space: in fact, we may and do assume that \( \mathcal{H} = \ell^2 \). The corresponding norm and scalar product will be denoted by \( | \cdot | \) and \( \cdot, \cdot \), respectively. Introduce \( U : \mathcal{H} \times [0, \infty) \to \mathbb{R} \) by the formula

\[
U(x, y) = \begin{cases} 
1 - (1 - y^2)^{1/2} \exp\left(\frac{|x|^2}{2(1 - y^2)}\right) & \text{if } |x|^2 + y^2 < 1, \\
1 - \sqrt{e} |x| & \text{if } |x|^2 + y^2 \geq 1.
\end{cases}
\]

In the lemma below, we study the properties of \( U \) and \( V \).

**Lemma 2.1.** The function \( U \) satisfies the conditions 1°, 2° and 3°.

**Proof.** To show the majorization, we may assume that \( |x|^2 + y^2 < 1 \). Then the inequality takes the form

\[
\exp\left(\frac{|x|^2}{2(1 - y^2)}\right) \leq \sqrt{e} \frac{|x|}{\sqrt{1 - y^2}} + \frac{1}{\sqrt{1 - y^2}}
\]

and follows immediately from an elementary bound \( \exp(s^2/2) \leq \sqrt{e}s + 1, s \in [0, 1], \) applied to \( s = |x|/\sqrt{1 - y^2} \). To check 2°, we introduce an auxiliary function

\[
A(x, y) = \begin{cases} 
-x(1 - y^2)^{-1/2} \exp\left(\frac{|x|^2}{2(1 - y^2)}\right) & \text{if } |x|^2 + y^2 < 1, \\
-\sqrt{e}x & \text{if } |x|^2 + y^2 \geq 1,
\end{cases}
\]
where \( x' = x/|x| \) for \( x \neq 0 \), and \( x' = 0 \) otherwise. We shall establish a pointwise estimate

\[
U(x + d, \sqrt{y^2 + |d|^2}) \leq U(x, y) + A(x, y) \cdot d
\]

for all \( x, d \in \mathcal{H} \) and \( y \geq 0 \). Observe that this inequality immediately yields \( 2^0 \), simply by putting \( d = T \) and taking expectation of both sides.

To prove (2.2), note first that

\[
U(x, y) \leq 1 - \sqrt{e}|x| \text{ for all } x \in \mathcal{H} \text{ and } y \geq 0.
\]

This is trivial for \( |x|^2 + y^2 \geq 1 \), while for the remaining pairs \((x, y)\) it can be transformed into the equivalent inequality:

\[
\frac{|x|^2}{1 - y^2} \leq \exp \left( \frac{|x|^2}{1 - y^2} - 1 \right),
\]

which is obvious. Consequently, when \( |x|^2 + y^2 \geq 1 \), we have

\[
U(x + d, \sqrt{y^2 + |d|^2}) \leq 1 - \sqrt{e}|x + d| \leq 1 - \sqrt{e}|x| + A(x, y) \cdot d = U(x, y) + A(x, y) \cdot d.
\]

Now suppose that \( |x|^2 + y^2 < 1 \) and \( |x + d|^2 + y^2 + |d|^2 \leq 1 \). Observe that for \( X, D \in \mathcal{H} \) with \( |D| < 1 \) we have

\[
\exp \left( \frac{|D|^2|X|^2 + 2X \cdot D + |D|^2}{1 - |D|^2} \right) \geq \exp \left( \frac{(X \cdot D)^2 + 2X \cdot D + |D|^2}{1 - |D|^2} \right)
\]

\[
\geq \frac{(X \cdot D)^2 + 2X \cdot D + |D|^2}{1 - |D|^2} + 1
\]

\[
= \frac{(1 + X \cdot D)^2}{1 - |D|^2}.
\]

It suffices to plug \( X = x/\sqrt{1 - y^2} \) and \( D = d/\sqrt{1 - y^2} \) to obtain (2.2). Finally, if \( |x|^2 + y^2 < 1 < |x + d|^2 + y^2 + |d|^2 \), then substituting \( X \) and \( D \) as previously, we have \( |X| < 1 \), \( |X + D|^2 + |D|^2 > 1 \) and (2.2) can be written in the form

\[
\exp \left( \frac{|X|^2 - 1}{2} \right) (1 + X \cdot D) \leq |X + D|,
\]

or

\[
\exp \left( \frac{|X|^2 - 1}{2} \right) \left( 1 + \frac{|X + D|^2 - |X|^2 - |D|^2}{2} \right) \leq |X + D|.
\]

Now we fix \( |X|, |X + D| \) and maximize the left-hand side over \( D \). Let us consider two cases. If \( |X + D|^2 + (|X + D| - |X|)^2 < 1 \), then there is \( D' \in \mathcal{H} \) satisfying
\[ |X + D| = |X + D'| \text{ and } |X + D'|^2 + |D'|^2 = 1. \text{ Consequently,} \]
\[
\exp \left( \frac{|X|^2 - 1}{2} \right) \left( 1 + \frac{|X + D|^2 - |X|^2 - |D|^2}{2} \right) \\
\leq \exp \left( \frac{|X|^2 - 1}{2} \right) \left( 1 + \frac{|X + D'|^2 - |X|^2 - |D'|^2}{2} \right) \leq |X + D'| = |X + D|. \]

Here the first passage is due to \(|D'| < |D|\), while in the second we have applied (2,2) to \(x = X, y = 0\) and \(d = D'\) (for these \(x, y\) and \(d\) we have already established the bound). Suppose, then, that \(|X + D|^2 + (|X + D| - |X|)^2 \geq 1\). This inequality is equivalent to
\[
|X + D| \geq \frac{1 - |X|^2}{\sqrt{2 - |X|^2 - |X|}},
\]
and hence
\[
\exp \left( \frac{|X|^2 - 1}{2} \right) \left( 1 + \frac{|X + D|^2 - |X|^2 - |D|^2}{2} \right) - |X + D| \\
\leq \exp \left( \frac{|X|^2 - 1}{2} \right) \left( 1 + \frac{|X + D|^2 - |X|^2 - (|X + D| - |X|)^2}{2} \right) - |X + D| \\
= \exp \left( \frac{|X|^2 - 1}{2} \right) \left( 1 - |X|^2 \right) + \left\{ \exp \left( \frac{|X|^2 - 1}{2} \right) |X| - 1 \right\} |X + D| \\
\leq \frac{1 - |X|^2}{\sqrt{2 - |X|^2 - |X|}} \left[ \exp \left( \frac{|X|^2 - 1}{2} \right) \sqrt{2 - |X|^2 - 1} \right].
\]

It suffices to observe that the expression in the square brackets is nonpositive, which follows from the estimate \(\exp(1 - |X|^2) \geq 2 - |X|^2\). This completes the proof of 2\(^o\). Finally, 3\(^o\) is a consequence of the inequality (2.2): \(U(x, |x|) \leq U(0, 0) + A(0, 0) \cdot x = 0\). \(\blacksquare\)

Thus, by the reasoning presented in the Introduction, the inequality (1.2) holds true. The constant \(\sqrt{c}\) is optimal even in the real case; see Cox [5]. In fact, we shall reprove this in the next section: see Remark 3.1 below.

### 3. Characterization of Hilbert Spaces

Let \((B, \| \cdot \|)\) be a separable Banach space and recall the number \(\beta(B)\) defined in the first section. Thus, for any \(B\)-valued martingale \(f\) we have
\[
\mathbb{P}\left( S(f) \geq 1 \right) \leq \beta(B) \|f\|_1.
\]

For \(x \in B\) and \(y \geq 0\), let \(M(x, y)\) denote the class of all simple martingales \(f\) given on the probability space \(([0, 1], \mathbb{B}(0, 1), \cdot, \cdot)\), such that \(f\) is \(B\)-valued, \(f_0 \equiv x\) and
\[
y^2 - \|x\|^2 + S^2(f) \geq 1 \text{ almost surely.}
\]
The function $U^0 : \mathcal{B} \times [0, \infty) \to \mathbb{R}$ given by

$$U^0(x, y) = \inf \{ \mathbb{E} \| f_n \| \},$$

where the infimum is taken over all $n$ and all $f \in M(x, y)$. We will prove that $U^0$ satisfies appropriate versions of the conditions 1o–3o.

**Lemma 3.1.** The function $U^0$ satisfies the following conditions:

1o For any $x \in \mathcal{B}$ and $y \geq 0$ we have $U^0(x, y) \geq \| x \|$.

2o For any $x \in \mathcal{B}$, $y \geq 0$ and any simple centered $\mathcal{B}$-valued random variable $T$,

$$\mathbb{E}U^0(x + T, \sqrt{y^2 + \| T \|^2}) \geq U^0(x, y).$$

3o For any $x \in \mathcal{B}$ we have $U^0(x, \| x \|) \geq \beta(\mathcal{B})^{-1}$.

**Proof.** The property 1o is obvious: when $f \in M(x, y)$, then it follows that $\| f_n \|_1 \geq \| f_0 \|_1 = \| x \|$ for all $n$. To establish 2o, we use a modification of the so-called “splicing argument”: see e.g. [1]. Let $T$ be as in the statement and let $\{x_1, x_2, \ldots, x_k\}$ be the set of its values: $\mathbb{P}(T = x_j) = p_j > 0$, $\sum_{j=1}^{k} p_j = 1$. For any $1 \leq j \leq k$, pick a martingale $f^j$ from the class $M(x + x_j, \sqrt{y^2 + \| x_j \|^2})$. Let $a_0 = 0$ and $a_j = \sum_{t=1}^{j} p_t$, $j = 1, 2, \ldots, k$. Define a simple sequence $f$ on $([0, 1], \mathcal{B}(0, 1), | \cdot |)$ by $f_0 \equiv x$ and

$$f_n(\omega) = f^j_{n-1}(\omega/(a_j - a_{j-1})), \quad n \geq 1,$$

when $\omega \in (a_{j-1}, a_j]$. Then $f$ is a martingale with respect to its natural filtration and, when $\omega \in (a_{j-1}, a_j]$,

$$y^2 - \| x \|^2 + S^2(f)(\omega) = y^2 + \| x_j \|^2 - \| x + x_j \|^2 + S^2(f^j)(\omega/(a_j - a_{j-1})) \geq 1,$$

unless $\omega$ belongs to a set of measure zero. Therefore (3.2) holds, so by the definition of $U^0$ we get

$$\| f_n \|_1 \geq U^0(x, y).$$

However, the left-hand side equals

$$\sum_{j=1}^{k} \int_{a_{j-1}}^{a_j} | f_n(\omega) | d\omega = \sum_{j=1}^{k} p_j \int_{0}^{1} | f^j_{n-1}(\omega) | d\omega,$$

which, by the proper choice of $n$ and $f^j$, $j = 1, 2, \ldots, k$, can be made arbitrarily close to $\sum_{j=1}^{k} p_j U^0(x + x_j, \sqrt{y^2 + \| x_j \|^2}) = \mathbb{E}U^0(x + T, \sqrt{y^2 + \| T \|^2})$. This gives 2o. Finally, the condition 3o follows immediately from (3.1) and the definition of $U^0$. □
The further properties of $U^0$ are described in the next lemma.

**LEMMA 3.2.** (i) The function $U^0$ satisfies the symmetry condition

$$U^0(x, y) = U^0(-x, y)$$

for all $x \in B$ and $y \geq 0$.

(ii) The function $U^0$ has the homogeneity-type property

$$U^0(x, y) = \sqrt{1 - y^2} U^0(\frac{x}{\sqrt{1 - y^2}}, 0)$$

for all $x \in B$ and $y \in [0, 1)$.

(iii) If $z \in B$ satisfies $\|z\| = 1$ and $0 \leq s < t \leq 1$, then

$$U^0(sz, 0) \leq U^0(tz, 0) \exp\left((s^2 - t^2)\|z\|^2/2\right).$$

**Proof.** (i) It is sufficient to use the equivalence $f \in M(x, y)$ if and only if $-f \in M(-x, y)$.

(ii) This follows immediately from the fact that $f \in M(x, y)$ if and only if $f/\sqrt{1 - y^2} \in M(x/\sqrt{1 - y^2}, 0)$.

(iii) Fix $x \in B$ with $0 < \|x\| < 1$ and $\delta > 0$ such that $\|x + \delta x\| \leq 1$. Apply $2\sigma$ to $y = 0$ and a centered random variable $T$ which takes two values: $\delta x$ and $-2x/(1 + \|x\|^2)$. We get

$$U^0(x, 0) \leq \frac{\delta \|x\|(1 + \|x\|^2)}{2\|x\| + \delta \|x\|(1 + \|x\|^2)} U^0\left(-\frac{x(1 - \|x\|^2)}{1 + \|x\|^2}, \frac{2\|x\|}{1 + \|x\|^2}\right) + \frac{2\|x\|}{2\|x\| + \delta \|x\|(1 + \|x\|^2)} U^0(\delta x, 0).$$

By (i) and (ii), the first term on the right equals

$$\frac{\delta \|x\|(1 - \|x\|^2)}{2\|x\| + \delta \|x\|(1 + \|x\|^2)} U^0(x, 0).$$

The second summand can be bounded from above by

$$\frac{2\|x\|}{2\|x\| + \delta \|x\|(1 + \|x\|^2)} U^0(\delta x, 0),$$

because $M(\delta x, 0) \subset M(\delta x, \delta \|x\|)$. Plugging these two facts into the inequality above yields

$$(3.4) \quad \frac{U^0(x + \delta x, 0)}{U^0(x, 0)} \geq 1 + \delta \|x\|^2.$$
This gives
\[
\frac{U^0(x(1 + k\delta), 0)}{U^0(x(1 + (k - 1)\delta), 0)} \geq 1 + \delta(1 + (k - 1)\delta)\|x\|^2,
\]
provided \(\|x(1 + k\delta)\| \leq 1\). Consequently, if \(N\) is an integer such that the condition \(\|x(1 + N\delta)\| \leq 1\) holds true, then
\[
(3.5) \quad \frac{U^0(x(1 + N\delta), 0)}{U^0(x, 0)} \geq \prod_{k=1}^{N} \left( 1 + \delta(1 + (k - 1)\delta)\|x\|^2 \right).
\]

Now we turn to (3.3). Assume first that \(s > 0\). Put \(x = sz\), \(\delta = (t/s - 1)/N\) and let \(N \to \infty\) in the inequality above to obtain
\[
U^0(tz, 0)/U^0(sz, 0) \geq \exp\left( \frac{1}{2} \|z\|^2 (t^2 - s^2) \right),
\]
which is the claim. Next, suppose that \(s = 0\). For any \(0 < s' < t\) we have, by \(2^{s'}\),
\[
U^0(0, 0) \leq \frac{1}{2} U^0(s'z, \|s'z\|) + \frac{1}{2} U^0(-s'z, \|s'z\|) = U^0(s'z, \|s'z\|) \leq U^0(s'z, 0),
\]
where in the latter passage we have used the inclusion \(M(s'z, 0) \subset M(s'z, \|s'z\|)\). Thus,
\[
\frac{U^0(tz, 0)}{U^0(0, 0)} \geq \frac{U^0(tz, 0)}{U^0(s'z, 0)} \geq \exp\left( \frac{1}{2} \|z\|^2 (t^2 - (s')^2) \right)
\]
and it remains to let \(s' \to 0\).

**Remark 3.1.** Suppose that \(B = \mathbb{R}\). It is easy to see that \(U^0(1, 0) \leq 1\): consider \(f\) starting from 1 and satisfying \(\mathbb{P}(df_1 = -1) = \mathbb{P}(df_1 = 1) = 1/2, df_2 = df_3 = \ldots = 0\). Thus, by \(3^{s'}\) and (3.3), we have
\[
\beta(\mathbb{R})^{-1} \leq U^0(0, 0) \leq U^0(1, 0)/\sqrt{e} \leq 1/\sqrt{e},
\]
that is, \(\beta(\mathbb{R}) \geq \sqrt{e}\). This implies the sharpness of (1.2) in the Hilbert-space-valued setting.

Now we will work under the assumption \(\beta(B) = \sqrt{e}\). Then we are able to derive the explicit formula for \(U^0\).

**Lemma 3.3.** If \(\beta(B) = \sqrt{e}\), then
\[
U^0(x, y) = \begin{cases} 
\sqrt{1 - y^2} \exp\left(\|x\|^2/[2(1 - y^2)] - \frac{1}{2}\right) & \text{if } \|x\|^2 + y^2 < 1, \\
\|x\| & \text{if } \|x\|^2 + y^2 \geq 1.
\end{cases}
\]
First let us focus on the set \( \{(x, y) : \|x\|^2 + y^2 \geq 1\} \). By 1’’ we have \( U^0(x, y) \geq \|x\| \). To get the reverse estimate, consider a martingale \( f \) such that \( f_0 = x \), \( df_1 \) takes values \(-x\) and \( x \), and \( df_2 = df_3 = \ldots = 0 \). Then \( y^2 - \|x\|^2 + S^2(f) = y^2 + \|x\|^2 \geq 1 \) (so \( f \in M(x, y) \)) and \( \|f\|_1 = \|x\| \), which implies \( U^0(x, y) \leq \|x\| \) by the definition of \( U^0 \). Now suppose that \( \|x\|^2 + y^2 < 1 \). Using the second and third part of the previous lemma, we may write

\[
U^0(x, y) = \sqrt{1 - y^2}U^0\left(\frac{x}{\sqrt{1 - y^2}}, 0\right) \geq \sqrt{1 - y^2} \exp\left(\frac{\|x\|^2}{2(1 - y^2)}\right),
\]

so, by 3’’,

\[
U^0(x, y) \geq \sqrt{1 - y^2} \exp\left(\frac{\|x\|^2}{2(1 - y^2)} - \frac{1}{2}\right).
\]

To get the reverse bound, we use the homogeneity of \( U^0 \) and (3.3) again:

\[
U^0(x, y) = \sqrt{1 - y^2}U^0\left(\frac{x}{\sqrt{1 - y^2}}, 0\right) \\
\leq \sqrt{1 - y^2}U^0\left(\frac{x}{|x|}, 0\right) \exp\left(\frac{1}{2} \left(\frac{\|x\|^2}{1 - y^2} - 1\right)\right) \\
= \sqrt{1 - y^2} \exp\left(\frac{\|x\|^2}{2(1 - y^2)} - \frac{1}{2}\right),
\]

where in the last line we have used the equality \( U^0(\pi, 0) = \|\pi\| \) valid for \( \pi \) of norm one (we have just established this in the first part of the proof). For completeness, let us mention here that if \( x = 0 \), then \( x/|x| \) should be replaced above by any vector of norm one. ■

**Lemma 3.4.** Suppose that \( \beta(B) = \sqrt{e} \) and let us assume that \( x, y \in B \) and \( \alpha > 0 \) satisfy \( \|x\| < 1 \), \( \|x + \alpha x + y\|^2 + \|\alpha x + y\|^2 < 1 \) and \( \|x + \alpha x - y\|^2 + \|\alpha x - y\|^2 < 1 \). Then

\[
2 + 2\alpha\|x\|^2 \leq \sqrt{1 - \|\alpha x + y\|^2} \exp\left(\frac{|x + \alpha x + y|^2}{2(1 - \|\alpha x + y\|^2)} - \frac{|x|^2}{2}\right) \\
+ \sqrt{1 - \|\alpha x - y\|^2} \exp\left(\frac{|x + \alpha x - y|^2}{2(1 - \|\alpha x - y\|^2)} - \frac{|x|^2}{2}\right).
\]

**Proof.** Consider a random variable \( T \) such that

\[
P\left(T = \alpha x + y\right) = P(T = \alpha x - y) = \frac{1 - p}{2},
\]

where \( p \in (0, 1) \) is chosen so that \( ET = 0 \). That is,

\[
p = \frac{\alpha(1 + \|x\|^2)}{2 + \alpha(1 + \|x\|^2)}.
\]
By 2", we have \( U^0(x, 0) \leq \mathbb{E} f^0(x + T, ||T||) \). Since \( ||x + T||^2 + ||T||^2 < 1 \) almost surely, the previous lemma implies that this can be rewritten in the equivalent form:

\[
\exp \left( \frac{||x||^2}{2} \right) \leq p \sqrt{1 - \left( \frac{2||x||}{1 + ||x||^2} \right)^2} \exp \left( \frac{||x((-1 + ||x||^2)/(1 + ||x||^2))^2}{2(1 - (2||x||/(1 + ||x||^2))^2)} \right) + \frac{1 - p}{2} \sqrt{1 - \|\alpha x + y\|^2} \exp \left( \frac{||x + \alpha x + y\|^2}{2(1 - \|\alpha x + y\|^2)} \right) + \frac{1 - p}{2} \sqrt{1 - \|\alpha x - y\|^2} \exp \left( \frac{||x + \alpha x - y\|^2}{2(1 - \|\alpha x - y\|^2)} \right).
\]

However, the first term on the right equals

\[
\frac{\alpha(1 - ||x||^2)}{2 + \alpha(1 + ||x||^2)} \exp \left( \frac{||x||^2}{2} \right)
\]

and, in addition, \((1 - p)/2 = \left(2 + \alpha(1 + ||x||^2)\right)^{-1} \). Consequently, it suffices to multiply both sides of the inequality above by \((2 + \alpha(1 + ||x||^2)) \exp(-||x||^2/2)\); the claim follows.

Now we are ready to complete the proof of Theorem 1.1. Suppose that \( a, b \) belong to the unit ball \( K \) of \( B \) and take \( \varepsilon \in (0, 1/2) \). Applying (3.6) to \( x = \varepsilon a, y = \varepsilon^2 b \) and \( \alpha = \varepsilon \) gives

\[
2 + 2\varepsilon^3 ||a||^2 \leq \sqrt{1 - \varepsilon^4 ||a + b||^2} \exp \left( m(a, b) \right) + \sqrt{1 - \varepsilon^4 ||a - b||^2} \exp \left( m(a, -b) \right),
\]

where

\[
m(a, b) = \frac{\varepsilon^2 ||a + \varepsilon(a + b)||^2}{2(1 - \varepsilon^4 ||a + b||^2)} - \frac{\varepsilon^2 ||a||^2}{2} = \frac{\varepsilon^2}{2} (||a + \varepsilon(a + b)||^2 - ||a||^2) + \frac{\varepsilon^6 ||a + \varepsilon(a + b)||^2 ||a + b||^2}{2(1 - \varepsilon^4 ||a + b||^2)}.
\]

It is easy to see that there exists an absolute constant \( M_1 \) such that

\[
\sup_{a, b \in K} |m(a, b)| \leq M_1 \varepsilon^3.
\]

Consequently, there is a universal \( M_2 > 0 \) such that if \( \varepsilon \) is sufficiently small, then

\[
\exp (m(a, b)) \leq 1 + m(a, b) + m(a, b)^2 \\
\leq 1 + \frac{\varepsilon^2}{2} (||a + \varepsilon(a + b)||^2 - ||a||^2) + M_2 \varepsilon^6
\]
for any $a, b \in K$. Since $\sqrt{1 - x} \leq 1 - x/2$ for $x \in (0, 1)$, the inequality (3.7) implies

$$2 + 2\varepsilon^3 \|a\|^2 \leq (1 - \varepsilon^4 \|a + b\|^2/2) \left(1 + \frac{\varepsilon^2}{2} \left(\|a + \varepsilon(a + b)\|^2 - \|a\|^2\right) + M_3 \varepsilon^6\right) + (1 - \varepsilon^4 \|a - b\|^2/2) \left(1 + \frac{\varepsilon^2}{2} \left(\|a + \varepsilon(a - b)\|^2 - \|a\|^2\right) + M_3 \varepsilon^6\right).$$

This, after some manipulations, leads to

$$\|a + \varepsilon(a + b)\|^2 + \|a + \varepsilon(a - b)\|^2 - 2\|a(1 + \varepsilon)\|^2 \geq \varepsilon^2 \left(\|a + b\|^2 + \|a - b\|^2 - 2\|a\|^2\right) - 2\varepsilon \|a\|^2 - 2\varepsilon M_3,$$

where $M_3$ is a positive constant not depending on $\varepsilon$, $a$ and $b$. Equivalently,

$$\left\|a + \frac{\varepsilon}{1 + \varepsilon} b\right\|^2 + \left\|a - \frac{\varepsilon}{1 + \varepsilon} b\right\|^2 - 2\|a\|^2 - 2\left\|\frac{\varepsilon}{1 + \varepsilon} b\right\|^2 \geq \frac{\varepsilon^2}{(1 + \varepsilon)^2} \left(\|a + b\|^2 + \|a - b\|^2 - 2\|a\|^2 - 2\|b\|^2\right) - 2\frac{\varepsilon^4}{(1 + \varepsilon)^2} M_3.$$ 

Next, let $c \in B$, $\gamma > 0$ and substitute $a = \gamma c$; we assume that $\gamma$ is small enough to ensure that $a \in K$. If we divide both sides by $\gamma^2$ and substitute $\delta = \varepsilon(1 + \varepsilon)^{-1}/\gamma^{-1}$, we obtain

$$\|c + \delta b\|^2 + \|c - \delta b\|^2 - 2\|c\|^2 - 2\|\delta b\|^2 \geq \delta^2 (\|\gamma c + b\|^2 + \|\gamma c - b\|^2 - 2\|\gamma c\|^2 - 2\|b\|^2) - 2\varepsilon \|\gamma c\|^2 - 2\|b\|^2 - 2\delta^4 M_3.$$ 

Let $\gamma$ and $\varepsilon$ go to 0 so that $\delta$ remains fixed. As the result, we infer that, for any $\delta > 0$, $b \in K$ and $c \in B$,

$$(3.8) \quad \|c + \delta b\|^2 + \|c - \delta b\|^2 - 2\|c\|^2 - 2\|\delta b\|^2 \geq -2\delta^4 M_3.$$ 

Now, let $N$ be a large positive integer and consider a symmetric random walk $(g_n)_{n \geq 0}$ over integers, starting from 0. Let $\tau = \inf\{n : |g_n| = N\}$. The inequality (3.8), applied to $\delta = N^{-1}$, implies that for any $a \in B$ and $b \in K$ the process

$$(\xi_n)_{n \geq 0} = \left(\left\|a + \frac{b g_{\tau \wedge n}}{N}\right\|^2 - \left\{\frac{\|b\|^2}{N^2} - \frac{M_3}{N^4}\right\} (\tau \wedge n)\right)_{n \geq 0}$$

is a submartingale. Since $E(\tau \wedge n) = E g_{\tau \wedge n}^2$, we obtain

$$E\left(\left\|a + \frac{b g_{\tau \wedge n}}{N}\right\|^2 - \left\{\frac{\|b\|^2}{N^2} - \frac{M_3}{N^4}\right\} g_{\tau \wedge n}^2\right) = E \xi_n \geq E \xi_0 = \|a\|^2.$$
Letting $n \to \infty$ and using Lebesgue’s dominated convergence theorem gives

$$\frac{1}{2}(\|a + b\|^2 + \|a - b\|^2) - \|b\|^2 + \frac{M_3}{N^2} \geq \|a\|^2.$$ 

It suffices to let $N$ go to $\infty$ to obtain

$$\|a + b\|^2 + \|a - b\|^2 \geq 2\|a\|^2 + 2\|b\|^2.$$ 

We have assumed that $b$ belongs to the unit ball $K$, but, by homogeneity, the above estimate extends to any $b \in B$. Putting $a + b$ and $a - b$ in the place of $a$ and $b$, respectively, we obtain the reverse estimate

$$\|a + b\|^2 + \|a - b\|^2 \leq 2\|a\|^2 + 2\|b\|^2.$$ 

This implies that the parallelogram identity is satisfied, and hence $B$ is a Hilbert space.

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