CLASSICAL METHOD OF CONSTRUCTING A COMPLETE SET
OF IRREDUCIBLE REPRESENTATIONS OF SEMIDIRECT PRODUCT
OF A COMPACT GROUP WITH A FINITE GROUP

BY

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Abstract. Let $G = U \rtimes S$ be a group of semidirect product of $U$ compact and $S$ finite. For an irreducible representation (IR) $\rho$ of $U$, let $S([\rho])$ be the stationary subgroup in $S$ of the equivalence class $[\rho] \in \hat{U}$. Intertwining operators $J_\rho(s)$ ($s \in S([\rho])$) between $\rho$ and $s^-1 \rho$ gives in general a spin (= projective) representation of $S([\rho])$, which is lifted up to a linear representation $J'_\rho$ of a covering group $S([\rho])'$ of $S([\rho])$. Put $\pi^0 := \rho \cdot J'_\rho$, and take a spin representation $\pi^1$ of $S([\rho])$ corresponding to the factor set inverse to that of $J_\rho$, and put $\Pi(\pi^0, \pi^1) = \text{Ind}_{G \times S([\rho])}(\pi^0 \boxtimes \pi^1)$. We give a simple proof that $\Pi(\pi^0, \pi^1)$ is irreducible and that any IR of $G$ is equivalent to some of $\Pi(\pi^0, \pi^1)$.

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1. INTRODUCTION

Let $G = U \rtimes S$ be a semidirect product group, with $U$ compact and $S$ finite. For an irreducible representation (IR) $\rho$ of $U$, let $S([\rho])$ be the stationary subgroup in $S$ of the equivalence class $[\rho] \in \hat{U}$. Intertwining operators $J_\rho(s)$ ($s \in S([\rho])$), defined by $\rho(s(u)) = J_\rho(s) \rho(u) J_\rho(s)^{-1}$ ($u \in U$), gives in general a spin (= projective) representation of $S([\rho])$, which is lifted up to a linear representation $J'_\rho$ of a certain covering group $S([\rho])'$ of $S([\rho])$ (cf. Lemma 2.2). Put $\pi^0 := \rho \cdot J'_\rho$. Take a spin representation $\pi^1$ of $S([\rho])$ corresponding to the factor set inverse to that of $J'_\rho$, then the tensor product $\pi^0 \boxtimes \pi^1$ is a non-spin IR of $U \rtimes S([\rho])$. Inducing $\pi^0 \boxtimes \pi^1$ up to $G$, we get

$$\Pi(\pi^0, \pi^1) = \text{Ind}^G_{U \rtimes S([\rho])}(\pi^0 \boxtimes \pi^1).$$

We give a simple proof for the following theorem (cf. Theorem 5.1).
THEOREM 1.1. Each \( \Pi(\pi_0, \pi_1) \) is irreducible and the set of \( \Pi(\pi_0, \pi_1) \) given above is complete in the sense that the dual \( \hat{G} \) of \( G \) has a complete set of representatives in it, or that any irreducible representation of \( G \) is equivalent to some of \( \Pi(\pi_0, \pi_1) \)'s.

Our proof is elementary and needs only the least minimum on projective representations of groups, given in Section 2. (See [8] or [10] for more knowledge on general theory of such representations.) For proofs of irreducibility and completeness, we utilize fully characters of induced representations. Thus, our proof is completely independent of the results of Clifford in [1], with which we can give another proof (cf. [4], Section 3). It is also independent of Mackey’s works [6], [7], in particular on construction of IRs of semidirect product groups.

2. PREPARATORY LEMMAS

2.1. Central extension. Let \( G' \) be a topological group and \( Z \) its closed central subgroup. Put \( G := G'/Z \). Then the following sequence is exact:

\[ 1 \to Z \to G' \to G \to 1 \] (exact),

and \( G' \) is called a central extension of \( G \) by \( Z \). We call \( G' \) also a covering group of \( G \), and \( G \) the base group of \( G' \).

A representation \( \Pi \) of \( G' \) is said to be of spin type \( \chi \in \hat{Z} \) if \( \Pi(zg') = \chi(z)I \) \((z \in Z, g' \in G')\). Assume \( G' \) is compact. For \( \chi \in \hat{Z} \), denote by \( L^2(G'; \chi) \) the subspace of \( L^2(G') \) consisting of all \( f \in L^2(G') \) of spin type \( \chi \). Then \( L^2(G') \) is an orthogonal direct sum of \( L^2(G'; \chi) \) over \( \chi \in \hat{Z} \). A matrix element \( f \) of a representation \( \Pi \) of spin type \( \chi \) is a spin function of the same type. Take a complete set of representatives \( \Omega^\chi = \{ \Pi \} \) of \( \hat{G}^\chi \), and denote by \( M(\Pi) \) the space spanned by matrix elements of \( \Pi \). Then \( L^2(G'; \chi) \) is an orthogonal direct sum of \( M(\Pi) \) over \( \Pi \in \Omega^\chi \).

LEMMMA 2.1. (i) Let \( G' \) be finite. Then for each \( \chi \in \hat{Z} \)

\[ \sum_{[\Pi] \in \hat{G}^\chi} (\dim \Pi)^2 = \frac{1}{|Z|} |G'| = |G|. \]

(ii) Let \( G' \) be infinite compact. Then for each \( \chi \in \hat{Z} \) the number of equivalence classes in \( \hat{G}^\chi \) is infinite.
P r o o f. The dimension of $\ell^2(G'; \chi)$ is equal to the sum of $\dim M([\Pi])$ over $[\Pi] \in \widehat{G}^X$. Matrix elements of $\Pi$ span $M([\Pi])$, and $\dim M([\Pi]) = (\dim \Pi)^2$. This gives us $\dim \ell^2(G'; \chi) = \sum_{[\Pi] \in \Omega} (\dim \Pi)^2$. On the other hand, a function $f$ on $G'$ belongs to $\ell^2(G'; \chi)$ if and only if it satisfies $f(\pi g') = \chi(z) f(g')$ ($z \in Z$, $g' \in G'$). Hence $\dim \ell^2(G'; \chi) = |G'| / |Z| = |G|$. ■

2.2. Cocycle and central extension. Let $G$ be a topological group. A projective or spin representation $\pi$ of $G$ is defined as a map assigning to each $g \in G$ a linear map $\pi(g)$ on a vector space $V(\pi)$ which satisfies

$$\begin{align*}
\pi(g)\pi(h) &= r_{g,h}\pi(gh) \quad (g, h \in G),
\end{align*}$$

where $r_{g,h} \in C^\times := \{a \in C : a \neq 0\}$. The function $r_{g,h}$ on $G \times G$ is called the factor set associated with $\pi$. If we replace $\pi(g)$ by its scalar multiple $\pi'(g) := \lambda g \pi(g)$, then $\pi'(g)\pi'(h) = r'_{g,h}\pi'(gh)$ with $r'_{g,h} = (\lambda g \lambda h / \lambda gh)r_{g,h}$. When $\pi$ is unitary, $r_{g,h} \in T^1 := \{a \in C : |a| = 1\}$.

On the other hand, a function $r_{g,h} \in C^\times ((g, h) \in G \times G)$ is called a two-cocycle of $G$ (with values in $C^\times$) if it satisfies

$$\begin{align*}
r_{g,h}r_{gh,k} &= r_{g,hk}r_{h,k} \quad (g, h, k \in G).
\end{align*}$$

Defining $r_{g,h}$ to be equivalent to $r'_{g,h}$ given above, we have the second cohomology group $H^2(G, C^\times)$ with multiplication product, called the Schur multiplier of $G$.

We assume that a cocycle $r_{g,h}$ is $T^1$-valued, continuous, and normalized as $r_{e,e} = 1$. Let $Z$ be the closed subgroup of $T^1$ generated by the set of values $r_{g,h}$. Then, since $Z$ is closed, we have only the following two cases:

- Case 1. $Z = \langle e^{2\pi i / n} \rangle \cong Z_n$, a cyclic group of order $n$.
- Case 2. $Z = T^1$.

Starting with a cocycle $r_{g,h}$, we can define a central extension $G'$ of $G$ as follows.

**Lemma 2.2.** Let $G$ be a topological group, $r_{g,h} \ (g, h \in G)$ a cocycle, and let $Z \subset T^1$ be the closed subgroup generated by the set of values $r_{g,h}$.

(i) Introduce in the set $Z \times G$ the following product rule:

$$\begin{align*}
(z, g)(z', h) := (zz'r_{g,h}, gh) \quad (z, z' \in Z, g, h \in G).
\end{align*}$$

Then we get a central extension $G'$ of $G$ by $Z$ as in (2.1).

(ii) Let $\pi$ be a projective (or spin) representation of $G$ whose factor set is $r_{g,h}$ (resp. $r_{g,h}^{-1}$). Then it can be lifted up to a linear representation $\pi'$ of $G'$ acting on the same representation space $V(\pi)$ in such a way that $\pi'( (z, g) ) := z \pi(g)$ (resp. $z^{-1} \pi(g)$) for $(z, g) \in G'$.

We say that the central extension $G'$ in (i) is associated with the cocycle $r_{g,h}$, and that the representation $\pi'$ in (ii) is called a spin representation of $G'$ (and also a spin representation of $G$). When we apply this lemma later in Section 3, it is for a finite group such as $S(\rho)$, and so the central subgroup $Z$ is finite in that case (cf. Section 3).
2.3. Central character of $G'$ and spin representation of $G$. Let $G'$ be a central extension of $G$ by a closed central subgroup $Z$ as in (2.1). Take a section $s : G \to S_G \subset G'$ for the canonical homomorphism $G' \to G$. Then, for $g, h \in G$, we have $s(g) s(h) = z_{g,h} s(gh)$ with a $z_{g,h} \in Z$.

**Lemma 2.3.** For a representation $\Pi$ of $G'$ of spin type $\chi \in \hat{Z}$, put $\pi(g) := \Pi(s(g)) (g \in G)$. Then $\pi$ is a spin representation of $G$ with factor set $t_{g,h} = \chi(z_{g,h})$.

**Lemma 2.4.** Let $G$ be a compact group, and $r_{g,h}$ a continuous cocycle of $G$ with values in $T^1$. Take a central extension $G'$ of $G$ associated with $r_{g,h}$. Then there exist unitary IRs of $G'$ of a certain spin type $\chi$ (resp. $\chi_0$) such that $\chi(z_{g,h}) = r_{g,h}$ (resp. $\chi_0(z_{g,h}) = r_{g,h}^{-1}$).

For the existence of such a $\pi$, we apply Lemma 2.1.

Let $\pi$ and $\pi''$ be spin representations of $G$ with factor sets $r_{g,h}$ and $r''_{g,h}$, respectively. Then the tensor product $\pi \otimes \pi''$ is a spin representation with factor set $r_{g,h}r''_{g,h}$. Therefore, if $r''_{g,h}$ is the inverse of $r_{g,h}$, that is, $r''_{g,h} = r_{g,h}^{-1}$ ($g, h \in G$), then $\pi \otimes \pi''$ is non-spin or is reduced from $G'$ to a linear representation of the base group $G$.

### 3. Case of Semidirect Product Groups

Let $G$ be a compact group of semidirect product type $G = U \rtimes S$, where $U$ is a compact group, normal in $G$, and $S$ is a finite group. Here the action of $s \in S$ on $u \in U$ is denoted by $s(u)$.

Take an IR $\rho$ of $U$ and consider its equivalence class $[\rho] \in \hat{U}$. Every $s \in S$ acts on $\rho$ as $\rho(u) := \rho(s^{-1}(u))$ ($u \in U$), and on equivalence classes as $[\rho] \mapsto [\rho']$.

Denote by $\hat{U}/S$ the set of $S$-orbits in the dual $\hat{U}$ of $U$. Take a stationary subgroup $S([\rho])$ of $[\rho]$ in $S$, that is, $S([\rho]) = \{s \in S : s \rho \cong \rho\}$. Put $H := U \rtimes S([\rho])$. For $s \in S([\rho])$, we determine explicitly an intertwining operator $J_\rho(s)$ as

\[
\rho(s(u)) = J_\rho(s) \rho(u) J_\rho(s)^{-1} \quad (u \in U).
\]

Then it is determined up to a non-zero scalar factor. Hence we have a projective representation $S([\rho]) \ni s \mapsto J_\rho(s)$. Let $\alpha_{s,t}$ be its factor set given as

\[
J_\rho(s) J_\rho(t) = \alpha_{s,t} J_\rho(st) \quad (s, t \in S([\rho])).
\]

Let $S([\rho])'$ be a central extension of $S([\rho])$ associated with the cocycle $\alpha_{s,t} : 1 \to Z \to S([\rho])' \xrightarrow{\Phi_S} S([\rho]) \to 1$ (exact), where $\Phi_S$ denotes the canonical homomorphism. Then, by Lemma 2.4, $J_\rho$ can be lifted up to a linear representation $J'_\rho$ of $S([\rho])'$. Put $H' := U \rtimes S([\rho])'$ with the action $s'(u) := s(u)$, $s' \in S([\rho])'$, $s = \Phi_S(s')$. Put also

\[
\pi^0((u, s')) := \rho(u) \cdot J'_\rho(s') \quad (u \in U, s' \in S([\rho])').
\]
Let \( \pi^0 = \rho \cdot J'_\rho \) be an IR of \( H' \). Take an IR \( \pi^1 \) of \( S([\rho])' \) and consider it as a representation of \( H' \) through the homomorphism \( H' \to S([\rho])' \cong H'/U \), and consider the inner tensor product \( \pi := \pi^0 \boxtimes \pi^1 \) as a representation of \( H' \). Let the factor set of \( \pi^1 \), viewed as a spin representation of the base group \( S([\rho]) \), be \( \beta_{s,t} \); then that of \( \pi \) is \( \alpha_{s,t} \beta_{s,t} \).

To get an IR of \( G \), we pick up \( \pi^1 \) with the factor set \( \beta_{s,t} = \alpha_{s,t}^{-1} \) (this is possible by Lemmas 2.1 and 2.4). Then \( \pi \) becomes a linear representation of the base group \( H = U \rtimes S([\rho]) \). Thus we obtain a representation of \( G \) by inducing it up as
\[
\Pi(\pi^0, \pi^1) := \text{Ind}_H^G \pi = \text{Ind}_H^G (\pi^0 \boxtimes \pi^1).
\]

**Lemma 3.1.** Let \( \rho \) be an IR of \( U \), and \( J'_\rho \) of \( S([\rho])' \) and \( \pi^0 = \rho \cdot J'_\rho \) of \( H' = U \rtimes S([\rho])' \) be as above. Let \( \pi^1 \) and \( \pi^1_o \) be IRs of \( S([\rho])' \), mutually inequivalent, with the factor set inverse to that of \( J'_\rho \). Then \( \pi = \pi^0 \boxtimes \pi^1 \) and \( \pi_o := \pi^0 \boxtimes \pi^1_o \) are irreducible and mutually inequivalent as representations of \( H = U \rtimes S([\rho]) \).

4. Character and Irreducibility of \( \Pi(\pi^0, \pi^1) \)

**4.1. Character of \( \Pi(\pi^0, \pi^1) \).** Put \( \Pi = \Pi(\pi^0, \pi^1) \), and let \( \chi_\Pi \) be the character of \( \Pi \). Since \( \Pi = \text{Ind}_H^G \pi \), we have the following expression of \( \chi_\Pi \) from the general formula for induced representations:
\[
\chi_\Pi(g) = \int_{H \backslash G} \chi_\pi(kgk^{-1}) \, dv_{H \backslash G}(k).
\]

Here the character \( \chi_\pi \) of \( \pi \) is extended from \( H \) to \( G \) by putting 0 outside \( H \), and \( v_{H \backslash G} \) is the invariant measure on \( H \backslash G \) giving mass one to each point, and \( k = Hk \).

Since \( H \backslash G \cong S([\rho]) \backslash S \) is finite, (4.1) can be rewritten, using the normalized Haar measure \( \mu_G \) on \( G \), as
\[
\chi_\Pi(g) = |H \backslash G| \int_G \chi_\pi(kgk^{-1}) \, d\mu_G(k).
\]

Note that, for \( (u, s) \in H = U \rtimes S([\rho]) \), \( \chi_\pi((u, s)) = \chi_{\pi^0}((u, s')) \chi_{\pi^1}(s') \), with a preimage \( s' \in S([\rho])' \) of \( s: s = \Phi_S(s') \).

**4.2. Irreducibility of \( \Pi(\pi^0, \pi^1) \).**

**Theorem 4.1.** Let \( G = U \rtimes S \) with \( U \) compact and \( S \) finite. Then the induced representation \( \Pi(\pi^0, \pi^1) = \text{Ind}_H^G (\pi^0 \boxtimes \pi^1) \) of \( G \) in (3.2) is irreducible.

To prove this, we use the following lemma.

**Lemma 4.1.** Let \( \rho_o \) be an IR of \( U \), and define an IR \( \pi_o^0 := \rho_o \cdot J'_\rho \) of \( H' := U \rtimes S([\rho])' \) in a similar way to \( \pi^0 = \rho \cdot J'_\rho \) of \( H' = U \rtimes S([\rho])' \). Assume that \( \rho_o \) is not equivalent to \( \rho \). Then, for any \( s' \in S([\rho])' \), \( s'' \in S([\rho])' \),
\[
\int_U \chi_{\pi^0}((u, s')) \chi_{\pi^0}((u, s'')) \, d\mu_U(u) = 0.
\]
Proof. Note that the character \( \chi_{\pi^0}(u,s') = \text{tr}(\rho(u)J_{\rho}^\prime(s')) \) is, as a function in \( u \in U \), a linear combination of matrix elements of \( \rho \). Similarly, \( \chi_{\pi^0}(u,s'') = \text{tr}(\rho_0(u)J_{\rho_0}^\prime(s'')) \) is a linear combination of matrix elements of \( \rho_0 \). On the other hand, any matrix element of \( \rho \) is orthogonal in \( L^2(U) \) to any such one of \( \rho_0 \). Hence the assertion of the lemma follows.

Proof of Theorem 4.1. Put \( \Pi = \Pi(\pi^0,\pi^1) \). Note that \( \Pi \) is irreducible if and only if

\[
\| \chi_{\Pi} \|^2 = \int_G |\chi_{\Pi}(g)|^2 \, d\mu_G(g) = 1.
\]

Therefore, it is enough for us to calculate the integral \( \int_G |\chi_{\Pi}(g)|^2 \, d\mu_G(g) \). It is equal to

\[
|H\backslash G|^2 \int_G \int_{G \times G} \chi_{\pi}(k_1gk_1^{-1}) \overline{\chi_{\pi}(k_2gk_2^{-1})} \, d\mu_G(k_1) \, d\mu_G(k_2) \, d\mu_G(g) = |H\backslash G|^2 \int_G \chi_{\pi}(g) \chi_{\pi}(kgk^{-1}) \, d\mu_G(k) \, d\mu_G(g) = \int_H \int_H \chi_{\pi}(h) \overline{\chi_{\pi}(khk^{-1})} \, dv_{H\backslash G}(k) \, d\mu_H(h) =: I_{\pi} \quad (\text{put}).
\]

Take a complete set of representatives of \( H\backslash G \cong S([\rho])\backslash S \) as \( \{s_q \in S \mid q \in Q\} \) with \( s_{q_0} = e \). Then

\[
I_{\pi} = \sum_{q \in Q} \int_H \chi_{\pi}(h) \overline{\chi_{\pi}(s_qhs_q^{-1})} \, d\mu_H(h).
\]

On the other hand, for \( h' = (u,s') \in H' = U \ltimes S([\rho])' \) with \( h = (u,s), s = \Phi_S(s') \), we have \( s_qhs_q^{-1} = (s_qus_q^{-1}, s_qss_q^{-1}) \). Hence

\[
\begin{align*}
\chi_{\pi}(h) &= \text{tr}(\rho(u)J_{\rho}^\prime(s')) \cdot \chi_{\pi^1}(s'), \\
\chi_{\pi}(s_qhs_q^{-1}) &= \text{tr}(\rho(s_qus_q^{-1})J_{\rho}^\prime(s_q's_q's_q^{-1})) \cdot \chi_{\pi^1}(s_q's_q's_q^{-1}),
\end{align*}
\]

where \( s'_q \in S([\rho])' \) is a preimage of \( s_q \); \( \Phi_S(s'_q) = s_q \). For any \( s_q, q \neq q_0 \), since \( s_q \notin S([\rho])' \), \( \Phi_S(s'_q) \) is not equivalent to \( \rho \), where \( \Phi_S(\rho)(u) = \rho(s_qus_q^{-1}) \). Therefore, taking into account \( d\mu_H(h) = d\mu_U(u) \, d\mu_{S([\rho])}(s) \) for \( h = (u,s) \in U \ltimes S([\rho]) \), and the function form in (4.6), we can apply Lemma 4.1 to the integral term in (4.5) for \( q \neq q_0 \), and get \( I_{\pi} = 0 \). Thus, applying Lemma 3.1 to the case of \( q = q_0 \), we obtain, as desired,

\[
I_{\pi} = \int_H \chi_{\pi}(h) \overline{\chi_{\pi}(h)} \, d\mu_H(h) = 1. \quad \blacksquare
\]
For $\chi$, $\chi$ The case of $I$ apply Lemma 4.1 again. Then we see that the integral for any $d\mu$ note that $s$ exists no $s$. Assume $(U := U \times S([\rho_i]), H_i := U \times S([\rho_i]))', \pi_{i,j} = \pi_i^0 \sqcup \pi_j^0$, and $\Pi_{i,j} = \Pi(\pi_i^0, \pi_j^0) = \text{Ind}_{H_i}^{G} \pi_{i,j}$. Define a set of IRs of $G$ as

$$\Omega(G) := \{\Pi_{i,j} := \Pi(\pi_i^0, \pi_j^0) ; i \in I_{U,S}, j \in J_i\}. \tag{4.7}$$

**Theorem 4.2.** For characters $\chi_{\Pi_{i,j}}, \Pi_{i,j} \in \Omega(G)$, the following orthogonality relations in $L^2(G)$ hold true:

$$\langle \chi_{\Pi_{i,j}}, \chi_{\Pi'_{i',j'}}>_{L^2(G)} = \begin{cases} 1 & \text{if } (i,j) = (i',j'), \\ 0 & \text{if } (i,j) \neq (i',j'). \end{cases} \tag{4.8}$$

**Proof.** The case of $(i,j) = (i',j')$ has been shown in the proof of Theorem 4.1. Assume $(i,j) \neq (i',j')$, and put $I_{i',j'}^{i,j} := \langle \chi_{\Pi_{i,j}}, \chi_{\Pi'_{i',j'}}>_{L^2(G)}$. Then, as in the proof of Theorem 4.1, we have

$$I_{i',j'}^{i,j} = |H_i \setminus G| \cdot |H_{i'} \setminus G| \int_{G} \int_{G} \chi_{\pi_{i,j}}(g) \chi_{\pi'_{i',j'}}(kgk^{-1}) \, d\mu_G(k) \, d\mu_G(g).$$

Take a complete set of representatives of $H_{i'} \setminus G \cong S([\rho_{i'}]) \setminus S$ as $\{s_q \in S ; q \in Q\}$ with $s_{q_0} = e$. Then

$$I_{i',j'}^{i,j} = \sum_{q \in Q} \int_{H_i} \chi_{\pi_{i,j}}(h) \chi_{\pi'_{i',j'}}(sh_q s_q^{-1}) \, d\mu_{H_i}(h). \tag{4.9}$$

(1) For $i = i'$, according to the reasoning in the paragraph following the formula (4.5), by Lemma 4.1 we have to consider only the term for $q = q_0$: $I_{i,j}^{i,j} = \int_{H_i} \chi_{\pi_{i,j}}(h) \chi_{\pi'_{i',j'}}(h) \, d\mu_{H_i}(h)$. By Lemma 3.1, we know that $\pi_{i,j}$ is irreducible, and that $\pi_{i,j} \not\cong \pi_{i',j'}$. Therefore $I_{i,j}^{i,j} = \delta_{j,j'}$, as desired.

(2) Assume that $i \neq i'$. Since $S$-orbits of $[\rho_i]$ and $[\rho_{i'}]$ are different, there exists no $s$ such that $s \rho_i \rho_{i'}$. In the sum over $q \in Q$ in (4.9), note that $d\mu_{H_i}(h) = d\mu_U(u) \, d\mu_{S([\rho_i])}(s)$ for $h = (u,s)$, $u \in U$, $s \in S([\rho_i])$, and apply Lemma 4.1 again. Then we see that the integral for any $q$ is equal to zero. So $I_{i,j}^{i,j}$ is zero, as desired. \blacksquare

**Corollary 4.1.** The set $\Omega(G)$ of IRs of $G$ defined in (4.7) consists of mutually inequivalent IRs.
5. COMPLETENESS OF THE SET $\Omega(G)$ OF IRs

Let us prove that the set $\Omega(G)$ in (4.7) of IRs $\Pi_{i,j} = \Pi(\pi^0_i, \pi^1_{i,j})$ is complete, or that our method of induced representations gives essentially all IRs of $G$.

**Theorem 5.1.** Let $G = U \rtimes S$ be such that $U$ is compact and $S$ is finite. Let $\Omega(G)$ be the set of IRs of $G$ defined in (4.7). Then $\Omega(G)$ gives a complete set of representatives of the dual $\hat{G}$.

For the proof, first note that

$$\dim \Pi_{i,j} = \dim \pi^1_{i,j} \cdot |H_i \backslash G| = \dim \rho_i \cdot \dim \pi^1_{i,j} \cdot |S([\rho_i]) \backslash S|. $$

Recall that $\{\pi^1_{i,j}, j \in J_i\}$ is a complete set of representatives of spin IRs of $S([\rho_i])'$, viewed from the base group $S([\rho_i])$, of a fixed factor set (the inverse of that of $J_\rho$). Then we have, by Lemma 2.1 (i),

$$\sum_{j \in J_i} (\dim \pi^1_{i,j})^2 = |S([\rho_i]) \backslash S| \times |S|. $$

(5.1)

**5.1. Proof in the case where $G$ is finite.** Assume that $G = U \rtimes S$ is finite. By Theorem 4.2, to prove the completeness, it is enough to establish the equality

(5.2)

$$\sum_{\Pi_{i,j} \in \Omega(G)} (\dim \Pi_{i,j})^2 = |G|. $$

Note that $(\dim \rho_i)^2 \cdot |S([\rho_i]) \backslash S|$ is equal to the sum of $(\dim \rho)^2$ over $[\rho]$ in the $S$-orbit of $[\rho_i]$. Since $\{\rho_i : I_U, S\}$ is a complete set of representatives of $\hat{U}/S$, we have

$$\sum_{i \in I_U, S} (\dim \rho_i)^2 = \sum_{[\rho] \in \hat{U}} (\dim \rho)^2 = |U|. $$

By (5.1), this gives the desired equality (5.2), because $|U| \cdot |S| = |G|$. 

**5.2. Proof in the case where $G$ is compact.** Let $G = U \rtimes S$ with $U$ compact and $S$ finite. In this case, to prove the completeness of the set $\Omega(G)$ of $\Pi_{i,j}$’s, first we give the following lemma, which corresponds to Lemma 2.1 for $G$ finite.

**Lemma 5.1.** Let $\rho$ be an IR of $U$. Then the number of equivalence classes $[\Pi] \in \hat{G}$ of IRs $\Pi$ of $G$ such that $\Pi|_U$ contains $\rho$, or $\Pi|_U \supset \rho$, is finite, and

(5.3)

$$\sum_{[\Pi] \in \hat{G} : \Pi|_U \supset \rho} (\dim \Pi)^2 = (\dim \rho)^2 \cdot |S([\rho]) \backslash S| \cdot |S|. $$

**Proof.** Denote by $\mathcal{M}_\rho(G)$ the space spanned by matrix elements of $\text{Ind}_U^G \rho$. Then it is a direct sum of spaces $\mathcal{M}(\Pi)$ spanned by matrices of $\Pi$, which appear in $\text{Ind}_U^G \rho$ or $[\text{Ind}_U^G \rho : \Pi] > 0$. By the Frobenius reciprocity law we have $[\Pi|_U : \rho] = \ldots$
ψ

\[ \text{norm in } V \] for \( \Pi \) (5.5)

and so (5.4)

\[ \dim M_\rho(G) = \sum_{[\Pi] \in G : \Pi |_U \supset \rho} (\dim \Pi)^2. \]

On the other hand, the space \( V(\text{Ind}_U^G \rho) \) is spanned by \( \rho \)-valued functions \( f \) on \( G \) such that \( f(ug) = \rho(u)f(g) (u \in U, g \in G) \). Therefore \( f \) corresponds one-to-one to \( \varphi := f|_S \) in \( \mathcal{F}(S; V(\rho)) \), the space of \( \rho \)-valued \( L^2 \)-functions on \( S \) for which the norm is \( \| \varphi \|^2 = \int_S \| \varphi(s) \|^2_{V(\rho)} d\mu_S(s) \), where \( \| \cdot \|_{V(\rho)} \) denotes the norm in \( V(\rho) \). Denote by \( \Pi_\rho \) the realization of \( \text{Ind}_U^G \rho \) on \( \mathcal{F}(S; V(\rho)) \). Note that, for \( s \in S \) and \( g_0 = (u_0, s_0) \in U \times S \), we have \( s g_0 = (e, s) g_0 = (su_0^{-1}, ss_0) \), and so

\[ \Pi_\rho((g_0)) \varphi(s) = \rho(su_0^{-1})(\varphi(ss_0)). \]

The space \( \mathcal{F}(S; V(\rho)) \) is spanned by functions of the form \( \varphi_v, \psi(s) := v \cdot \psi(s) (s \in S) \), where \( v \in V(\rho) \), \( \psi \in L^2(S) \). Take \( \varphi_1, \varphi_2 \in \mathcal{F}(S; V(\rho)) \) as \( \varphi_i(s) = v_i \cdot \psi_i(s) (s \in S) \) with \( v_i \in V(\rho) \) and \( \psi_i \in L^2(S) \). Calculate the matrix element for \( \Pi_\rho \) as

\[
\langle \Pi_\rho(g_0) \varphi_1, \varphi_2 \rangle = \int_S \langle \Pi_\rho(g_0) \varphi_1(s), \varphi_2(s) \rangle_{V(\rho)} d\mu_S(s) = \int_S \langle \rho(su_0^{-1})v_1, v_2 \rangle_{V(\rho)} \psi_1(ss_0) \psi_2(s) d\mu_S(s) =: F(g_0) \text{ (put)}. \]

For \( t \in S \), denote by \( \delta_t \) the delta functions on \( S \) given as \( \delta_t(s) = 1 \) or 0 according to \( s = t \) or not. Put \( \psi_i = \delta_{t_i} \) for \( t_i \in S \). Then

\[ F(g_0) = |S|^{-1} \cdot \langle \rho(t_2u_0t_2^{-1})v_1, v_2 \rangle_{V(\rho)} \cdot \delta_{t_2^{-1}t_1}(s_0). \]

Here, the second factor, as a function in \( u_0 \in U \), spans the space \( \mathcal{M}(\rho) \) of matrix elements of an \( \text{IR} \) \( \rho \) of \( U \). The third factor, as a function in \( s_0 \in S \), i.e. \( \delta_{t_2^{-1}t_1} \) with \( t = t_2^{-1}t_1 \), spans the space \( \mathcal{F}(S) \) of all functions on \( S \). Thus we obtain

\[ \dim M_\rho(G) = \sum_{[\rho]} \dim M(\rho) \cdot \dim \mathcal{F}(S) = \sum_{[\rho]} (\dim \rho)^2 \cdot |S| = (\dim \rho)^2 \cdot |S(\rho)| \cdot |S|. \]

From (5.4) and (5.7) we obtain the desired equality (5.3). 

Applying Lemma 5.1 we see that the completeness of the set \( \Omega(G) \) of \( \Pi_{i,j} \) is equivalent to the following equality, given already in (5.1): for each \( i \in I_{U,S} \),

\[ \sum_{j \in J_i} (\dim \Pi_{i,j})^2 = (\dim \rho_i)^2 \cdot |S(\rho_i)| \cdot |S|. \]
REMARK 5.1. In his Chicago lecture note [6], Mackey discussed construction of irreducible representation or factor representations of semidirect product groups \( U \rtimes S \) mainly in a locally compact case. However, an explicit statement such as Theorem 1.1 or Theorem 5.1 in the present paper cannot be found for the case where \( U \) is compact and \( S \) is finite. It seems that his discussion is beyond this classical case (cf. also [7]).

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REFERENCES


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