A CHAOTIC DECOMPOSITION FOR GENERALIZED STOCHASTIC PROCESSES WITH INDEPENDENT VALUES

BY

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Abstract. We extend the result of Nualart and Schoutens on chaotic decomposition of the $L^2$-space of a Lévy process to the case of a generalized stochastic processes with independent values.

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1. INTRODUCTION

Among all stochastic processes with independent increments, essentially only Brownian motion and Poisson process have a chaotic representation property. The latter property means that, by using multiple stochastic integrals with respect to the centered stochastic process, one can construct a unitary isomorphism between the $L^2$-space of the process and a symmetric Fock space. In the case of a Lévy process, several approaches have been proposed in order to construct a Fock space-type realization of the corresponding $L^2$-space. In this paper, we will be concerned with the approach of Nualart and Schoutens [9], who constructed a representation of every square integrable functional of a Lévy process in terms of orthogonalized Teugels martingales. Recall that, for a given Lévy process $(X_t)_{t\geq 0}$, its $k$-th order Teugels martingale is defined by centering the power jump process

$$X_t^{(k)} := \sum_{0<s\leq t} (\Delta X_s)^k, \quad k \in \mathbb{N}.\$$

For numerous applications of this result, see e.g. [6] and [10]. We also refer to [7] for an extension of this result to the case of a Lévy process taking values in $\mathbb{R}^d$, and

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The aim of this note is to extend the Nualart–Schoutens decomposition to the case of a generalized stochastic process with independent values. Consider a standard triple \( D \subset L^2(\mathbb{R}^d, dx) \subset D' \), where \( D = C_0^\infty(\mathbb{R}^d) \) is the nuclear space of all smooth, compactly supported functions on \( \mathbb{R}^d \), and \( D' \) is the dual space of \( D \) with respect to the center space \( L^2(\mathbb{R}^d, dx) \), see e.g. [2] for detail. For \( \omega \in D' \) and \( \varphi \in D \), we denote by \( \langle \omega, \varphi \rangle \) the dual pairing of \( \omega \) and \( \varphi \). Denote by \( C(D') \) the cylinder \(\sigma\)-algebra on \( D' \). A generalized stochastic process is a probability measure \( \mu \) on \( (D', C(D')) \). Thus, a generalized stochastic process is a random generalized function \( \omega \in D' \). One says that a generalized stochastic process has independent values if for any \( \varphi_1, \ldots, \varphi_n \in D \) which have mutually disjoint support, the random variables \( \langle \omega, \varphi_1 \rangle, \ldots, \langle \omega, \varphi_n \rangle \) are independent. So, heuristically, we infer that, for any \( x_1, \ldots, x_n \in \mathbb{R}^d \), the random variables \( \omega(x_1), \ldots, \omega(x_n) \) are independent. In the case where \( d = 1 \), one can (at least heuristically) interpret \( \omega(t) \) as the time \( t \) derivative of a classical stochastic process \( X = (X(t))_{t \in \mathbb{R}} \) with independent increments, so that, for \( t \geq 0 \), \( X(t) = \int_0^t \omega(s) \, ds \).

If a generalized stochastic process with independent values, \( \mu \), has the property that the measure \( \mu \) remains invariant under each transformation \( x \mapsto x + a \) \( (a \in \mathbb{R}^d) \) of the underlying space, then one calls \( \mu \) a Lévy process (which is, for \( d = 1 \), the time derivative of a classical Lévy process.) So, below, for a certain class of generalized stochastic processes with independent values, we will construct an orthogonal decomposition of the space \( L^2(D', \mu) \), which, in the case of a classical Lévy process, will be exactly the Nualart–Schoutens decomposition from [9]. This paper will also extend the results of [8] for generalized stochastic processes being Lévy processes.

2. PRELIMINARIES

We start by briefly recalling some results from [5]. Assume that, for each \( x \in \mathbb{R}^d \), \( \sigma(x, ds) \) is a probability measure on \( (\mathbb{R}, B(\mathbb{R})) \). We also assume that, for each \( \Delta \in B(\mathbb{R}), \mathbb{R}^d \ni x \mapsto \sigma(x, \Delta) \) is a measurable mapping. Hence, we can define a \(\sigma\)-finite measure \( dx \sigma(x, ds) \) on \( (\mathbb{R}^d \times \mathbb{R}, B(\mathbb{R}^d \times \mathbb{R})) \). Let \( \mathcal{B}_0(\mathbb{R}^d) \) denote the collection of all sets \( \Lambda \in B(\mathbb{R}^d) \) which are bounded. We will additionally assume that, for each \( \Lambda \in \mathcal{B}_0(\mathbb{R}^d) \), there exists \( C_\Lambda > 0 \) such that

\[
(2.1) \quad \int_\mathbb{R} |s|^n \sigma(x, ds) \leq C_\Lambda n!, \quad n \in \mathbb{N},
\]

for all \( x \in \Lambda \). We fix the Hilbert space \( H = L^2(\mathbb{R}^d \times \mathbb{R}, dx \sigma(x, ds)) \). We denote by \( \mathcal{F}(H) = \bigoplus_{n=0}^\infty H^\otimes n \) the symmetric Fock space over \( H \). Here \( \otimes \) denotes symmetric tensor product. We denote by \( \mathcal{D} \) the subset of \( \mathcal{F}(H) \) which consists of all finite vectors \( f = (f^{(0)}, f^{(1)}, \ldots, f^{(n)}, 0, 0, \ldots) \), where each \( f^{(k)} \) is a symmet-
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ric function on \((\mathbb{R}^d \times \mathbb{R})^k\) which is obtained as the symmetrization of a finite sum of functions of the form

\[
g^{(k)}(x_1, s_1, \ldots, x_k, s_k) = \phi(x_1, \ldots, x_k)s_1^{i_1} \ldots s_k^{i_k},
\]

where \(\phi \in D^ \otimes k = C^\infty_0((\mathbb{R}^d)^k)\) and \(i_1, \ldots, i_k \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}\). For each \(\varphi \in D\), we define an operator \(A(\varphi)\) in \(\mathcal{F}(H)\) with domain \(\mathcal{D}\) by

\[
A(\varphi) := a^+(\varphi \otimes m_0) + a^-(\varphi \otimes m_0) + a^0(\varphi \otimes m_1).
\]

Here and below, for \(i \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}\),

\[
(\varphi \otimes m_i)(x, s) := \varphi(x)s^i;
\]

\(a^+(\varphi \otimes m_i)\) is the creation operator corresponding to \(\varphi \otimes m_i\),

\[
a^+(\varphi \otimes m_i)f^{(k)} = f^{(k)} \otimes (\varphi \otimes m_i), \quad f^{(k)} \in H^ \otimes k;
\]

\(a^- (\varphi \otimes m_i)\) is the corresponding annihilation operator,

\[
a^- (\varphi \otimes m_i)f^{(k)} = k \int_{\mathbb{R}^d \times \mathbb{R}} dy \sigma(y, du) \varphi(y) u^1 f^{(k)}(y, u, \cdot);
\]

and \(a^0(\varphi \otimes m_i)\) is the neutral operator corresponding to \(\varphi \otimes m_i\),

\[
(a^0(\varphi \otimes m_i)f^{(k)})(x_1, s_1, \ldots, x_k, s_k)
= (\varphi(x_1)s_1^i \ldots \varphi(x_k)s_k^i) f^{(k)}(x_1, s_1, \ldots, x_k, s_k).
\]

Note that \(A(\varphi)\) maps \(\mathcal{D}\) into itself, and it is a symmetric operator in \(\mathcal{F}(H)\).

**Theorem 2.1.** For each \(\varphi \in \mathcal{D}\), the operator \(A(\varphi)\) is essentially self-adjoint on \(\mathcal{D}\). Furthermore, there exists a unique probability measure \(\mu\) on \(\mathcal{D}\) such that the linear operator \(I : \mathcal{F}(H) \to L^2(\mathcal{D}', \mu)\) given through \(I(\Omega) = 1\), \(\Omega\) being the vacuum vector \((1, 0, 0, \ldots)\), and

\[
I(A(\varphi_1) \ldots A(\varphi_n)\Omega) = \langle \omega, \varphi_1 \rangle \ldots \langle \omega, \varphi_n \rangle,
\]

is a unitary operator. The Fourier transform of the measure \(\mu\) is given by

\[
(2.3) \quad \int_{\mathcal{D}'} e^{i\langle \varphi, \omega \rangle} \mu(d\omega) = \exp\left[-\frac{1}{2} \int_{\mathbb{R}^d} dx \sigma(x, \{0\}) \varphi(x)^2 \right. \right.
\]

\[
+ \left. \left. \int_{\mathbb{R}^d} \int_{\mathbb{R}^*} dx ds \frac{1}{s^2} (e^{is\varphi(x) - i\varphi(x)s} - 1) \right],
\]

where \(\mathbb{R}^* := \mathbb{R} \setminus \{0\}\). In particular, \(\mu\) is a generalized stochastic process with independent values.

Note that, if the measure \(\sigma(ds) = \sigma(x, ds)\) is the same for all \(x \in \mathbb{R}^d\), then \(\mu\) is a Lévy process.
3. AN ORTHOGONAL DECOMPOSITION OF A FOCK SPACE

We will now discuss an orthogonal decomposition of a general symmetric Fock space. This decomposition generalizes the well-known basis of occupation numbers in the Fock space, see e.g. [2].

In this section, we will denote by $H$ any real separable Hilbert space. Let $(H_k)_{k=0}^\infty$ be a sequence of closed subspaces of $H$ such that $H = \bigoplus_{k=0}^\infty H_k$. Let $n \geq 2$. Then clearly

$$H^\otimes n = \bigoplus_{k_1=0}^\infty H_{k_1} \otimes \bigoplus_{k_2=0}^\infty H_{k_2} \otimes \ldots \otimes \bigoplus_{k_n=0}^\infty H_{k_n}$$

(3.1)

Denote by $\text{Sym}_n$ the orthogonal projection of $H^\otimes n$ onto $H^\otimes n$. Recall that, for any $f_1, f_2, \ldots, f_n \in H$,

$$f_1 \otimes \ldots \otimes f_n = \text{Sym}_n f_1 \otimes \ldots \otimes f_n = \frac{1}{n!} \sum_{\sigma \in S_n} f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)}.$$  

(3.2)

(Here, $S_n$ denotes the symmetric group of order $n$.) For each $(k_1, k_2, \ldots, k_n) \in \mathbb{Z}_+^n$, let us assume that $H_{k_1} \otimes H_{k_2} \otimes \ldots \otimes H_{k_n}$ denote the Hilbert space $\text{Sym}_n(H_{k_1} \otimes H_{k_2} \otimes \ldots \otimes H_{k_n})$, i.e., the space of all $\text{Sym}_n$-projections of elements of $H_{k_1} \otimes H_{k_2} \otimes \ldots \otimes H_{k_n}$.

Assume that $(k_1, k_2, \ldots, k_n) \in \mathbb{Z}_+^n$ and $(l_1, l_2, \ldots, l_n) \in \mathbb{Z}_+^n$ are such that there exists a permutation $\sigma \in S_n$ such that

$$(k_1, k_2, \ldots, k_n) = (l_{\sigma(1)}, l_{\sigma(2)}, \ldots, l_{\sigma(n)}).$$

Then

$$H_{k_1} \otimes H_{k_2} \otimes \ldots \otimes H_{k_n} = H_{l_1} \otimes H_{l_2} \otimes \ldots \otimes H_{l_n}.$$  

(3.3)

Indeed, take any $f_1 \in H_{l_1}, f_2 \in H_{l_2}, \ldots, f_n \in H_{l_n}$. Then

$$f_1 \otimes f_2 \otimes \ldots \otimes f_n = f_{\sigma(1)} \otimes f_{\sigma(2)} \otimes \ldots \otimes f_{\sigma(n)}.$$  

(3.4)

We have $f_{\sigma(i)} \in H_{l_{\sigma(i)}} = H_{k_i}$. Therefore, the vector in (3.5) belongs to $H_{k_1} \otimes H_{k_2} \otimes \ldots \otimes H_{k_n}$. Since the set of all vectors of the form $f_1 \otimes f_2 \otimes \ldots \otimes f_n$ with $f_i \in H_{k_i}$ is total in $H_{l_1} \otimes H_{l_2} \otimes \ldots \otimes H_{l_n}$, we conclude that

$$H_{l_1} \otimes H_{l_2} \otimes \ldots \otimes H_{l_n} \subset H_{k_1} \otimes H_{k_2} \otimes \ldots \otimes H_{k_n}.$$  

(3.5)

By inverting the argument, we obtain the inverse conclusion, and so formula (3.4) holds.
If no permutation $\sigma \in S_n$ exists which satisfies (3.3), then

$$H_{k_1} \odot H_{k_2} \odot \ldots \odot H_{k_n} \perp H_{l_1} \odot H_{l_2} \odot \ldots \odot H_{l_n}. \quad (3.6)$$

Indeed, take any $f_i \in H_{k_i}, \; g_i \in H_{l_i}, \; i = 1, 2, \ldots, n$. Then, since $\text{Sym}_n$ is an orthogonal projection,

$$\left( f_1 \odot f_2 \odot \ldots \odot f_n, g_1 \odot g_2 \odot \ldots \odot g_n \right)_{H^n}$$

$$= \left( \text{Sym}_n(f_1 \odot f_2 \odot \ldots \odot f_n), g_1 \odot g_2 \odot \ldots \odot g_n \right)_{H^n}$$

$$= \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^n (f_{\sigma(i)}, g_i)_H = \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^n (f_i, g_{\sigma(i)})_H = 0. \quad (3.6)$$

Since the vectors of the form $f_1 \odot f_2 \odot \ldots \odot f_n$ with $f_i \in H_{k_i}$ and $g_1 \odot g_2 \odot \ldots \odot g_n$ with $g_i \in H_{l_i}$ form a total set in $H_{k_1} \odot H_{k_2} \odot \ldots \odot H_{k_n}$ and $H_{l_1} \odot H_{l_2} \odot \ldots \odot H_{l_n}$, respectively, we get (3.6).

By (3.1), the closed linear span of the spaces $H_{k_1} \odot H_{k_2} \odot \ldots \odot H_{k_n}$ with $(k_1, k_2, \ldots, k_n) \in \mathbb{Z}_+^n$ coincides with $H^\otimes n$. Hence, by (3.4) and (3.6), we get the orthogonal decomposition

$$H^\otimes n = \bigoplus_{\alpha \in \mathbb{Z}^\infty_+, |\alpha| = n} H_0^{\otimes \alpha_0} \odot H_1^{\otimes \alpha_1} \odot H_2^{\otimes \alpha_2} \odot \ldots \quad (3.7)$$

Here $\mathbb{Z}^\infty_+$ denotes the set of indices $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots)$ such that all $\alpha_i \in \mathbb{Z}_+$ and $|\alpha| := \alpha_0 + \alpha_1 + \alpha_2 + \ldots < \infty$. Hence, by (3.7), we get the following

**Lemma 3.1.** We have the orthogonal decomposition of the symmetric Fock space $\mathcal{F}(H) = \bigoplus_{n=0}^{\infty} H^\otimes n$, i.e.,

$$\mathcal{F}(H) = \bigoplus_{\alpha \in \mathbb{Z}^\infty_+, |\alpha| = n} (H_0^{\otimes \alpha_0} \odot H_1^{\otimes \alpha_1} \odot H_2^{\otimes \alpha_2} \odot \ldots) |\alpha|!. \quad (3.8)$$

Next, we have

**Lemma 3.2.** Let $\alpha \in \mathbb{Z}^\infty_+, 0$. Then

$$\text{Sym}_{|\alpha|} : (H_0^{\otimes \alpha_0} \odot H_1^{\otimes \alpha_1} \odot H_2^{\otimes \alpha_2} \odot \ldots) |\alpha_0|! |\alpha_1|! |\alpha_2|! \ldots \rightarrow (H_0^{\otimes \alpha_0} \odot H_1^{\otimes \alpha_1} \odot H_2^{\otimes \alpha_2} \odot \ldots) |\alpha|! \quad (3.9)$$

is a unitary operator.

**Proof.** We start the proof with the following well-known observation. Let $k, l \geq 1, \; n := k + l$. Then $\text{Sym}_n = \text{Sym}_k \otimes \text{Sym}_l$. Hence, for any $\alpha \in$
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Let us assume that each Hilbert space isometry.

\[ \text{Sym} : H \rightarrow \mathcal{F}(H) \]

Since the set of all vectors of the form \( f \) we conclude that the operator in (3.9) is indeed an isometry.

Fix any \( f_i, g_i \in H_i \) with \( i \in \mathbb{Z}_+ \), and any \( \alpha \in \mathbb{Z}_{\alpha_0,0}^\infty \). Then, by (3.2),

\[
\left( \text{Sym}_n(f_0^{\alpha_0} \otimes f_1^{\alpha_1} \otimes f_2^{\alpha_2} \otimes \ldots), \text{Sym}_n(g_0^{\alpha_0} \otimes g_1^{\alpha_1} \otimes g_2^{\alpha_2} \otimes \ldots) \right)_{H^\otimes n}
\]

\[
= \frac{1}{n!} \sum_{\sigma_0 \in S_{\alpha_0}} (f_0, g_0)_{H_0}^\alpha_0 \sum_{\sigma_1 \in S_{\alpha_1}} (f_1, g_1)_{H_1}^\alpha_1 \ldots
\]

\[
= \frac{1}{n!} (f_0^{\alpha_0} \otimes g_0^{\alpha_0})_{H_0}^{\alpha_0} (f_1^{\alpha_1} \otimes g_1^{\alpha_1})_{H_1}^{\alpha_1} \ldots
\]

Since the set of all vectors of the form \( f_i^{\alpha_i} \) with \( f_i \in H_i \) is a total subset of \( H_i^{\otimes \alpha_i} \), we conclude that the operator in (3.9) is indeed an isometry.

We define the symmetrization operator

\[
\text{Sym} : \bigoplus_{\alpha \in \mathbb{Z}_{\alpha_0,0}^\infty} (H_0^{\otimes \alpha_0} \otimes H_1^{\otimes \alpha_1} \otimes H_2^{\otimes \alpha_2} \otimes \ldots)_{\alpha_0!\alpha_1!\alpha_2!} \ldots \rightarrow \mathcal{F}(H)
\]

so that the restriction of Sym to each space

\[
(H_0^{\otimes \alpha_0} \otimes H_1^{\otimes \alpha_1} \otimes H_2^{\otimes \alpha_2} \otimes \ldots)_{\alpha_0!\alpha_1!\alpha_2!} \ldots
\]
is equal to \( \text{Sym}_{|\alpha|} \). By Lemmas 3.1 and 3.2, we get

**Lemma 3.3.** The symmetrization operator \( \text{Sym} \) is a unitary operator.

**Remark 3.1.** Let us assume that each Hilbert space \( H_k \) is one-dimensional and in each \( H_k \) we fix a vector \( e_k \in H_k \) such that \( \|e_k\| = 1 \). Thus, \( (e_k)^{\otimes \infty}_{k=0} \) is an orthonormal basis of \( H \). By Lemma 3.3, the set of the vectors

\[
(\alpha_0!\alpha_1!\alpha_2! \ldots)^{-1/2} e_0^{\otimes \alpha_0} \otimes e_1^{\otimes \alpha_1} \otimes e_2^{\otimes \alpha_2} \otimes \ldots \)_{\alpha \in \mathbb{Z}_{\alpha_0,0}^\infty}
\]
is an orthonormal basis of \( \mathcal{F}(H) \). This basis is called a basis of occupation numbers.
4. AN ORTHOGONAL DECOMPOSITION OF $L^2(D', \mu)$

We want to apply the general result about the orthogonal decomposition of the Fock space to the case of $F(H)$, where $H = L^2(\mathbb{R}^d \times \mathbb{R}, dx \sigma(x, ds))$. We note that, by (2.1), for each $x \in \mathbb{R}^d$, the set of polynomials is dense in $L^2(\mathbb{R}, \sigma(x, ds))$.

We denote by $(q^{(n)}(x, s))_{n \geq 0}$ the sequence of monic polynomials which are orthogonal with respect to the measure $\sigma(x, ds)$. These polynomials satisfy the following recursive formula:

\begin{align*}
q^{(n)}(x, s) &= q^{(n+1)}(x, s) + b_n(x)q^{(n)}(x, s) + a_n(x)q^{(n-1)}(x, s), \quad n \geq 1, \\
q^{(0)}(x, s) &= q^{(1)}(x, s) + b_0(x)
\end{align*}

with some $b_n(x) \in \mathbb{R}$ and $a_n(x) > 0$. (Note that if the support of $\sigma(x, ds)$ consists of $k < \infty$ points, then, for $n \geq k$, we set $q^{(n)}(x, s) = 0$, $a_n(x) = 0$ with $b_n(x) \in \mathbb{R}$ being arbitrary.)

From now on, we will assume that the following condition is satisfied:

(A) For each $n \in \mathbb{N}$, the function $a_n(x)$ from (4.1) is locally bounded on $\mathbb{R}^d$, i.e., for each $\Lambda \in B(\mathbb{R}^d)$, $\sup_{x \in \Lambda} a_n(x) < \infty$.

Denote by $\mathcal{L}$ the linear space of all functions on $\mathbb{R}^d \times \mathbb{R}$ which have the form

\begin{equation}
\sum_{k=0}^{n} a_k(x)q^{(k)}(x, s),
\end{equation}

where $n \in \mathbb{N}$, $a_k \in \mathcal{D}$, $k = 0, 1, \ldots, n$.

**Lemma 4.1.** The space $\mathcal{L}$ is densely embedded into $H$.

**Proof.** Let $f(x, s) = a(x)q^{(k)}(x, s)$, where $a \in \mathcal{D}$. Let us show that $f \in H$. Put $\Lambda := \text{supp}(a)$. We have, for some $C > 0$,

\begin{equation}
\int_{\mathbb{R}^d} \int_{\mathbb{R}} dx \sigma(x, ds) f(x, s)^2 \leq C \int_{\Lambda} dx \int_{\mathbb{R}} \sigma(x, ds) q^{(k)}(x, s)^2.
\end{equation}

If $k = 0$, then $q^{(0)}(x, s) = 1$, and the right-hand side of (4.3) is evidently finite. By the theory of orthogonal polynomials (see e.g. [4])

\begin{equation}
\int_{\mathbb{R}} \sigma(x, ds) q^{(k)}(x, s)^2 = a_1(x)a_2(x)\ldots a_k(x), \quad k \geq 1.
\end{equation}

Hence we continue (4.3) and obtain

\begin{equation}
\int_{\mathbb{R}^d} \int_{\mathbb{R}} dx \sigma(x, ds)f(x, s)^2 \leq C \int_{\Lambda} dx a_1(x)a_2(x)\ldots a_k(x) < \infty
\end{equation}

by (A). Thus, $\mathcal{L} \subset H$. 

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We now have to show that \( \mathcal{L} \) is a dense subset of \( H \). Let \( g \in H \) be such that 
\[
(g, f)_H = 0 \quad \text{for all} \quad f \in \mathcal{L}.
\]
Hence for any \( a \in \mathcal{D} \) and \( k \geq 0 \)
\[
\int_{\mathbb{R}^d} dx \int_{\mathbb{R}} \sigma(x, ds) g(x, s) a(x) q^{(k)}(x, s) = 0.
\]
Fix any compact set \( \Lambda \) in \( \mathbb{R}^d \) and let \( a \in \mathcal{D} \) be such that the support of \( a \) is a subset of \( \Lambda \). Then,
\[
\int_{\mathbb{R}^d} dx a(x) \left( \int_{\mathbb{R}} \sigma(x, ds) g(x, s) q^{(k)}(x, s) \right) = 0.
\]
Hence
\[
\int_{\Lambda} dx a(x) \left( \int_{\mathbb{R}} \sigma(x, ds) g(x, s) q^{(k)}(x, s) \right) = 0. \tag{4.5}
\]
We state that the function
\[
\Lambda \ni x \mapsto \int_{\mathbb{R}} \sigma(x, ds) g(x, s) q^{(k)}(x, s)
\]
belongs to \( L^2(\Lambda, dx) \). Indeed, if \( k = 0 \), then \( q^{(0)}(x, s) = 1 \), and this statement evidently follows from Cauchy's inequality. Assume that \( k \geq 1 \). Then, by Cauchy's inequality, (4.3), and condition (A),
\[
\int_{\Lambda} dx \left( \int_{\mathbb{R}} \sigma(x, ds) g(x, s) q^{(k)}(x, s) \right)^2 \\
\leq \int_{\Lambda} dx \int_{\mathbb{R}} \sigma(x, ds_1) g(x, s_1)^2 \int_{\mathbb{R}} \sigma(x, ds_2) q^{(k)}(x, s_2)^2 \\
= \int_{\Lambda} dx \int_{\mathbb{R}} \sigma(x, ds) g(x, s)^2 a_1(x) a_2(x) \ldots a_k(x) \\
\leq \left( \prod_{i=1}^k \sup_{x \in \Lambda} a_i(x) \right) \int_{\Lambda} dx \int_{\mathbb{R}} \sigma(x, ds) g(x, s)^2 < \infty.
\]
Since the set of all functions \( a \in \mathcal{D} \) with support in \( \Lambda \) is dense in \( L^2(\Lambda, dx) \), we therefore conclude from (4.5) that, for \( dx \)-a.a. \( x \in \Lambda \),
\[
\int_{\mathbb{R}} \sigma(x, ds) g(x, s) q^{(k)}(x, s) = 0 \quad \text{for all} \quad k \geq 0. \tag{4.6}
\]
Since \( g \in H \), we infer that, for \( dx \)-a.a. \( x \in \mathbb{R}^d \), \( g(x, \cdot) \in L^2(\mathbb{R}, \sigma(x, ds)) \). Since \( \{q^{(k)}(x, \cdot)\}_{k=0}^\infty \) form an orthogonal basis in \( L^2(\mathbb{R}, \sigma(x, ds)) \), we conclude from (4.6) that, for \( dx \)-a.a. \( x \in \mathbb{R}^d \), \( g(x, s) = 0 \) for \( \sigma(x, ds) \)-a.a. \( s \in \mathbb{R} \). From here, we easily conclude that \( g = 0 \) as an element of \( H \). Hence \( \mathcal{L} \) is indeed dense in \( H \). \( \blacksquare \)
For each $n \in \mathbb{Z}_+$, we define
\[ \mathcal{L}_n := \{ g_n(x, s) = f(x) q^{(n)}(x, s) \mid f \in D \}. \]
We have $\mathcal{L}_n \subset \mathcal{L}$, and so the linear span of the $\mathcal{L}_n$ spaces coincides with $\mathcal{L}$. For any $g_n(x, s) = f_n(x) q^{(n)}(x, s) \in \mathcal{L}_n$ and $g_m(x, s) = f_m(x) q^{(m)}(x, s) \in \mathcal{L}_m$, $n, m \in \mathbb{Z}_+$, we have
\[
(4.7) \quad (g_n, g_m)_H = \int_{\mathbb{R}^d \times \mathbb{R}} g_n(x, s) g_m(x, s) dx \sigma(x, ds) = \int_{\mathbb{R}^d} f_n(x) f_m(x) \left( \int_{\mathbb{R}} q^{(n)}(x, s) q^{(m)}(x, s) \sigma(x, ds) \right) dx.
\]
Hence, if $n \neq m$, then
\[
(g_n, g_m)_H = 0,
\]
which implies that the linear spaces $\{\mathcal{L}_n\}_{n=0}^{\infty}$ are mutually orthogonal in $H$. Denote by $H_n$ the closure of $\mathcal{L}_n$ in $H$. Then, by Lemma 4.1, $H = \bigoplus_{n=0}^{\infty} H_n$.

By (4.7), setting $n = m$, we get
\[
(4.8) \quad ||g_n||_{H_n}^2 = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}} q^{(n)}(x, s)^2 \sigma(x, ds) \right) dx = \int_{\mathbb{R}^d} f_n^2(x) \rho_n(dx),
\]
where
\[
\rho_n(dx) = \left( \int_{\mathbb{R}} q^{(n)}(x, s)^2 \sigma(x, ds) \right) dx
\]
is a measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Consider a linear operator
\[ \mathcal{D} \ni f_n \mapsto (J_n f_n)(x, s) := f_n(x) q^{(n)}(x, s) \in \mathcal{L}_n. \]
The image of $J_n$ is clearly the whole $\mathcal{L}_n$. Now, $\mathcal{L}_n$ is dense in $H_n$, while $\mathcal{D}$ is evidently dense in $L^2(\mathbb{R}^d, \rho_n(dx))$. By (4.8), for each $f_n \in \mathcal{D}$,
\[
||J_n f_n||_{H_n} = ||f_n||_{L^2(\mathbb{R}^d, \rho_n(dx))}.
\]
Therefore, we can extend the operator $J_n$ by continuity to a unitary operator
\[
(4.9) \quad J_n : L^2(\mathbb{R}^d, \rho_n(dx)) \rightarrow H_n.
\]
In particular,
\[ H_n = \{ f_n(x) q^{(n)}(x, s) \mid f_n \in L^2(\mathbb{R}^d, \rho_n(dx)) \}. \]
Therefore, for each $k \geq 2$,
\[ H_n^\otimes k = \{ f_n^{(k)}(x_1, \ldots, x_k) q^{(n)}(x_1, s_1) \ldots q^{(n)}(x_k, s_k) \mid f_n^{(k)} \in L^2(\mathbb{R}^d, \rho_n(dx)) \} = L^2(\mathbb{R}^d^k, \rho_n(dx_1) \ldots \rho_n(dx_k)). \]
Since the operator $J_n$ in (4.9) is unitary, we infer that the operator

$$J_n^{\otimes k} : L^2(\mathbb{R}^d, \rho_n(dx))^{\otimes k} \rightarrow H_n^{\otimes k}$$

is also unitary. The restriction of $J_n^{\otimes k}$ to $L^2(\mathbb{R}^d, \rho_n(dx))^{\otimes k}$ is a unitary operator

(4.10) $$J_n^{\otimes k} : L^2(\mathbb{R}^d, \rho_n(dx))^{\otimes k} \rightarrow H_n^{\otimes k}.$$ 

Indeed, take any $f_n \in L^2(\mathbb{R}^d, \rho_n(dx))$. Then $f_n^{\otimes k} \in L^2(\mathbb{R}^d, \rho_n(dx))^{\otimes k}$ and the set of all such vectors is total in $L^2(\mathbb{R}^d, \rho_n(dx))^{\otimes k}$. Now, by the definition of $J_n^{\otimes k}$, we get

$$J_n^{\otimes k} f_n^{\otimes k} = (J_n f_n)^{\otimes k} \in H_n^{\otimes k},$$

and furthermore the set of all vectors of the form $(J_n f_n)^{\otimes k}$ is total in $H_n^{\otimes k}$. Hence, the statement follows.

For any $f_n^{(k)} \in L^2(\mathbb{R}^d, \rho_n(dx))^{\otimes k}$,

$$(J_n^{\otimes k} f_n^{(k)})(x_1, s_1, \ldots, x_k, s_k) = f_n^{(k)}(x_1, \ldots, x_k) q^{(n)}(x_1, s_1) \ldots q^{(n)}(x_k, s_k).$$

Hence, the unitary operator (4.10) acts as follows:

$$L^2(\mathbb{R}^d, \rho_n(dx))^{\otimes k} \ni f_n^{(k)}(x_1, \ldots, x_k)$$

$$\mapsto (J_n^{\otimes k} f_n^{(k)})(x_1, s_1, \ldots, x_k, s_k) = f_n^{(k)}(x_1, \ldots, x_k) q^{(n)}(x_1, s_1) \ldots q^{(n)}(x_k, s_k).$$

Thus, each function $g_n^{(k)} \in H_n^{\otimes k}$ has a representation

$$g_n^{(k)}(x_1, s_1, \ldots, x_k, s_k) = f_n^{(k)}(x_1, \ldots, x_k) q^{(n)}(x_1, s_1) \ldots q^{(n)}(x_k, s_k),$$

where $f_n^{(k)} \in L^2(\mathbb{R}^d, \rho_n(dx))^{\otimes k}$ and $\|g_n^{(k)}\|_{H_n^{\otimes k}} = \|f_n^{(k)}\|_{L^2(\mathbb{R}^d, \rho_n(dx))^{\otimes k}}.$

For each $\alpha \in \mathbb{Z}^\infty_{\geq 0}$, we consider the Hilbert space

(4.11) $$L^2_{\alpha}((\mathbb{R}^d)^{|\alpha|}) := L^2(\mathbb{R}^d, \rho_0(dx))^{\otimes |\alpha|} \otimes L^2(\mathbb{R}^d, \rho_1(dx))^{\otimes |\alpha| + 1} \otimes \ldots$$

We now define a unitary operator

$$J_{\alpha} : L^2_{\alpha}((\mathbb{R}^d)^{|\alpha|}) \rightarrow H_0^{\otimes |\alpha|} \otimes H_1^{\otimes |\alpha|} \otimes \ldots,$$

where

$$J_{\alpha} = J_0^{\otimes |\alpha|} \otimes J_1^{\otimes |\alpha|} \otimes \ldots$$

We evidently have, for each $f_\alpha \in L^2_{\alpha}((\mathbb{R}^d)^{|\alpha|}),$

$$(J_{\alpha} f_\alpha)(x_1, s_1, x_2, s_2, \ldots, x_{|\alpha|}, s_{|\alpha|})$$

$$= f_\alpha(x_1, x_2, \ldots, x_{|\alpha|}) q^{(0)}(x_1, s_1) \ldots q^{(0)}(x_{|\alpha|}, s_{|\alpha|})$$

$$\times q^{(1)}(x_{|\alpha|+1}, s_{|\alpha|+1}) \ldots q^{(1)}(x_{|\alpha|+\alpha}, s_{|\alpha|+\alpha}) \ldots$$
For each $\alpha \in \mathbb{Z}_{+0}^\infty$, we define a Hilbert space

$$G_\alpha := L^2_a((\mathbb{R}^d)^{\mid \alpha \mid}) \alpha_0! \alpha_1! \ldots$$

The $J_\alpha$ is evidently a unitary operator

$$J_\alpha : G_\alpha \to (H_0^{\circ \alpha_0} \otimes H_1^{\circ \alpha_1} \otimes \ldots) \alpha_0! \alpha_1! \ldots$$

Put $G := \bigoplus_{\alpha \in \mathbb{Z}_{+0}^\infty} G_\alpha$. Hence, we can construct a unitary operator

$$J : G \to \bigoplus_{\alpha \in \mathbb{Z}_{+0}^\infty} (H_0^{\circ \alpha_0} \otimes H_1^{\circ \alpha_1} \otimes \ldots) \alpha_0! \alpha_1! \ldots$$

by setting $J := \bigoplus_{\alpha \in \mathbb{Z}_{+0}^\infty} J_\alpha$. By Lemma 3.3, we get a unitary operator $R : G \to \mathcal{F}(H)$ by setting $R := \text{Sym} J$. Thus, by Theorem 2.1, we get

**THEOREM 4.1.** Let condition (A) be satisfied. We have a unitary isomorphism $\mathcal{K} : G \to L^2(\mathcal{D}', \mu)$ given by $\mathcal{K} := IR$, where the unitary operator $I : \mathcal{F}(H) \to L^2(\mathcal{D}', \mu)$ is from Theorem 2.1.

5. THE UNITARY ISOMORPHISM $\mathcal{K}$ THROUGH MULTIPLE STOCHASTIC INTEGRALS

We will now give an interpretation of the unitary isomorphism $\mathcal{K}$ in terms of multiple stochastic integrals. We will only present a sketch of the proof, omitting some technical details.

Let us recall the operators $A(\varphi)$ in $\mathcal{F}(H)$ defined by (2.2). Now, for each $k \in \mathbb{N}$, we define operators

$$A^{(k)}(\varphi) := a^+(\varphi \otimes m_{k-1}) + a^0(\varphi \otimes m_k) + a^-(\varphi \otimes m_{k-1}).$$

In particular, $A^{(1)}(\varphi) = A(\varphi)$. The operator $A^{(k)}(\varphi)$ being symmetric, we denote by $A^{(k)}(\varphi)$ the closure of $A^{(k)}(\varphi)$. For each $k \in \mathbb{N}$ and $\varphi \in \mathcal{D}$, we define

$$Y^{(k-1)}(\varphi) := I(\varphi \otimes m_{k-1}).$$

It can be shown that, for each $k \in \mathbb{N}$, $IA^{(k)}(\varphi)^-I^{-1}$ is the operator of multiplication by the function $Y^{(k-1)}$.

Suppose, for a moment, that the measures $\sigma(x, ds)$ do not depend on $x \in \mathbb{R}^d$.

For a fixed $\varphi \in \mathcal{D}$, let us orthogonalize in $L^2(\mathcal{D}', \mu)$ the functions $(Y^{(k)}(\varphi))_{k=0}^\infty$.

This is of course equivalent to the orthogonalization of the monomials $(s^k)_{k=0}^\infty$ in $L^2(\mathbb{R}, \sigma)$. Denote by $(q^{(k)})_{k=0}^\infty$ the system of monic orthogonal polynomials with respect to the measure $\sigma$. Let us put $$(\varphi \otimes q^{(k)})(x, s) := \varphi(x)q^{(k)}(s).$$

Thus, the random variables

$$Z^{(k)}(\varphi) := I(\varphi \otimes q^{(k)}), \quad k \in \mathbb{Z}_+,$$

appear as a result of the orthogonalization of $(Y^{(k)}(\varphi))_{k=0}^\infty$. Since $q^{(0)}(s) = 1$, we have

$$Z^{(0)}(\varphi) = Y^{(0)}(\varphi) = \langle \cdot, \varphi \rangle.$$
For each \( k \geq 1 \), we have a representation of \( q^{(k)}(s) \) as follows:

\[
q^{(k)}(s) = \sum_{i=0}^{k} b_i^{(k)} s^i.
\]

Thus,

\[
Z^{(k)}(\varphi) = I(\varphi \otimes q^{(k)}) = \sum_{i=0}^{k} b_i^{(k)} I(\varphi \otimes m_i) = \sum_{i=0}^{k} b_i^{(k)} Y^{(i)}(\varphi).
\]

Hence, under \( I^{-1} \), the image of the operator of multiplication by \( Z^{(k)}(\varphi) \) is the operator

\[
R^{(k)}(\varphi) := \sum_{i=0}^{k} b_i^{(k)} \left( a^+(\varphi \otimes m_i) + a^-(\varphi \otimes m_i) + a^0(\varphi \otimes m_{i+1}) \right) = a^+(\varphi \otimes q^{(k)}) + a^-(\varphi \otimes q^{(k)}) + a^0(\varphi \otimes \rho^{(k)}),
\]

where \( \rho^{(k)}(s) := sq^{(k)}(s) \).

Let us now consider the general case, i.e., the case where the measure \( \sigma(x, ds) \) does depend on \( x \in \mathbb{R}^d \). We are using the monic polynomials \( \left( q^{(k)}(x, \cdot) \right)_{k=0}^{\infty} \) which are orthogonal with respect to the measure \( \sigma(x, ds) \). We have

\[
q^{(k)}(x, s) = \sum_{i=0}^{k} b_i^{(k)}(x) s^i.
\]

We define

\[
Z^{(k)}(\varphi) := I(\varphi q^{(k)}) = \sum_{i=0}^{k} Y^{(i)}(\varphi b_i^{(k)}),
\]

where \( (\varphi q^{(k)})(x, s) := \varphi(x)q^{(k)}(x, s) \). Hence, under \( I^{-1} \), the image of the operator of multiplication by \( Z^{(k)}(\varphi) \) is the operator

\[
R^{(k)}(\varphi) := \sum_{i=0}^{k} \left( a^+((\varphi b_i^{(k)}) \otimes m_i) + a^-((\varphi b_i^{(k)}) \otimes m_i) + a^0((\varphi b_i^{(k)}) \otimes m_{i+1}) \right) = a^+((\varphi \sum_{i=0}^{k} b_i^{(k)}) \otimes m_i) + a^-(\varphi \sum_{i=0}^{k} b_i^{(k)}) \otimes m_i) + a^0((\varphi b_i^{(k)}) \otimes m_{i+1}) = a^+((\varphi q^{(k)}) + a^-((\varphi q^{(k)}) + a^0(\varphi \rho^{(k)}),
\]

where \( \rho^{(k)}(x, s) := sq^{(k)}(x, s) \).
It is not hard to see that the above definitions and formulas can be easily extended to the case where the function \( \varphi : \mathbb{R}^d \to \mathbb{R} \) is just measurable, bounded, and has compact support. In particular, for each \( \Delta \in \mathcal{B}_0(\mathbb{R}^d) \), we will use the operators \( Z^{(k)}(\Delta) := Z^{(k)}(\chi_{\Delta}) \).

We will now introduce a multiple Wiener–Itô integral with respect to \( Z^{(k)} \)'s. So, we fix any \( \alpha \in \mathbb{Z}_{\geq 0}^d, |\alpha| = n, n \in \mathbb{N} \). Take any \( \Delta_1, \ldots, \Delta_n \in \mathcal{B}_0(\mathbb{R}^d) \), mutually disjoint. Then we define

\[
\int_{\Delta_1 \times \Delta_2 \times \ldots \times \Delta_n} dZ^{(0)}(x_1) \ldots dZ^{(0)}(x_{\alpha_0})dZ^{(1)}(x_{\alpha_0+1}) \ldots dZ^{(1)}(x_{\alpha_0+\alpha_1}) \\
\times dZ^{(2)}(x_{\alpha_0+\alpha_1+1}) \ldots \\
= \int_{(\mathbb{R}^d)^n} \chi_{\Delta_1}(x_1) \chi_{\Delta_2}(x_2) \ldots \chi_{\Delta_n}(x_n) dZ^{(0)}(x_1) \ldots dZ^{(0)}(x_{\alpha_0}) \\
\times dZ^{(1)}(x_{\alpha_0+1}) \ldots dZ^{(1)}(x_{\alpha_0+\alpha_1})dZ^{(2)}(x_{\alpha_0+\alpha_1+1}) \ldots \\
:= Z^{(0)}(\Delta_1) \ldots Z^{(0)}(\Delta_{\alpha_0})Z^{(1)}(\Delta_{\alpha_0+1}) \ldots Z^{(1)}(\Delta_{\alpha_0+\alpha_1})Z^{(2)}(\Delta_{\alpha_0+\alpha_1+1}) \ldots
\]

Using the fact that the sets \( \Delta_1, \ldots, \Delta_n \) are mutually disjoint, we get

\[
I^{-1} \bigl( Z^{(0)}(\Delta_1) \ldots Z^{(0)}(\Delta_{\alpha_0})Z^{(1)}(\Delta_{\alpha_0+1}) \ldots Z^{(1)}(\Delta_{\alpha_0+\alpha_1})Z^{(2)}(\Delta_{\alpha_0+\alpha_1+1}) \ldots \bigr) \\
= R^{(0)}(\chi_{\Delta_1}) \ldots R^{(0)}(\chi_{\Delta_{\alpha_0}})R^{(1)}(\chi_{\Delta_{\alpha_0+1}}) \ldots R^{(1)}(\chi_{\Delta_{\alpha_0+\alpha_1}})R^{(2)}(\chi_{\Delta_{\alpha_0+\alpha_1+1}}) \ldots \\
= a^+(\chi_{\Delta_1} q^{(0)}) \ldots a^+(\chi_{\Delta_{\alpha_0}} q^{(0)})a^+(\chi_{\Delta_{\alpha_0+1}} q^{(1)}) \ldots a^+(\chi_{\Delta_{\alpha_0+\alpha_1}} q^{(1)}) \\
\times a^+(\chi_{\Delta_{\alpha_0+\alpha_1+1}} q^{(2)}) \ldots \Omega \\
= (\chi_{\Delta_1} q^{(0)}) \odot \ldots \odot (\chi_{\Delta_{\alpha_0}} q^{(0)}) \odot (\chi_{\Delta_{\alpha_0+1}} q^{(1)}) \odot \ldots \odot (\chi_{\Delta_{\alpha_0+\alpha_1}} q^{(1)}) \\
\odot (\chi_{\Delta_{\alpha_0+\alpha_1+1}} q^{(2)}) \odot \ldots \\
= \text{Sym}_n \left[ (\chi_{\Delta_1} q^{(0)}) \odot \ldots \odot (\chi_{\Delta_{\alpha_0}} q^{(0)}) \right] \\
\odot \left[ (\chi_{\Delta_{\alpha_0+1}} q^{(1)}) \odot \ldots \odot (\chi_{\Delta_{\alpha_0+\alpha_1}} q^{(1)}) \right] \odot \ldots \\
= \text{Sym}_n \left[ (\chi_{\Delta_1} \odot \ldots \odot \chi_{\Delta_{\alpha_0}})(x_1, \ldots, x_{\alpha_0}) q^{(0)}(x_{\alpha_0}, s_{\alpha_0}) \right] \\
\odot \left[ (\chi_{\Delta_{\alpha_0+1}} \odot \ldots \odot \chi_{\Delta_{\alpha_0+\alpha_1}})(x_{\alpha_0+1}, \ldots, x_{\alpha_0+\alpha_1}) q^{(1)}(x_{\alpha_0+1}, s_{\alpha_0+1}) \right. \\
\left. \ldots q^{(1)}(x_{\alpha_0+\alpha_1}, s_{\alpha_0+1}) \right] \odot \ldots \\
= \mathcal{R} \left[ (\chi_{\Delta_1} \odot \ldots \odot \chi_{\Delta_{\alpha_0}} \odot (\chi_{\Delta_{\alpha_0+1}} \odot \ldots \odot \chi_{\Delta_{\alpha_0+\alpha_1}}) \odot \ldots \right].
\]

Hence

\[
Z^{(0)}(\Delta_1) \ldots Z^{(0)}(\Delta_{\alpha_0})Z^{(1)}(\Delta_{\alpha_0+1}) \ldots Z^{(1)}(\Delta_{\alpha_0+\alpha_1})Z^{(2)}(\Delta_{\alpha_0+\alpha_1+1}) \ldots \\
= \mathcal{K} \left[ (\chi_{\Delta_1} \odot \ldots \odot \chi_{\Delta_{\alpha_0}} \odot (\chi_{\Delta_{\alpha_0+1}} \odot \ldots \odot \chi_{\Delta_{\alpha_0+\alpha_1}}) \odot \ldots \right].
\]
The set of all vectors of the form

\[
\left((\chi_{\Delta_1} \odot \ldots \odot \chi_{\Delta_{n_0}}) \otimes (\chi_{\Delta_{n_0+1}} \odot \ldots \odot \chi_{\Delta_{n_0+n_1}}) \otimes \ldots \right)
\]

is total in \( G_{\alpha} \). Therefore, by linearity and continuity, we can extend the definition of the multiple Winner–Itô integral to the whole space \( G_{\alpha} \). Thus, we get, for each \( f_\alpha \in G_{\alpha} \),

\[
\int_{(\mathbb{R}^d)^{\mid \alpha \mid}} f_\alpha(x_1, \ldots, x_{\mid \alpha \mid}) dZ^{(0)}(x_1) \ldots dZ^{(0)}(x_{n_0}) dZ^{(1)}(x_{n_0+1}) \ldots dZ^{(1)}(x_{n_0+n_1}) \times dZ^{(2)}(x_{n_0+n_1+1}) \ldots = K f_\alpha.
\]

Thus, we have the following theorem.

**Theorem 5.1.** The unitary isomorphism \( K: G \to L^2(D', \mu) \) from Theorem 4.1 is given by

\[
G = \bigoplus_{\alpha \in \mathbb{Z}^+_\infty} G_{\alpha} \ni (f_\alpha)_{\alpha \in \mathbb{Z}^+_\infty} = f \mapsto K f
\]

\[
= \sum_{\alpha \in \mathbb{Z}^+_\infty} \int_{(\mathbb{R}^d)^{\mid \alpha \mid}} f_\alpha(x_1, \ldots, x_{\mid \alpha \mid}) dZ^{(0)}(x_1) \ldots dZ^{(0)}(x_{n_0}) \times dZ^{(1)}(x_{n_0+1}) \ldots dZ^{(1)}(x_{n_0+n_1}) dZ^{(2)}(x_{n_0+n_1+1}) \ldots
\]

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