HERZ–SCHUR MULTIPLIERS AND NON-UNIFORMLY BOUNDED REPRESENTATIONS OF LOCALLY COMPACT GROUPS

BY

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Abstract. Let \( G \) be a second countable, locally compact group and let \( \varphi \) be a continuous Herz–Schur multiplier on \( G \). Our main result gives the existence of a (not necessarily uniformly bounded) strongly continuous representation \( \pi \) of \( G \) on a Hilbert space \( H \), together with vectors \( \xi, \eta \in H \), such that
\[
\varphi(y^{-1}x) = \langle \pi(x)\xi, \pi(y^{-1})^*\eta \rangle \quad \text{for } x, y \in G.
\]
Moreover, we obtain control over the growth of the representation in the sense that
\[
\|\pi(g)\| \lesssim \exp(c_2d(g,e))
\]
for \( g \in G \), where \( e \in G \) is the identity element, \( c \) is a constant, and \( d \) is a metric on \( G \).

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1. INTRODUCTION

Let us assume that \( Y \) is a non-empty set. A function \( \psi : Y \times Y \to \mathbb{C} \) is called a Schur multiplier if for every operator \( A = (a_{x,y})_{x,y \in Y} \in B(\ell^2(Y)) \) the matrix \( (\psi(x,y)a_{x,y})_{x,y \in Y} \) represents an operator from \( B(\ell^2(Y)) \) (this operator is denoted by \( M_\psi A \)). If \( \psi \) is a Schur multiplier it follows from the closed graph theorem that \( M_\psi \in B(B(\ell^2(Y))) \), and one refers to \( \|M_\psi\| \) as the Schur norm of \( \psi \) and denotes it by \( \|\psi\|_S \).

Let \( G \) be a locally compact group. In [6], Herz introduced a class of functions on \( G \), which was later called the class of Herz–Schur multipliers on \( G \). By the introduction to [1], a continuous function \( \varphi : G \to \mathbb{C} \) is a Herz–Schur multiplier if and only if the function
\[
\hat{\varphi}(x, y) = \varphi(y^{-1}x) \quad (x, y \in G)
\]
is a Schur multiplier, and the Herz–Schur norm of \( \varphi \) is given by
\[
\|\varphi\|_{HS} = \|\hat{\varphi}\|_S.
\]
In [3] De Cannière and Haagerup introduced the Banach algebra $MA(G)$ of Fourier multipliers of $G$, consisting of functions $\varphi : G \to \mathbb{C}$ such that

$$\varphi \psi \in A(G) \quad (\psi \in A(G)),$$

where $A(G)$ is the Fourier algebra of $G$ as introduced by Eymard in [4]. The norm of $\varphi$ (denoted by $\|\varphi\|_{MA(G)}$) is given by considering $\varphi$ as an operator on $A(G)$. According to Proposition 1.2 in [3] a Fourier multiplier of $G$ can also be characterized as a continuous function $\varphi : G \to \mathbb{C}$ such that

$$\lambda(g) \Mapsto \varphi(g)\lambda(g) \quad (g \in G)$$

extends to a $\sigma$-weakly continuous operator (still denoted by $M_\varphi$) on the group von Neumann algebra ($\lambda : G \to B(L^2(G))$ is the left regular representation and the group von Neumann algebra is the closure of the span of $\lambda(G)$ in the weak operator topology). Moreover, one has $\|\varphi\|_{MA(G)} = \|M_\varphi\|$. The Banach algebra $M_0A(G)$ of completely bounded Fourier multipliers of $G$ consists of the Fourier multipliers $\varphi$ of $G$, for which $M_\varphi$ is completely bounded. Let $\|\varphi\|_{M_0A(G)} = \|M_\varphi\|_{cb}$.

In [1] Bożejko and Fendler show that the completely bounded Fourier multipliers coincide isometrically with the continuous Herz–Schur multipliers. In [7] Jolissaint gives a short and self-contained proof of this result in the following form.

**Proposition 1.1 ([1], [7]).** Let $G$ be a locally compact group and assume that $\varphi : G \to \mathbb{C}$ and $k \geq 0$ are given. Then the following are equivalent:

(i) $\varphi$ is a completely bounded Fourier multiplier of $G$ with $\|\varphi\|_{M_0A(G)} \leq k$.

(ii) $\varphi$ is a continuous Herz–Schur multiplier on $G$ with $\|\varphi\|_{HS} \leq k$.

(iii) There exist a Hilbert space $\mathcal{H}$ and two bounded, continuous maps $P, Q : G \to \mathcal{H}$ such that

$$\varphi(y^{-1}x) = \langle P(x), Q(y) \rangle \quad (x, y \in G)$$

and

$$\|P\|_\infty \|Q\|_\infty \leq k,$$

where

$$\|P\|_\infty = \sup_{x \in G} \|P(x)\| \quad \text{and} \quad \|Q\|_\infty = \sup_{y \in G} \|Q(y)\|.$$

By a representation $(\pi, \mathcal{H})$ of a locally compact group $G$ on a Hilbert space $\mathcal{H}$ we mean a homomorphism of $G$ into the invertible elements of $B(\mathcal{H})$. A representation $(\pi, \mathcal{H})$ of $G$ is said to be uniformly bounded if

$$\sup_{g \in G} \|\pi(g)\| < \infty.$$
and one usually writes $\|\pi\|$ for $\sup_{g \in G} \|\pi(g)\|$. If $g \mapsto \pi(g)$ is continuous with respect to the strong operator topology on $B(H)$ then we say that $(\pi, H)$ is strongly continuous. Let $(\pi, H)$ be a strongly continuous, uniformly bounded representation of $G$. Then, according to Theorem 2.2 of [3], any coefficient of $(\pi, H)$ is a continuous Herz–Schur multiplier, i.e.,

$$g \mapsto \langle \pi(g)\xi, \eta \rangle \quad (g \in G)$$

is a continuous Herz–Schur multiplier with

$$\|\varphi\|_{M_0A(G)} \leq \|\pi\|^2\|\xi\|\|\eta\|$$

for any $\xi, \eta \in H$ (note that this result also follows as a corollary to Proposition 1.1). U. Haagerup has shown that on the non-abelian free groups there are Herz–Schur multipliers which cannot be realized as coefficients of uniformly bounded representations. The proof by Haagerup has remained unpublished, but Pisier has later given a different proof, cf. [8]. Haagerup’s proof can be modified to prove the corresponding result for the connected, real rank one, simple Lie groups with finite center; cf. [9], Theorem 3.6.

Strictly speaking, the requirement that the above representations be uniformly bounded is not fully needed in order to construct a continuous Herz–Schur multiplier. From Proposition 1.1 it follows that it is enough to require that

$$(1.2) \quad \sup_{x \in G} \|\pi(x)\xi\| < \infty \quad \text{and} \quad \sup_{y \in G} \|\pi(y^{-1})^*\eta\| < \infty.$$  

In Theorem 1.1 of [2] Bożejko and Fendler show that for countable discrete groups all Herz–Schur multipliers can be realized as coefficients of representations satisfying a condition similar to (1.2). More specifically, they show that if $\varphi$ is a Hermitian Herz–Schur multiplier on a countable discrete group $\Gamma$, then there exist a representation $(\pi, H)$ and vectors $\xi, \eta \in H$ such that

$$\varphi(y^{-1}x) = \langle \pi(x)\xi, \pi(y)\eta \rangle \quad (x, y \in \Gamma)$$

with

$$\sup_{x \in \Gamma} \|\pi(x)\xi\| \leq \|\varphi\|_{M_0A(\Gamma)}^{1/2} \quad \text{and} \quad \sup_{y \in \Gamma} \|\pi(y)\eta\| \leq \|\varphi\|_{M_0A(\Gamma)}^{1/2}.$$  

Furthermore, they give a quantitative bound on $\|\pi(g)\|$ for $g \in \Gamma$ and note that the same holds for non-Hermitian Herz–Schur multipliers by including a $\sqrt{2}$ factor in the bound for $\sup_{x \in \Gamma} \|\pi(x)\xi\|$ and $\sup_{y \in \Gamma} \|\pi(y)\eta\|$. In Section 2 we present a generalization of the result by Bożejko and Fendler to second countable, locally compact groups (Theorem 2.1 and Corollary 2.1) and prove our main result, which is the following
**Theorem 1.1.** Let $G$ be a second countable, locally compact group and let $d$ be a proper, left invariant metric on $G$ which has at most exponential growth, i.e.,

$$\mu(B_n(e)) \leq a \cdot e^{bn} \quad (n \in \mathbb{N})$$

for some constants $a, b > 0$, where $\mu$ is a left invariant Haar measure on $G$ and $B_n(e) = \{g \in G : d(g, e) < n\}$ is the open ball of radius $n$ centered at the identity element $e \in G$. Then for any continuous Herz–Schur multiplier $\varphi$ on $G$ there exist a strongly continuous representation $(\pi, \mathcal{H})$ and vectors $\xi, \eta \in \mathcal{H}$ such that

$$\varphi(y^{-1}x) = \langle \pi(x)\xi, \pi(y^{-1})^*\eta \rangle \quad (x, y \in G)$$

with

$$\sup_{x \in G} \|\pi(x)\xi\| = \|\varphi\|_{M_0A(G)}^{1/2} \quad \text{and} \quad \sup_{y \in G} \|\pi(y^{-1})^*\eta\| = \|\varphi\|_{M_0A(G)}^{1/2}.$$ 

Moreover, for every fixed $c > b$, $(\pi, \mathcal{H})$ can be chosen such that

$$\|\pi(g)\| \leq e^{c^2d(g, e)} \quad (g \in G).$$

Note that the existence of a proper, left invariant metric with at most exponential growth on a second countable, locally compact group is guaranteed by [5].

**2. Coefficients of non-uniformly bounded representations**

Second countability guarantees the existence of a proper, left invariant metric, cf. the Theorem in [10]. Actually, according to Haagerup and Przybyszewska [5] one can choose this metric, $d$, to have at most exponential growth, i.e.,

$$\mu(B_n(e)) \leq a \cdot e^{bn} \quad (n \in \mathbb{N})$$

for some constants $a, b > 0$, where $\mu$ is a left invariant Haar measure on $G$.

Inspired by the proof of Theorem 1.1 in [2], we state and prove Theorem 2.1 for Hermitian Herz–Schur multipliers, i.e., Herz–Schur multipliers $\varphi$ for which $\varphi^* = \varphi$, where

$$\varphi^*(g) = \overline{\varphi(g^{-1})} \quad (g \in G).$$

The non-Hermitian case is treated in Corollary 2.1.

**Theorem 2.1.** If $\varphi$ is a continuous Hermitian Herz–Schur multiplier on a second countable, locally compact group $G$, and $d$ is a proper, left invariant metric on $G$ satisfying (2.1) for some $a, b > 0$ (which exists according to [5]), then there exist a strongly continuous representation $(\pi, \mathcal{H})$ and vectors $\xi, \eta \in \mathcal{H}$ such that

$$\varphi(y^{-1}x) = \langle \pi(x)\xi, \pi(y)\eta \rangle \quad (x, y \in G).$$
with
\[ \sup_{x \in G} \| \pi(x) \xi \| = \| \varphi \|_{M_0 A(G)}^{1/2} \quad \text{and} \quad \sup_{y \in G} \| \pi(y) \eta \| = \| \varphi \|_{M_0 A(G)}^{1/2}. \]

Moreover, for every fixed \( c > b \), \((\pi, \mathcal{H})\) can be chosen such that
\[ \| \pi(g) \| \leq e^{c \cdot d(g, e)} \quad (g \in G). \]

Before we proceed with the proof of Theorems 1.1 and 2.1 we need the following application of [5], which was communicated to us by Haagerup.

**Lemma 2.1.** If \( G \) is a second countable, locally compact group, then there exist a positive function \( h \in L^1(G) \) with \( \| h \|_1 = 1 \), and a positive function \( c \) on \( G \) such that
\[ \frac{1}{\| h \|} \int_G e^{-c \cdot d(z, e)} dz \leq \| f \| \leq \int_G e^{-c \cdot d(z, e)} dz \]
for \( g \in G \) and any positive \( f \in L^\infty(G) \), where \( \mu \) is the Haar measure on \( G \). Moreover, we may use
\[ h(g) = \frac{e^{-c \cdot d(g, e)}}{\int_G e^{-c \cdot d(x, e)} d\mu(x)} \quad \text{and} \quad c(g) = e^{c \cdot d(g, e)} \quad (g \in G) \]
for \( c > b \), when \( d \) is a proper, left invariant metric on \( G \) satisfying (2.1).

**Proof.** Let \( \mu \) be a left invariant Haar measure on \( G \) and let \( d \) be a proper, left invariant metric on \( G \) satisfying (2.1). We claim that
\[ 0 < \int_G e^{-c \cdot d(g, e)} d\mu(g) < \infty. \]
Put \( E_1 = B_1(e) \) and define inductively
\[ E_n = B_n(e) \setminus B_{n-1}(e) \quad (n \geq 2). \]
Then \( G \) is the disjoint union of \( E_n \) for \( n \in \mathbb{N} \) and
\[ e^{-cn} \leq e^{-c \cdot d(g, e)} \leq e^{-c(n-1)} \quad (g \in E_n). \]

Hence,
\[ \int_G e^{-c \cdot d(g, e)} d\mu(g) = \sum_{n=1}^{\infty} \int_{E_n} e^{-c \cdot d(g, e)} d\mu(g) \leq \sum_{n=1}^{\infty} e^{-c(n-1)} \mu(E_n) \leq e^c \sum_{n=1}^{\infty} e^{(b-c)n} < \infty \]
because \( c > b \).
By the reverse triangle inequality we see that
\[ |d(z, g^{-1}) - d(z, e)| \leq d(e, g^{-1}) \quad (g, z \in G). \]

Using left invariance of the metric one finds that
\[ |d(gz, e) - d(z, e)| \leq d(g, e) \quad (g, z \in G). \]

This implies
\[ \frac{1}{c(g)} e^{-c \cdot d(z, e)} \leq e^{-c \cdot d(gz, e)} \leq c(g) e^{-c \cdot d(z, e)} \quad (g, z \in G), \]

which is easily seen to complete the proof. ■

**Lemma 2.2.** Assume that $G$ is a second countable, locally compact group, that $\mathcal{H}$ is a Hilbert space, and $R : G \to \mathcal{H}$ is bounded and continuous. Let $R' : G \to L^2(G, \mathcal{H}, \mu)$ be given by
\[ R'(x)(z) = \sqrt{h(z)} R(z^{-1} x) \quad (x, z \in G), \]
where $h \in L^{-1}(G)$ is chosen as in Lemma 2.1. Then $R'$ is bounded and continuous, with $\|R'(x)\|_2 \leq \|R\|_\infty$ for all $x \in G$. Also, let $\mathcal{K}_R = \text{span}\{R'(x) : x \in G\}$ be a sub-Hilbert space of $L^2(G, \mathcal{H}, \mu)$. Then there exists a unique representation $(\pi_R, \mathcal{K}_R)$ such that
\[ \pi_R(g) R'(x) = R'(g x) \quad (g, x \in G). \]

Moreover,
\[ \|\pi_R(g)\| \leq e^{c \cdot d(g, e)} \quad (g \in G) \]
and the representation is strongly continuous.

**Proof.** From Lebesgue’s dominated convergence theorem it follows easily that $R'$ is continuous. To see that $R'$ is bounded, note that
\[ \|R'(x)\|_2^2 = \int_G h(z) \|R(z^{-1} x)\|^2 d\mu(z) \leq \|R\|_\infty^2 \quad (x \in G). \]

If $n \in \mathbb{N}$, $x_1, \ldots, x_n \in G$, and $c_1, \ldots, c_n \in \mathbb{C}$, then Lemma 2.1 implies that
\[ \int_G \| \sum_{i=1}^n c_i R(z^{-1} x_i) \|^2 h(gz) d\mu(z) \leq c(g) \int_G \| \sum_{i=1}^n c_i R(z^{-1} x_i) \|^2 h(z) d\mu(z) \]
for $g \in G$, where
\[ c(g) = e^{c \cdot d(g, e)} \quad (g \in G). \]

It follows that
\[ \| \sum_{i=1}^n c_i R'(g x_i) \|_2^2 \leq c(g) \| \sum_{i=1}^n c_i R'(x_i) \|_2^2 \quad (g \in G), \]
from which we conclude that there exists a unique representation \((\pi_R, \mathcal{H}_R)\) of \(G\) such that
\[
\pi_R(g)R'(x) = R'(gx) \quad (g, x \in G).
\]
Furthermore,
\[
\|\pi_R(g)\| \leq \sqrt{c(g)} \quad (g \in G).
\]

We proceed to show that the representation is strongly continuous. Since \(\text{span}\{R'(x) : x \in G\}\) is total in \(\mathcal{H}_R\) and \(\|\pi_R(g)\| \leq \sqrt{c(g)}\), where \(g \mapsto \sqrt{c(g)}\) is a continuous function, it is enough to show that
\[
\lim_{n \to \infty} \pi_R(g_n)R'(x) = R'(x)
\]
for \(x \in G\), when \((g_n)_{n \in \mathbb{N}}\) is a sequence converging to the identity \(e \in G\) (since \(G\) is second countable, we do not have to consider nets). But \(\pi_R(g_n)R'(x) = R'(g_nx)\), so this follows simply from continuity of \(R'\).

**Proof of Theorem 2.1.** Let us assume that \(\varphi\) is a continuous Hermitian Herz–Schur multiplier and use Proposition 1.1 to find a Hilbert space \(\mathcal{H}\) and bounded, continuous maps \(P, Q : G \to \mathcal{H}\) such that
\[
\varphi(y^{-1}x) = \langle P(x), Q(y) \rangle \quad (x, y \in G)
\]
and
\[
\|P\|_{\infty} = \|Q\|_{\infty} = \|\varphi\|_{M_b A(G)}^{1/2}.
\]
Define
\[
a_{\pm}(x, y) = \frac{1}{4}(P(x) \pm Q(x), P(y) \pm Q(y)) \quad (x, y \in G).
\]
This gives rise to two positive definite, bounded kernels on \(G \times G\) satisfying
\[
a_+(x, y) - a_-(x, y) = \frac{1}{2}\varphi(y^{-1}x) + \frac{1}{2}\varphi^*(y^{-1}x) = \varphi(y^{-1}x) \quad (x, y \in G)
\]
and
\[
a_+(x, x) + a_-(x, x) = \frac{1}{2}\|P(x)\|^2 + \frac{1}{2}\|Q(x)\|^2 \leq \|\varphi\|_{M_b A(G)} (x \in G).
\]
Let
\[
h(g) = \int_G \frac{e^{-c d(g, e)}}{\int_G e^{-c d(x, e)} d\mu(x)} \quad (g \in G)
\]
for some \(c > b\), when \(d\) is a proper, left invariant metric on \(G\) satisfying (2.1) (cf. Lemma 2.1). Define \((P \pm Q)' : G \to L^2(G, \mathcal{H}, \mu)\) by
\[
(P \pm Q)'(x)(z) = \sqrt{h(z)}(P \pm Q)(z^{-1}x) \quad (x, z \in G).
\]
By Lemma 2.2 there exist strongly continuous representations \((\pi_{P\pm Q}, \mathcal{K}_{P\pm Q})\), where \(\mathcal{K}_{P\pm Q} = \bigoplus \{(P' \pm Q')(x) : x \in G\}\) and \(\pi_{P\pm Q}(g)(P \pm Q)'(x) = (P \pm Q)'(gx)\) for \(g, x \in G\). Furthermore, these representations satisfy

\[
\|\pi_{P\pm Q}(g)\| \leq e^{\frac{C}{2} d(g,e)} \quad (g \in G).
\]

Put

\[
A_\pm(x, y) = \langle (P \pm Q)'(x), (P \pm Q)'(y)\rangle_{\mathcal{K}_{P\pm Q}} \quad (x, y \in G).
\]

Then \(A_\pm\) are positive definite, bounded kernels on \(G \times G\) satisfying

\[
A_+(x, y) - A_-(x, y) = \varphi(y^{-1}x) \quad (x, y \in G)
\]

and

\[
A_+(x, x) + A_-(x, x) \leq \|\varphi\|_{\mathcal{M}_0A(G)} \quad (x \in G).
\]

To make the notation less cumbersome, let \(\pi_\pm = \pi_{P\pm Q}\) and \(\mathcal{K}_\pm = \mathcal{K}_{P\pm Q}\) and define \(\xi_\pm = (P \pm Q)'(e)\). Notice that

\[
\langle \pi_\pm(x)\xi, \pi_\pm(y)\eta \rangle_{\mathcal{K}_\pm} = A_\pm(x, y) \quad (x, y \in G),
\]

and that (2.2) now reads

\[
\|\pi_\pm(g)\| \leq e^{\frac{C}{2} d(g,e)} \quad (g \in G).
\]

Put

\[
\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-, \quad \xi = \xi_+ \oplus \xi_-, \quad \eta = \xi_+ \oplus -\xi_-, \quad \text{and} \quad \pi = \pi_+ \oplus \pi_-.
\]

Observe that \(\pi\) is a strongly continuous representation such that

\[
\|\pi(g)\| \leq e^{\frac{C}{2} d(g,e)} \quad (g \in G)
\]

and

\[
\langle \pi(x)\xi, \pi(y)\eta \rangle_{\mathcal{K}} = \varphi(y^{-1}x) \quad (x, y \in G).
\]

Finally, observe that

\[
\|\pi(x)\xi\|^2 = \|\pi_+(x)\xi_+\|^2 + \|\pi_-(x)\xi_-\|^2 = A_+(x, x) + A_-(x, x) \leq \|\varphi\|_{\mathcal{M}_0A(G)}
\]

for \(x \in G\), and similarly

\[
\|\pi(y)\eta\|^2 \leq \|\varphi\|_{\mathcal{M}_0A(G)}
\]

for \(y \in G\). This completes the proof. \(\blacksquare\)
Corollary 2.1. If $\varphi$ is a continuous Herz–Schur multiplier on a second countable, locally compact group $G$, and $d$ is a proper, left invariant metric on $G$ satisfying (2.1) for some $a, b > 0$ (which exist according to [5]), then there exist a strongly continuous representation $(\pi, \mathcal{H})$ and vectors $\xi, \eta \in \mathcal{H}$ such that

$$\varphi(y^{-1}x) = \langle \pi(x)\xi, \pi(y)\eta \rangle \quad (x, y \in G)$$

with

$$\sup_{x \in G} \|\pi(x)\xi\| \leq \sqrt{2}\|\varphi\|_{M_0A(G)}^{1/2} \quad \text{and} \quad \sup_{y \in G} \|\pi(y)\eta\| \leq \sqrt{2}\|\varphi\|_{M_0A(G)}^{1/2}.$$ 

Moreover, for every fixed $c > b$, $(\pi, \mathcal{H})$ can be chosen such that

$$\|\pi(g)\| \leq e^{2c}d(g,e) \quad (g \in G).$$

Proof. This follows from Theorem 2.1 since

$$\varphi = \Re(\varphi) + i\Im(\varphi),$$

where

$$\Re(\varphi) = \frac{\varphi + \varphi^*}{2} \quad \text{and} \quad \Im(\varphi) = \frac{\varphi - \varphi^*}{2i}$$

are continuous Hermitian Herz–Schur multipliers with

$$\|\Re(\varphi)\|_{M_0A(G)} \leq \|\varphi\|_{M_0A(G)} \quad \text{and} \quad \||\Im(\varphi)\|_{M_0A(G)} \leq \|\varphi\|_{M_0A(G)}.$$ 

Thus the proof is complete. 

Proof of Theorem 1.1. Assume that $\varphi$ is a continuous Herz–Schur multiplier and use Proposition 1.1 to find a Hilbert space $\mathcal{H}$ and bounded, continuous maps $P, Q : G \to \mathcal{H}$ such that

$$\varphi(y^{-1}x) = \langle P(x), Q(y) \rangle \quad (x, y \in G)$$

and

$$\|P\|_{\infty} = \|Q\|_{\infty} = \|\varphi\|_{M_0A(G)}^{1/2}. $$

Let

$$h(g) = \int_G e^{-c\cdot d(g,e)}d\mu(x) \quad (g \in G)$$

for some $c > b$, when $d$ is a proper, left invariant metric on $G$ satisfying (2.1) (cf. Lemma 2.1). Define $P', Q' : G \to L^2(G, \mathcal{H}, \mu)$ by

$$P'(x)(z) = \sqrt{h(z)}P(z^{-1}x) \quad \text{and} \quad Q'(y)(z) = \sqrt{h(z)}Q(z^{-1}y) \quad (z \in G)$$

for some $c > b$, when $d$ is a proper, left invariant metric on $G$ satisfying (2.1) (cf. Lemma 2.1). Define $P', Q' : G \to L^2(G, \mathcal{H}, \mu)$ by

$$P'(x)(z) = \sqrt{h(z)}P(z^{-1}x) \quad \text{and} \quad Q'(y)(z) = \sqrt{h(z)}Q(z^{-1}y) \quad (z \in G)$$

for some $c > b$, when $d$ is a proper, left invariant metric on $G$ satisfying (2.1) (cf. Lemma 2.1). Define $P', Q' : G \to L^2(G, \mathcal{H}, \mu)$ by
for \(x, y \in G\). According to Lemma 2.2 there exists a strongly continuous representation \((\pi_P, \mathcal{K}_P)\), where \(\mathcal{K}_P = \mathcal{K} \mathcal{M}_G\{P'(x) : x \in G\}\) and \(\pi_P(g)P'(x) = P'(gx)\) for \(g, x \in G\). Furthermore, this representation satisfies

\[
\|\pi_P(g)\| \leq e^{\frac{c}{2}d(g, e)} \quad (g \in G).
\]

Observe that

\[
\|P'(x)\|_2^2, \|Q'(y)\|_2^2 \leq \|\varphi\|_{M_0A(G)}
\]

and

\[
\langle P'(x), Q'(y) \rangle_{L^2(G, \mathcal{K}_P))} = \int_G h(z)\langle P(z^{-1}x), Q(z^{-1}y) \rangle_{\mathcal{K}}d\mu(z) = \varphi(y^{-1}x)
\]

for \(x, y \in G\). Put \(\xi = P'(e)\) and \(\eta = P_{\mathcal{K}_P}Q'(e)\), where \(P_{\mathcal{K}_P}\) is the orthogonal projection on \(\mathcal{K}_P\). Note that \(\xi, \eta \in \mathcal{K}_P\) and

\[
\varphi(y^{-1}x) = \langle \pi_P(y^{-1}x)\xi, \eta \rangle_{\mathcal{K}_P} = \langle \pi_P(x)\xi, \pi_P(y^{-1})^*\eta \rangle_{\mathcal{K}_P}, \quad (x, y \in G).
\]

It is clear that \(\|\pi_P(x)\xi\|_{\mathcal{K}_P}^2 = \|P'(x)\|_2^2 \leq \|\varphi\|_{M_0A(G)}\). The corresponding result for \(\|\pi_P(y^{-1})^*\eta\|_{\mathcal{K}_P}^2\) requires more work. For \(x \in G\) arbitrary we find that

\[
\langle \pi_P(y^{-1})P'(x), P_{\mathcal{K}_P}Q'(e) \rangle_{\mathcal{K}_P} = \langle P'(y^{-1}x), P_{\mathcal{K}_P}Q'(e) \rangle_{\mathcal{K}_P} = \langle P'(y^{-1}x), Q'(e) \rangle_{\mathcal{K}} = \varphi(y^{-1}x)
\]

\[
= \langle P'(x), Q'(y) \rangle_{\mathcal{K}} = \langle P'(x), P_{\mathcal{K}_P}Q'(y) \rangle_{\mathcal{K}_P}
\]

from which we conclude that \(\pi_P(y^{-1})^*P_{\mathcal{K}_P}Q'(e) = P_{\mathcal{K}_P}Q'(y)\), and therefore

\[
\|\pi_P(y^{-1})^*\eta\|_{\mathcal{K}_P}^2 = \|P_{\mathcal{K}_P}Q'(y)\|_{\mathcal{K}_P}^2 \leq \|Q'(y)\|_2^2 \leq \|\varphi\|_{M_0A(G)}.
\]

Thus the proof is complete. \(\blacksquare\)

**Remark 2.1.** For the free group on \(N\) generators \((2 \leq N < \infty)\) the constants \(a, b\) in (2.1) may be chosen as

\[
a = \frac{N}{(N - 1)(2N - 1)} \quad \text{and} \quad b = \ln(2N - 1).
\]

This implies that for \(r > \sqrt{2N - 1}\) the representations \((\pi, \mathcal{K})\) from Theorems 1.1 and 2.1 and Corollary 2.1 may be chosen to satisfy \(\|\pi(g)\| \leq r^\delta(g, e)\) for all \(g \in G\).
REFERENCES


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