GELFAND–RAIKOV REPRESENTATIONS OF COXETER GROUPS ASSOCIATED WITH POSITIVE DEFINITE NORM FUNCTIONS

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Abstract. The main purpose of the paper is to study the type of Gelfand–Raikov representations of Coxeter groups \((W, S)\) for the special positive definite functions coming from the deformed Poisson (Haagerup) positive definite functions \(q^{L(w)}\) for some special length (norm) functions \(L\) on Coxeter groups \(W\).

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1. INTRODUCTION

In recent investigations of non-commutative, free probability and type \(B\)-free probability, important roles are taken by the representations and the geometry of Cayley graphs of the symmetry groups \(S_n\), type \(B\)-symmetry groups and, in general, by Coxeter groups \((W, S)\) — see the papers: [5]–[11], [12], [13], and [16].

In the last papers connected with the free infinite divisibility, the first main step was done looking for the length function on \(S_n\), a so-called block length function: for \(w \in S_n\) and \(s_k = (k, k+1)\), \(\|w\| := \text{cardinality of different Coxeter generators } s_{ij} \text{ in an irreducible decomposition } w = s_{i_1}s_{i_2}\ldots s_{i_k}; \text{ see [II], [12], and [5]–[7].}

The norm functions: \(\|w\|_s\) and \(\|w\| := \sum_{s \in S} \|w\|_s\), are the main objects of our paper. We will study the positive definite functions like \(f_{q,s}(w) = q^{\|w\|_s}, f_q(w) = q^{\|w\|}\) and related positive definite functions \(f_Q,F\) and \(f_Q\) (for the definition, cf. (2.3) below) of the Coxeter group \((W, S)\). We are interested in the following:

PROBLEM 1. Are Gelfand–Raikov representations associated with the non-central positive definite functions \(f_{q,s}, f_q, f_Q,F,\) and \(f_Q\) irreducible or factorial? What are their structures?

PROBLEM 2. What does it mean the infinite divisibility of \(f\) for Gelfand–Raikov representation \(\pi_f\) for \(f = f_{q,s}\)?
To answer these questions we will work with the Gelfand–Raikov (GNS) construction of representations coming from the given positive definite, normalized functions on $W$ and the structure of those representations like: inducing representations, cyclic representations, factoriality for special situations (see Sections 3–6 below). The main results for Coxeter groups are given in Section 7 and the next sections. One of the more interesting results is Theorem 8.1, saying that for finite Coxeter groups, the Gelfand–Raikov representation related to our length function, the function $f_q = \prod_{s \in S} f_{q,s}$, is equivalent to the left regular representation of the group $W$.

Also some structure results are presented in Theorem 10.1 in the case of the affine Weyl (Coxeter) group, and in Theorem 11.1 for compact hyperbolic Coxeter groups. Our representations in typical cases are: either the left regular ones $L_W$ or direct sums of trivial representations $1_W$ and those induced from certain parabolic subgroups (containing $L_W$).

2. THE $s$-SEMINORM $\|w\|_s$ ON A COXETER GROUP AND PROBLEM SETTING

Definition 2.1. A Coxeter group $(W, S)$ is defined by a set $S$, $|S| \leq \infty$, of generators and a set of fundamental relations of the form

\[(ss')^{m(s,s')} = e \quad (s, s' \in S),\]

\[m(s, s') = m(s', s) \in \{\infty, 1, 2, \ldots\}, \quad m(s, s') = 1 \text{ if and only if } s = s',\]

where $e$ denotes the identity element of $W$. The order $|S|$ is called a rank of $W$. A subgroup $\langle J \rangle$ generated by a subset $J$ of $S$ is called a parabolic subgroup of $W$.

Our fundamental reference for Coxeter groups is the book of Humphreys [21].

Prepare an undirected graph with $S$ as vertex set, joining vertices $s$ and $s'$ by an edge whenever $m(s, s') \geq 3$, labelled $m(s, s')$ if $m(s, s') \geq 4$. We call $\Gamma$ a Coxeter graph of $(W, S)$. A Coxeter group is called irreducible if its Coxeter graph is connected. In the case where $S$ is finite, as a geometric representation of $(W, S)$, we prepare a vector space $V$ over $\mathbb{R}$ with basis $\{\alpha_s; s \in S\}$, and a symmetric bilinear form $B$ on $V$ given by

\[B(\alpha_s, \alpha_{s'}) := -\cos \frac{\pi}{m(s, s')} \cdot \]

Then $B(\alpha_s, \alpha_s) = 1$, $B(\alpha_s, \alpha_{s'}) \leq 0$ ($s \neq s'$). For each $s \in S$, define a reflection $\sigma_s$ on $V$ by $\sigma_s(\lambda) := \lambda - 2B(\alpha_s, \lambda)\alpha_s$ ($\lambda \in V$). Then the order of $\sigma_s\sigma_{s'}$ is exactly $m(s, s')$ for $s, s' \in S$. This geometric representation $\sigma$ is not necessarily faithful. When $B$ is positive definite (resp. positive semidefinite), we call $\Gamma$ positive definite (resp. positive semidefinite). For an irreducible $(W, S)$, its graph $\Gamma$ is positive definite if and only if $W$ is finite, and $\Gamma$ is positive semidefinite (but not positive definite) if and only if $W$ is an affine Weyl group. All finite irreducible Coxeter groups are given in [21], Sections 2.4 and 6.4.
Let \( \psi_f \) be a positive definite function on the space of all functions on \( G \).

In particular, a positive definite function \( f \) is \( G \)-invariant, non-negative definite inner product on \( \mathfrak{g} \) with respect to \( S \) and a seminorm \( \| \cdot \|_S \), in connection with asymptotic theory of characters or invariant positive definite functions (cf., e.g., [18]–[20] and [22]).

Now we are concerned with functions defined in (2.3).

For a fixed \( q \), such that \( 0 < q_i < 1 \), we have \( f_{q,v}(w) = f_{q,v_1}(w) \cdot f_{q,v_2}(w) \). In particular, if \( 0 < q < 1 \), then \( f_{q,v} = (f_{q,v/m})^m \) for any \( m \geq 2 \), and so it is infinitely divisible.

We are interested in Problems 1 and 2 stated in the Introduction.

### 3. GELFAND–RAIKOV REPRESENTATIONS, POSITIVE DEFINITE FUNCTIONS


Let \( G \) be a discrete group, \( \mathcal{P}(G) \) the set of all positive definite functions on \( G \), and \( \mathcal{P}_1(G) \) the subset consisting of \( f \in \mathcal{P}(G) \) normalized as \( f(e) = 1 \) at the unit element \( e \in G \). Let \( \mathfrak{g}(G) \) be the space of all functions on \( G \) which vanish outside of a finite number of elements, and consider it as a \( G \)-module by left translations of \( g \in G \). A function \( F \) on \( G \) defines a linear functional on \( \mathfrak{g}(G) \) by \( F(\psi) := \sum_{g \in G} F(g) \psi(g) \) \( (\psi \in \mathfrak{g}(G)) \). In particular, a positive definite function \( f \) gives a positive semidefinite inner product on \( \mathfrak{g}(G) \) by \( \langle \varphi, \psi \rangle_f := f(\varphi \ast \psi^* \rangle \), where \( (\varphi \ast \psi)(g) := \sum_{h \in G} \varphi(h^{-1}g) \psi(h) \), \( \psi^*(g) = \overline{\psi(g^{-1})} \).

Then this gives a \( G \)-invariant, non-negative definite inner product in \( \mathfrak{g}(G) \), and a seminorm \( \| \cdot \|_f := \sqrt{\langle \cdot, \cdot \rangle_f} \). Let \( J_f \) be the kernel of this inner product. Then we get on the quotient space \( \mathfrak{g}(G)/J_f \) a positive definite inner product \( \langle \cdot, \cdot \rangle_f \), and then completing it, we obtain a Hilbert space \( \mathcal{H}_f \). For \( \varphi \in \mathfrak{g}(G) \), its image in \( \mathcal{H}_f \) is denoted by \( \varphi_f \); then \( \| \varphi_f \|_f = \| \varphi \|_f \).

On \( \mathcal{H}_f \), we get a unitary representation \( \pi_f \) of \( G \) with a cyclic vector \( \psi_f := \delta_e \) the image of the delta function \( \delta_e \in \mathfrak{g}(G) \) such that

\[
(3.1) \quad f(g) = \langle \pi_f(g)\psi_f, \psi_f \rangle_f, \quad (\pi_f(g)\psi_f, \pi_f(h)\psi_f)_f = f(h^{-1}g).
\]
We call \( \pi_f \) a Gelfand–Raikov representation (GR representation in short) associated with \( f \), since Gelfand and Raikov [LS] gave this construction in 1943, and thus proved for the first time that any locally compact group has sufficiently many unitary representations.

3.2. Cyclic vectors corresponding to a positive definite function. In this paper, a representation of a discrete group \( G \) is usually assumed to be unitary. A cyclic representation \( \pi \) with a specified cyclic vector \( v \) is denoted by \( (\pi, v) \), similarly to \( (\pi_f, v_f) \). This means that we have also specified the positive definite function \( f(g) = \langle \pi(g)v, v \rangle \) associated with \( v \). When \( \pi_i \) \((i = 1, 2)\) are both cyclic, and unit vectors \( v_i \in V(\pi_i) \) \((i = 1, 2)\) define the same positive definite function \( \langle \pi_1(g)v_1, v_1 \rangle_{V(\pi_1)} = \langle \pi_2(g)v_2, v_2 \rangle_{V(\pi_2)} \), then we write \( (\pi_1, v_1) \cong (\pi_2, v_2) \).

**Lemma 3.1.** Let \( \pi_i \) \((i = 1, 2)\) be a cyclic representation of \( G \) with cyclic vectors \( v_i \), respectively. Assume that there exists a \( G \)-module homomorphism \( \Phi : V(\pi_1) \to V(\pi_2) \) such that \( \Phi(v_1) = v_2 \) and that the associated positive definite functions \( f_i \) \((i = 1, 2)\), defined by \( f_i(g) := \langle \pi_i(g)v_i, v_i \rangle_{V(\pi_i)} \) \((i = 1, 2; g \in G)\), coincide with each other. Then \( \Phi \) is necessarily unitary.

**Lemma 3.2.** Let \( \pi \) be a cyclic representation of \( G \). For two unit cyclic vectors \( v_\alpha \) and \( v_\beta \) for \( \pi \), the corresponding positive definite functions are the same as \( f(g) = \langle \pi(g)v_\alpha, v_\alpha \rangle = \langle \pi(g)v_\beta, v_\beta \rangle \) if and only if there exists a unitary intertwining operator \( U \) of \( \pi \) such that \( Uv_\alpha = v_\beta \).

3.3. Partial orders for positive definite functions and subrepresentations. For \( f_1, f_2 \in \mathcal{P}(G) \), we define a partial order \( f_1 \lesssim f_2 \) if \( \langle \varphi, \varphi \rangle_{f_1} \leq \langle \varphi, \varphi \rangle_{f_2} \) for any \( \varphi = \sum_{1 \leq i \leq n} c_i g_i \in \mathfrak{g}(G) \) with \( c_i \in \mathbb{C} \), or

\[
\sum_{1 \leq i, j \leq n} c_i c_j f_1(g_j^{-1} g_i) \leq \sum_{1 \leq i, j \leq n} c_i c_j f_2(g_j^{-1} g_i).
\]

Note that \( f_1 \lesssim f_2 \) is equivalent to that \( f_2 - f_1 \) is again positive definite. We also define \( f_1 \preceq f_2 \) if \( f_1 \lesssim a f_2 \) for some \( a > 0 \).

**Lemma 3.3.** Suppose that \( f_1 \lesssim f_2 \) for \( f_1, f_2 \in \mathcal{P}(G) \). Then there exists a natural \( G \)-module homomorphism \( P_{f_1, f_2} \) from \( \pi_{f_1} \) onto \( \pi_{f_2} \) such that \( \varphi_{f_2} \rightarrow \varphi_{f_1} \) \((\varphi \in \mathfrak{g}(G))\).

**Proof.** Since \( \langle \varphi, \varphi \rangle_{f_1} \leq \langle \varphi, \varphi \rangle_{f_2} \) \((\varphi \in \mathfrak{g}(G))\), we have \( J_{f_1} \supset J_{f_2} \), and there exists a self-adjoint positive operator \( A \) on \( \mathfrak{g}_{f_2} \) such that \( \langle A \varphi_{f_2}, \varphi_{f_2} \rangle_{f_2} = \langle \varphi, \varphi \rangle_{f_1} \) for all \( \varphi \in \mathfrak{g}_{f_2} \), and that \( A \) commutes with \( \pi_{f_2}(g) \) \((g \in G)\). Take \( B = \sqrt{A} \); then it also commutes with \( \pi_{f_2}(g) \) and \( \langle B \varphi_{f_2}, B \varphi_{f_2} \rangle_{f_2} = \langle \varphi, \varphi \rangle_{f_1} \). From this we see that the image \( B(\mathfrak{g}_{f_2}) \) is isomorphic to \( \mathfrak{g}_{f_1} \). Denote by \( Q \) this natural isomorphism; then \( P_{f_1, f_2} := Q B \) satisfies \( P_{f_1, f_2}(\varphi_{f_2}) = \varphi_{f_1} \) \((\varphi \in \mathfrak{g}(G))\), and gives a surjective map from \( \mathfrak{g}_{f_2} \) onto \( \mathfrak{g}_{f_1} \). It intertwines \( \pi_{f_2} \) with \( \pi_{f_1} \). \( \blacksquare \)
Suppose \( f_1 \preceq f_2 \) for \( f_1, f_2 \in P(G) \). Then there exists a natural \( G \)-homomorphic map \( P_{f_1, f_2} \) from \( S_{f_2} \) onto \( S_{f_1} \) defined by \( P_{f_1, f_2}(\phi^{f_2}) = \phi^{f_1} \), which intertwines \( \pi_{f_2} \) with \( \pi_{f_1} \). Furthermore, suppose \( f_1 \preceq f_2 \) and \( f_2 \preceq f_1 \) at the same time. Then \( P_{f_1, f_2} \) gives a natural \( G \)-module isomorphism between \( \pi_{f_2} \) and \( \pi_{f_1} \).

4. THE CASE OF \( s \)-SEMINORM AND POSITIVE DEFINITE FUNCTION \( f_{q_s, s} \)

4.1. Decomposition of positive definite functions \( f_{q_s, s} \). Denote by \( W_s \) the subgroup of \( W \) consisting of elements \( w \) with \( \|w\|_s = 0 \). Then it is also a Coxeter group with the set of generators \( S_s := S \setminus \{s\} \), and for \( 0 < q_s < 1 \),

\[
(4.1) \quad f_{q_s, s}(w) = q_s^{\|w\|_s} = \begin{cases} 
1 & \text{if } w \in W_s, \\
q_s & \text{if } w \in W \setminus W_s,
\end{cases}
\]

\[
(4.2) \quad f_{q_s, s} = q_s 1_W + (1 - q_s) X_{W_s},
\]

where \( 1_W \) denotes the constant function equal to 1 on \( W \), and \( X_{W_s} \) is the trivial character of \( W_s \) extended as 0 on \( W \setminus W_s \). These formulas are valid for \( q_s = 0 \) with \( 0^0 = 1 \).

Note that the formula (4.2) proves that \( f_{q_s, s} \) is positive definite on \( W \), since the function \( X_{W_s} \) is a diagonal matrix element of the induced representation \( \Pi_s := \text{Ind}_{W_s}^W 1_{W_s} \) of the trivial representation \( 1_{W_s} \) of \( W_s \), as seen in Lemma 4.1 below.

4.2. Induced representations and GR representations. In general, let \( G \) be a discrete group and \( H \) its subgroup, and denote by \( \mu_{G/H} \) a \( G \)-invariant measure on \( G/H \). For a unitary representation \( \rho_H \) of \( H \), its induced representation \( \Pi := \text{Ind}_H^G \rho_H \) is defined as follows. Let \( V' \) be a space of functions \( \varphi \) on \( G \) with values in the space \( V(\rho_H) \) of \( \rho_H \) satisfying

\[
(1-1) \quad \varphi(gh) = \rho_H(h)^{-1}(\varphi(g)) \quad (h \in H, g \in G),
\]

\[
(1-2) \quad \|\varphi\|^2 = \int_{G/H} |\varphi(g)|^2 d\mu_{G/H}(gH) < \infty.
\]

Dividing \( V' \) by the kernel of the inner product and then completing, we get a Hilbert space \( V \). The operator \( \Pi(g_0) \) is defined by \( \Pi(g_0)\varphi(g) := \varphi(g_0^{-1}g) \) \( (g \in G) \).

As positive definite functions on \( W \), \( 1_W \) is the character of the trivial representation \( 1_W \), and \( X_{W_s} \) is a matrix element of \( \Pi_s \).

**Lemma 4.1.** The function \( X_{W_s} \) on \( W \) is a diagonal matrix element of \( \Pi_s := \text{Ind}_{W_s}^W 1_{W_s} \), corresponding to a cyclic vector \( X_{W_s} \) in \( V(\Pi_s) \).

Let us note that the GR representation \( \pi_f \) associated with \( f \) is characterized, modulo equivalence, as a cyclic representation containing a unit vector \( \mathbf{v}_f \) such that \( \langle \pi_f(g)\mathbf{v}_f, \mathbf{v}_f \rangle = f(g) \).
Consider the direct sum $\pi_s := 1_W \oplus \Pi_s$ of two representations, and take a unit vector $v_{1,s} \in V(\pi_s)$ given by
\begin{equation}
(4.3) \quad v_{1,s} = \sqrt{q_s} \cdot 1_W \oplus \sqrt{1-q_s} \cdot X_{W_s}.
\end{equation}

**Lemma 4.2.** The cyclic subrepresentation $\pi'_s$ of $\pi_s = 1_W \oplus \Pi_s$ on the subspace $\langle \pi_s(W)v_{1,s} \rangle$ generated by $v_{1,s}$ is unitarily equivalent to the GR representation $\pi_{f_{q_s,s}}$. A $W$-isomorphism from $\pi'_s$ to $\pi_{f_{q_s,s}}$, which maps the unit cyclic vector $v_{1,s} \in V(\pi'_s)$ to the unit cyclic vector $v_{f_{q_s,s}} \in V(\pi_{f_{q_s,s}})$, is unique.

### 5. GR Representations $\pi_{f_{q,F}}$ and Induced Ones

#### 5.1. Isomorphism of $\pi_{f_{q,F}}$ into the Direct Sum $\bigoplus_{F' \subset F} \text{Ind}_{W_{F'}}^W 1_{W_{F'}}$.

For a subset $F$ of $S$, put $W_F := \langle S \setminus F \rangle$ the subgroup generated by $S \setminus F$. From a property of Coxeter groups we have the following.

**Lemma 5.1.** For two subsets $F_1, F_2$ of $S$, we have $W_{F_1} \cap W_{F_2} = W_{F_1 \cup F_2}$. For a subset $B$ of $W$, denote by $X_B$ the indicator function of $B$. Then $X_{W_{F_1}} \cdot X_{W_{F_2}} = X_{W_{F_1 \cup F_2}}$.

Let $Q = (q_s)_{s \in S}, 0 \leq q_s \leq 1 \ (s \in S)$. For a subset $F \subset S$, we have a product formula as $f_Q = f_{Q,F} : f_{Q,S \setminus F}$. If $F$ is finite, applying Lemma 5.1. we have the following expression of the positive definite function $f_{Q,F}$:

\begin{equation}
(5.1) \quad f_{Q,F} = \prod_{s \in F} (q_s 1_W + (1 - q_s) X_{W_s}) = \sum_{F' \subset F} c_{Q,F,F'} X_{W_{F'}} ,
\end{equation}

\begin{equation}
(5.2) \quad c_{Q,F,F'} := \prod_{s \in F \setminus F'} q_s \prod_{t \in F'} (1 - q_t).
\end{equation}

For a finite subset $F \subset S$, consider the direct sum $\pi_{(F)}$ of quasi-regular representations $\Pi_{F'} := \text{Ind}_{W_{F'}}^W 1_{W_{F'}}$, induced from subgroups $W_{F'} = \langle S \setminus F' \rangle$ with $F' \subset F$, as

\begin{equation}
(5.2) \quad \pi_{(F)} := \bigoplus_{F' \subset F} \Pi_{F'} , \quad V(\pi_{(F)}) := \bigoplus_{F' \subset F} V(\Pi_{F'}),
\end{equation}

and take a unit vector $w_{Q,F}$ of $V(\pi_{(F)})$ depending on $Q$ as

\begin{equation}
(5.3) \quad w_{Q,F} := \bigoplus_{F' \subset F} d_{Q,F,F'} X_{W_{F'}} .
\end{equation}

Here, $\Pi_{\emptyset} = 1_W$ for $F' = \emptyset$, since $W_{\emptyset} = \langle S \rangle = W$. The diagonal matrix element of $\pi_{(F)}$ with respect to $w_{Q,F}$ is

\begin{equation}
(5.4) \quad \langle \pi_{(F)}(g) w_{Q,F}, w_{Q,F} \rangle_{V(\pi_{(F)})} = \sum_{F' \subset F} (d_{Q,F,F'})^2 X_{W_{F'}}(g) \quad (g \in W).
\end{equation}
5.1. Orders among positive definite functions on \(W\).

**Lemma 5.2.** (i) For a subset \(F \subset S\), assume that \(W_F = \langle S \setminus F \rangle\) is finite. Then the positive definite function \(X_{W_F}\) on \(W\) is dominated by \(X_{W_F} = X_{\{e\}} = \delta_e\) as \(X_{W_F} \lesssim |W_F| \cdot \delta_e\).

(ii) For two subsets \(F_1 \subset F_2 \subset S\), assume that \(|W_{F_1} : W_{F_2}| < \infty\). Then \(X_{W_{F_1}} \lesssim X_{W_{F_2}}\) or, more exactly, \(X_{W_{F_1}} \lesssim |W_{F_1}|/|W_{F_2}| \cdot X_{W_{F_2}}\).

(iii) For \(F_1 \subset F_2 \subset S\), let us assume that \(F_2 \setminus F_1\) is finite. If \(q_s \neq 0\) for \(s \in F_2 \setminus F_1\), then \(f_{Q,F_1} \lesssim f_{Q,F_2}\). More generally, let \(F_{21} = q_{F_2} : \{s \in F_2 \setminus F_1: q_s = 0\}\). Then \(f_{Q,F_2} = f_{Q,F_2} \setminus F_{21}\) and \(f_{Q,F_1} \cdot X_{W_{F_2} \setminus (F_1 \cup F_{21})} \lesssim f_{Q,F_2}\).

**Proof.** (i) For \(\varphi \in \mathfrak{F}(W)\),

\[
\langle \varphi, \varphi \rangle_{X_{W_F}} = \sum_{g,h \in W} \varphi(g) X_{W_F} (h^{-1} g) \varphi(h) \leq |W_F| \sum_{g \in W} |\varphi(g)|^2 = |W_F| \langle \varphi, \varphi \rangle_{\delta_e}.
\]

(ii) Similarly to (i).

(iii) This comes from the following:

\[
f_{Q,F_2} = f_{Q,F_1} \prod_{s \in F_2 \setminus F_1} \left( q_s 1_W + (1 - q_s) X_{W_s} \right) = f_{Q,F_1} \sum_{F' \subset F_2 \setminus F_1} c_{Q,F;F_2,F'} X_{W_{F'}}.
\]

Thus the proof is complete. \(\blacksquare\)

**Lemma 5.3.** Assume \(W\) is finite. Then any positive definite function \(f\) on \(W\) is dominated by \(f_0 := \delta_e\). More exactly,

\[
f \lesssim f(e) |W| \cdot f_0, \quad \|\varphi\|_f \leq \sqrt{f(e)|W|} \cdot \|\varphi\|_{L^2(W)} \quad (\varphi \in \mathfrak{F}(W)).
\]

5.3. Isomorphism of \(\pi_{f_{Q,F}}\) with \(\text{Ind}_{W_F}^W 1_{W_F}\) for an \(F\) finite.

**Theorem 5.1.** Let \(F\) be a finite subset of \(S\). Assume that \(0 \leq q_s < 1 (s \in F)\) for \(Q = (q_s)_{s \in S}\) and that the subgroup \(\langle F \rangle \subset W\) generated by \(F\) is finite.
(i) The GR representation $(\pi_{f_{Q,F}},\nu_{f_{Q,F}})$ is isomorphic to $(\Pi_{F},\nu_{Q,F})$, where $\Pi_{F} = \text{Ind}_{W}^{W_{F}} 1_{W_{F}}$ and $\nu_{Q,F}$ is a cyclic vector which corresponds to $\nu_{f_{Q,F}}$ under a surjective $W$-module isomorphism $\Psi_{F}$ from $V(\pi_{f_{Q,F}})$ onto $V(\text{Ind}_{W}^{W_{F}} 1_{W_{F}})$.

(ii) Let us consider $(F)$-cyclic subspaces spanned by $\pi_{f_{Q,F}}(\langle F \rangle)\nu_{f_{Q,F}}$ and $\Pi_{F}(\langle F \rangle)\nu_{Q,F}$ respectively. Then, as cyclic representations of $(F)$,

$$\langle f_{Q,F}, \nu_{f_{Q,F}} \rangle \cong (\Pi_{F}|_{\langle F \rangle}, \nu_{Q,F}) \cong (\mathcal{L}_{(F)}, \nu_{Q,F}).$$

Here the second $\cong$ means that $\Pi_{F}|_{\langle F \rangle}$ on $\langle \Pi_{F}(\langle F \rangle)\nu_{Q,F} \rangle$, with the cyclic vector $\nu_{Q,F}$, is equivalent to the left regular representation $\mathcal{L}_{(F)} := \text{Ind}_{\{e\}}^{\langle F \rangle} 1_{\langle F \rangle}$ on $\ell^{2}(\langle F \rangle)$, with a cyclic vector $\nu_{Q,F}' \in \ell^{2}(\langle F \rangle)$ corresponding to $\nu_{Q,F}$, such that $\langle \mathcal{L}_{(F)}(g)\nu_{Q,F}', \nu_{Q,F} \rangle = f_{Q,F}(g) (g \in \langle F \rangle)$.

**Proof.** (i) By Proposition 5.11 we have $(\pi_{f_{Q,F}},\nu_{f_{Q,F}}) \cong (\pi_{Q,F},\nu_{Q,F})$, where $\pi_{Q,F}$ is the cyclic subrepresentation of $\pi_{F} = \bigoplus_{F' \subset F} \Pi_{F'}$ generated by $\nu_{Q,F} = \sum_{F' \subset F} d_{Q,F,F'} \cdot \nu_{W_{F'}}$. Note that, under the assumption on $(Q,F)$, we have $d_{Q,F,F'} = (\prod_{s \in F}(1 - q_{s}))^{1/2} \neq 0$, and so each component $d_{Q,F,F'} \cdot \nu_{W_{F'}}$ of $\nu_{Q,F}$ is dominated by the principal component $d_{Q,F,F'} \cdot \nu_{W_{F}}$ by Lemma 5.2(ii). Hence, by Lemma 6.1 below, the cyclic representation $(\pi_{Q,F}',\nu_{Q,F})$ is isomorphic to $(\Pi_{F},\nu_{Q,F})$ with a certain cyclic vector $\nu_{Q,F} \in V(\Pi_{F})$ corresponding to $\nu_{Q,F}$.

(ii) Denote by $f_{(F)}$ the restriction $f_{Q,F}|_{\langle F \rangle}$. We apply (i) replacing $W$ by $(F)$ and $f_{Q,F}$ by $f_{(F)}$. Then we see that $(\pi_{f_{(F)}},\nu_{f_{(F)}}) \cong (\mathcal{L}_{(F)}, \nu_{Q,F})$. Moreover, the former is naturally isomorphic to $(\pi_{f_{Q,F}}|_{\langle F \rangle}, \nu_{f_{Q,F}})$. \hfill \blacksquare

### 6. CYCLIC SUBREPRESENTATIONS AND GR REPRESENTATIONS

#### 6.1. Cyclic subrepresentation of a finite direct sum of representations.

Let $G$ be a discrete group, and $\pi_{i} (1 \leq i \leq N)$ be unitary representations of $G$. Let $\pi = \pi_{1} \oplus \ldots \oplus \pi_{N}$ be their direct sum, and take a non-zero element

$$\nu := \bigoplus_{1 \leq i \leq N} \nu_{i}, \quad \nu_{i} \in V(\pi_{i}) \quad (1 \leq i \leq N).$$

**Lemma 6.1.** Let $f_{i}(g) = \langle \pi_{i}(g)\nu_{i}, \nu_{i} \rangle_{V(\pi_{i})}$ and assume that $f_{i} \not\preceq f_{N}$ for any $i < N$. Then the cyclic subrepresentation $\pi_{\nu}$ of $\pi$ generated by $\nu$ is equivalent to $\pi_{N,\nu_{N}} : \pi_{\nu} \cong \pi_{N,\nu_{N}}$, where $\pi_{N,\nu_{N}}$ denotes the cyclic subrepresentation of $\pi_{N}$ generated by $\nu_{N}$. Put $f := f_{1} + f_{2} + \ldots + f_{N}$. Then there exists a cyclic vector $\nu_{\bigoplus_{1 \leq i \leq N}} \in V(\pi_{N,\nu_{N}})$ such that

$$\langle f(g), \nu_{\bigoplus_{1 \leq i \leq N}} \rangle = \langle \pi_{N,\nu_{N}}(g)\nu, \nu \rangle, \quad \langle \nu_{\nu}, \nu \rangle \cong \langle \pi_{N,\nu_{N}}, \nu \rangle.$$

The positive definite function associated with the cyclic representation \( \pi_v \) with the cyclic vector \( v \) is \( f = f_1 + \ldots + f_N \), and we have \( f \preceq f_N \) by assumption. On the other hand, \( f_N \preceq f \). Hence \( f_N \preceq f \preceq f_N \), and this implies that GR representations \( \pi_f \) and \( \pi_{fN} \) associated with \( f \) and \( f_N \), respectively, are mutually equivalent.

Moreover, \( (\pi_v, v) \cong (\pi_f, v_f) \) and \( (\pi_{N,v_N}, v_N) \cong (\pi_{fN}, v_{fN}) \). Hence \( \pi_v \) and \( \pi_{N,v_N} \) are mutually equivalent. To get a unitary intertwining operator from \( \pi_v \) to \( \pi_{fN} \), we need to find a cyclic vector \( v \in V(\pi_{fN}) \), as asserted in the lemma.

**Lemma 6.2.** Let \( \pi = \pi_1 \oplus \ldots \oplus \pi_N \) and \( v = v_1 \oplus \ldots \oplus v_N \) (\( v_i \in V(\pi_i) \)). Suppose that, for any pair \( \{i,j\}, i \neq j \), \( \text{Hom}_G(\pi_i, \pi_j) = \{0\} \), or there exists no intertwining operator except 0. Then the cyclic subrepresentation \( \pi_v \) of \( \pi \) generated by \( v \) is the direct sum of \( \pi_{i,v_i} \) (\( 1 \leq i \leq N \)) as \( \pi_v = \bigoplus_{1 \leq i \leq N} \pi_{i,v_i} \), where \( \pi_{i,v_i} \) denotes the cyclic subrepresentation of \( \pi_i \) generated by \( v_i \) for \( 1 \leq i \leq N \), and in the case \( v_i = 0 \), \( \pi_{i,v_i} = 0 \) by definition.

**Proof.** We may and do assume that \( \pi_{i,v_i} = \pi_i \) or \( v_i \) generates cyclically the whole \( \pi_i \) for any \( i \). Let \( P_i \) be the orthogonal projection of \( V(\pi) = \bigoplus_{1 \leq i \leq N} V(\pi_i) \) onto \( V(\pi_i) \), and \( R \) be the orthogonal projection onto \( V(\pi_u) = \langle \pi(G)v \rangle \). Then \( R_{ij} := P_j RP_i \) is essentially a \( G \)-module homomorphism from \( V(\pi_i) \) to \( V(\pi_j) \).

Therefore, by assumption, \( R_{ij} = 0 \) for \( i \neq j \), and \( R = \sum_{1 \leq i \leq N} R_{ii} \). This means that \( V(\pi_v) \) is the direct sum of \( V(\pi_{i,v_i}) \) (\( 1 \leq i \leq N \)).

In general, let \( \pi_i \) (\( i = 1, 2 \)) be cyclic unitary representations of \( G \) with specified cyclic vectors \( v_i \) respectively. We study the structure of the cyclic part \( \pi'_{12} \) of \( \pi = \pi_1 \oplus \pi_2 \) generated by \( v'_{12} := v_1 \oplus v_2 \in V(\pi) \). Its representation space \( V(\pi'_{12}) \) is spanned by \( \pi(g)v'_{12} = \pi_1(g)v_1 \oplus \pi_2(g)v_2 \) (\( g \in G \)). Put

\[
V_i^{(d)} := V(\pi_i) \cap V(\pi'_{12}), \quad V(\pi_i) = V_i^{(d)} \oplus V_i^{(c)} \quad (i = 1, 2),
\]

and \( \pi_i^{(d)} := \pi|_{V_i^{(d)}} \) and \( \pi_i^{(c)} := \pi|_{V_i^{(c)}} \); moreover, let \( v_i = v_i^{(d)} \oplus v_i^{(c)} \) (\( i = 1, 2 \)) be the decomposition of \( v_i \) according to (6.3). Then \( \pi_i = \pi_i^{(d)} \oplus \pi_i^{(c)} \).

Put \( V^{(d)} := V(\pi_1^{(d)}) \oplus V(\pi_2^{(d)}) \subset V(\pi'_{12}) \), let \( V^{(c)} \) be the orthogonal complement of \( V^{(d)} \) in \( V(\pi'_{12}) \), and define \( \pi^{(d)} := \pi|_{V^{(d)}} \) and \( \pi^{(c)} := \pi|_{V^{(c)}} \). Then, \( V^{(c)} \subset V(\pi_1^{(c)}) \oplus V(\pi_2^{(c)}) \) and

\[
V(\pi'_{12}) = V^{(d)} \oplus V^{(c)} \quad (\text{in } V(\pi)), \quad \pi'_{12} \cong \pi^{(d)} \oplus \pi^{(c)}.
\]

From the definition of \( V(\pi_i^{(d)}) \) it follows that, for any \( w_1 \oplus w_2 \in V(\pi) \subset V(\pi_1^{(c)}) \oplus V(\pi_2^{(c)}) \), the correspondence \( T : w_1 \rightarrow w_2 \), from the first component to the second, is bijective. Moreover, we have \( \pi_1^{(c)}(g)v_1^{(c)} \oplus \pi_2^{(c)}(g)v_2^{(c)} \in V^{(c)} \), and so \( T \) maps bijectively as

\[
T : \pi_1^{(c)}(g)v_1^{(c)} \rightarrow \pi_2^{(c)}(g)v_2^{(c)} \quad (g \in G).
\]
LEMMA 6.3. The restriction $\pi'(c)$ of $\pi'_{12}$ or of $\pi$ onto $V(c)$ is equivalent to each of the restrictions $\pi_i'(c)$ of $\pi_i$ onto $V_i(c)$ for $i = 1, 2$. The equivalences between them are given by the projections $V(c) \ni w_1 \oplus w_2 \mapsto w_i \in V_i(c)$. The cyclic part $\pi'_{12}$ of $\pi = \pi_1 \oplus \pi_2$ generated by $v_1 \oplus v_2$ is given as

$$\pi'_{12} = (\pi_1^{(d)} \oplus \pi_2^{(d)}) \oplus \pi(c), \quad \pi(c) \cong \pi_1(c) \cong \pi_2(c). \quad (6.5)$$

6.2. The cyclic part of $(\text{Ind}_{H_i}^G 1_{H_1}) \otimes (\text{Ind}_{H_2}^G 1_{H_2})$. Let $G$ be a discrete group, and $H_i$ be subgroups of $G$ for $i = 1, 2$. Put $\pi_i = \text{Ind}_{H_i}^G 1_{H_1}$ ($i = 1, 2$), and $\rho := \pi_1 \otimes \pi_2$.

LEMMA 6.4. Let $\rho'$ be the cyclic subrepresentation of $\rho = \pi_1 \otimes \pi_2$ generated by a cyclic vector $w_1 \otimes w_2$ with $w_i := X_{H_i} \in V_i$ ($i = 1, 2$). Then $\rho'$ is canonically equivalent to the induced representation $\pi := \text{Ind}_{H_1 \cap H_2}^G 1_{H_1 \cap H_2}$, and $w_1 \otimes w_2$ corresponds to the cyclic vector $X_{H_1 \cap H_2} \in V(\pi)$.

Proof. Consider $V(\pi_i)$ as the space of functions $\varphi_i$ on $G$ such that $\varphi_i(gh_i) = \varphi_i(g)$ ($h_i \in H_i, g \in G$) with the norm $\|\varphi_i\|^2 = \sum_{g \in G/H_i} |\varphi_i(g)|^2$, where $g \in G/H_i$ means that $g$ runs over a complete set of representatives of $G/H_i$. In $V(\pi_1) \otimes V(\pi_2)$, we take $w_{12} := w_1 \otimes w_2 = X_{H_1} \otimes X_{H_2}$ and consider the subspace $V'$ spanned by

$$\rho(g_0)w_{12} = X_{g_0H_1} \otimes X_{g_0H_2} \quad (g_0 \in G), \quad (6.6)$$

where $X_B$ denotes the indicator function of a subset $B$ of $G$.

On the other hand, consider a bilinear map $\Phi'$ from $V(\pi_1) \times V(\pi_2)$, which assigns to $(\varphi_1, \varphi_2)$ a function $\varphi$ on $G$ given by the product as $\varphi(g) := \varphi_1(g)\varphi_2(g) : \Phi'(\varphi_1, \varphi_2) = \varphi$. Then it induces uniquely a linear map $\Phi''$ from $V(\pi_1) \otimes V(\pi_2)$ into a space of functions $\psi$ on $G$ such that $\psi(gh) = \psi(g)$, where $g \in G, h \in H_{12} := H_1 \cap H_2$. Denote by $\Phi$ the restriction of $\Phi''$ on the subspace $V'$. Then $\varphi_{12} := \Phi(w_{12})$ is given by

$$\varphi_{12}(g) = X_{H_1}(g)X_{H_2}(g) = X_{H_1 \cap H_2}(g) \quad (g \in G), \quad (6.7)$$

and $\Phi(\rho(g_0)w_{12}) = \pi(g_0)\varphi_{12}$, where

$$\pi(g_0)\varphi_{12}(g) := \varphi_{12}(g_0^{-1}g) = X_{H_1 \cap H_2}(g_0^{-1}g) \quad (g \in G). \quad (6.8)$$

From (6.6) we see that the set of vectors $\{\rho(g_0)w_{12}: g_0 \in G/(H_1 \cap H_2)\}$ gives an orthogonal basis of $V'$. On the other hand, we see from (6.8) that the set of functions $\{\pi(g_0)\varphi_{12}: g_0 \in G/(H_1 \cap H_2)\}$ gives an orthogonal basis of $\Phi(V')$. This means that $\Phi$ is a linear isomorphism from $V'$ onto $\Phi(V')$. Moreover, we see from (6.7) and (6.8) that $\Phi(V') = V(\pi)$, and the representation $\pi$ on $\Phi(V')$ is nothing but $\text{Ind}_{H_1 \cap H_2}^G 1_{H_1 \cap H_2}$. ■
Note 6.1. On $V(\pi_1) \otimes V(\pi_2)$, $\Phi$ is isometric inside of the cyclic subspace $V'$ by Lemma 5.1. But it is not necessarily isometric outside of $V'$ even though it is $G$-homomorphic.

6.3. The cyclic part of tensor product $\pi_{f_1} \otimes \pi_{f_2}$. For $f_i \in P_1(G)$ $(i = 1, 2)$, we realize the GR representation $\pi_{f_i}$ as follows: prepare a standard cyclic vector $v_i$ and a symbolic $G$-span $\mathcal{B}_i := \{ \pi_{f_i}(g)v_i; g \in G \}$ such that

$$f_i(g) = \langle \pi_{f_i}(g)v_i, v_i \rangle \quad (g \in G),$$
$$\langle \pi_{f_i}(g)v_i, \pi_{f_i}(h)v_i \rangle := f_i(h^{-1}g) \quad (g, h \in G).$$

The representation space $V(\pi_{f_i})$ is a completion of the linear span $\langle \mathcal{B}_i \rangle$ modulo the kernel of the inner product, and the representation operator $\pi_{f_i}(g)$ is induced from the left translation by $g_0$ as $\pi_{f_i}(g_0)(\pi_{f_i}(g)v_i) := \pi_{f_i}(g_0g)v_i$.

Put $f := f_1f_2$. Consider the tensor product $\pi := \pi_{f_1} \otimes \pi_{f_2}$ on $V(\pi_{f_1}) \otimes V(\pi_{f_2})$, and take a unit vector $v_{12} := v_1 \otimes v_2$. The matrix element associated with $v_{12}$ is

$$\langle \pi(g)v_{12}, v_{12} \rangle = \langle \pi_{f_1}(g)v_1, v_1 \rangle \cdot \langle \pi_{f_2}(g)v_2, v_2 \rangle = f_1(g)f_2(g) = f(g).$$

Lemma 6.5. The cyclic part $\langle \pi(G)v_{12} \rangle \subset V(\pi_{f_1}) \otimes V(\pi_{f_2})$ carries GR representation $\pi_f$ associated with $f = f_1f_2$, where $\langle \pi(G)v_{12} \rangle$ denotes the closed linear span of $\pi(G)v_{12}$.

7. The Case of GR Representations of Coxeter Groups

7.1. GR representations and induced representations. Let $(W, S)$ be a Coxeter group. For a subset $S' \subset S$, denote by $(S')$ the subgroup of $W$ generated by $S'$, and for a subset $F \subset S$, put $W_F := \langle S' \rangle$ as before. Let us put $S_F := \{ s \in S; |W_s| < \infty \}$.

Definition 7.1. An $s \in S$ is called co-finite (resp. co-infinite) if $|W/W_s| < \infty$ (resp. $|W/W_s| = \infty$). A subset $F \subset S$ is said to be of infinite type if $|W_F| = |\langle S' \rangle| = \infty$.

Note that the induced representation $\Pi_s = \text{Ind}_{W_s}^W 1_{W_s}$ contains or not the trivial representation $1_W$ according as $s$ is co-finite or co-infinite.

Let $Q = (q_s)_{s \in S}$, $0 \leq q_s \leq 1$ $(s \in S)$. For a subset $F \subset S$, we have, by the definition in (2.2), $f_{Q,F} = \prod_{s \in F} f_{q_s,s}$ and $f_Q = f_{Q,F} f_{Q,S\setminus F}$.

Lemma 7.1. Suppose $F \subset S$ is finite. Then, for $Q = (q_s)_{s \in S}$,

(7.1) $\pi_{f_{Q,F}} \cong$ the cyclic part of $\bigotimes_{s \in F} \pi_{f_{q_s,s}}$ generated by $v_{F} := \bigotimes_{s \in F} v_{f_{q_s,s}}$;
(7.2) $\pi_{f_{Q}} \cong$ the cyclic part of $\bigotimes_{s \in F} \pi_{f_{q_s,s}} \otimes \pi_{f_{Q,S\setminus F}}$ generated by $v_{F} \otimes v_{f_{Q,S\setminus F}}$.  

Gelfand–Raikov representations of Coxeter groups
LEMMA 7.2. Suppose $F \subset S$ is finite, and let $Q = (q_s)_{s \in S}$.

(i) The GR representation $(\pi_{f_Q}, v_{f_Q})$ is isomorphic to a cyclic subrepresentation of

\[(7.3) \quad \bigotimes_{s \in F} (1_W + \Pi_s) \otimes \pi_{f_Q,S \setminus F}\]

with a cyclic vector $(\bigotimes_{s \in F} v_{1,s}) \otimes v_{f_Q,S \setminus F}$, where $v_{1,s} \in V(1_W + \Pi_s)$ is given in (7.3).

(ii) Put $F_f := F \cap S_f$. Then $(\pi_{f_Q}, v_{f_Q})$ is isomorphic to a cyclic subrepresentation of

\[(7.4) \quad \bigotimes_{s \in F \setminus F_f} (1_W + \Pi_s) \otimes (1_W + \text{Ind}_{W_f} W_f) \otimes \pi_{f_Q,S \setminus F_f},\]

with a cyclic vector $(\bigotimes_{s \in F \setminus F_f} v_{1,s}) \otimes w_{f_f} \otimes v_{f_Q,S \setminus F_f}$, where $w_{f_f}$ is a certain cyclic vector of $1_W + \text{Ind}_{W_f} W_f$.

(iii) In case of $W$ is finite, the middle term $1_W + \text{Ind}_{W_f} W_f$ in (7.4) can be replaced by $\text{Ind}_{W_f} W_f$ with a certain cyclic vector $w_{f_f}$.

The following is an extended version of Lemma 6.4 in the case of Coxeter groups. For a finite subset $F \subset S$, assume that $\Pi_f$ is the cyclic subrepresentation of $\bigotimes_{s \in F} \Pi_s$ generated by the vector $\bigotimes_{s \in F} X_{1,s}$. Consider a multilinear map

\[\Phi : \prod_{s \in F} V(\Pi_s) \ni (\varphi_s)_{s \in F} \to \prod_{s \in F} \varphi_s =: \varphi,\]

where $\varphi(w) = \prod_{s \in F} \varphi_s(w) (w \in W)$. Then we get a linear map $\Phi$ onto a space of functions on $W$ invariant from the right under $W_F = F \cap S_f$ as

\[(7.5) \quad \Phi : \bigotimes_{s \in F} V(\Pi_s) \ni \bigotimes_{s \in F} \varphi_s \to \prod_{s \in F} \varphi_s.\]

LEMMA 7.3. Let $F \subset S$ be finite. The linear map $\Phi$ gives a $W$-isomorphism from the cyclic subrepresentation $\Pi_F$ onto the quasi-regular representation $\Pi_F := \text{Ind}_{W_f} W_f$, and the cyclic vector $\bigotimes_{s \in F} X_{1,s}$ is mapped to the cyclic vector $X_{W_F}$.

7.2. Induced representation $\Pi_s$ and subrepresentation of $\pi_s = 1_W + \Pi_s$.

LEMMA 7.4. Suppose $s \in S$ is co-finite, or $W/W_s$ is finite, and $0 < q_s < 1$. Then GR representation $\pi_{f_s}$ associated with $f_s := f_{q_s,s}$ is equivalent to the induced representation $\Pi_s = \text{Ind}_{W_s} W_s$, and $\Pi_s$ contains the trivial representation $1_W$. 
exactly once, or \( \Pi_s = 1_W \oplus (1_W)^{-1} \), where \((1_W)^{-1}\) contains \(1_W\) no more. Under the isomorphism from \(\pi_{f_s}\) to \(\Pi_s\), the unit cyclic vector \(\mathbf{v}_{f_s} \in V(\pi_{f_s})\) is mapped to the following unit cyclic vector \(\mathbf{w}_s \in V(\Pi_s)\): with \(c_s := |W/W_s|^{-1}\).

\[
(7.6) \quad \mathbf{w}_s := \sqrt{q_s}c_s + (1 - q_s)c_s^2 \cdot 1_W \oplus \sqrt{1 - q_s}(X_{W_s} - c_s 1_W),
\]

so that \((\mathbf{f}_{f_s}, \mathbf{v}_{f_s}) \cong (\Pi_s, \mathbf{w}_s)\). Denote by \(P_s\) the orthogonal projection of \(V(\Pi_s)\) onto the subspace carrying the trivial representation \(1_W\). Then

\[
\|P_s\mathbf{w}_s\| = \sqrt{q_s + (1 - q_s)c_s}.
\]

**Lemma 7.5.** Let \(s \in S\) be co-infinite, or \(|W/W_s| = \infty\). Then GR representation \(\pi_{f_s}\) associated with \(f_s := f_{q_s,s}\) is equivalent to \(\pi_s = 1_W \oplus \Pi_s\) with \(\Pi_s = \text{Ind}_{W_s}^W 1_{W_s}\), and \(\Pi_s\) does not contain the trivial representation \(1_W\). The positive definite function \(f_s\) is the diagonal matrix element corresponding to the reference vector \(\mathbf{v}_{1,s} \in V(\pi_s)\) in (3.3):

\[
(7.7) \quad (\mathbf{f}_{f_s}, \mathbf{v}_{f_s}) \cong (1_W \oplus \Pi_s, \mathbf{v}_{1,s}).
\]

**8. The Case of Finite Coxeter Groups \((W, S)\)**

**Theorem 8.1.** Assume \(W\) is finite, and let \(Q = (q_s)_{s \in S}\).

(i) Suppose \(0 \leq q_s < 1\) \((s \in S)\). Then GR representation \(\pi_{f_Q}\) associated with \(f_Q\) is equivalent to the left regular representation \(L_W\) of \(W\), or \((\pi_{f_Q}, \mathbf{v}_{f_Q}) \cong (L_W, \mathbf{v}_{Q,S})\), where \(\mathbf{v}_{Q,S} \in \ell^2(W)\) corresponds to \(w_{Q,F} \in (\text{Ind}_W^W 1_F)\) and \((\text{Ind}_W^W 1_F)\) for \(F = S\).

(ii) Suppose \(q_s = 1\) \((s \in F_0)\), \(0 \leq q_s < 1\) \((s \in S \setminus F_0)\) for an \(F_0 \neq \emptyset\). Then

\[
\pi_{f_Q} \cong \text{Ind}_W^W 1_{F_0} \text{ on } \ell^2(W/(F_0)).
\]

**Proof.** (i) Apply Proposition 5.1 for \(F = S\). Then \(\pi_{f_Q}\) with the cyclic vector \(\mathbf{v}_{f_Q}\) is realized as the cyclic superrepresentation of \(\pi(S)\) associated with the vector \(w_{Q,S}\), where \(\pi(S) = \bigoplus_{F \subseteq S} \Pi_F = 1_W \oplus \bigoplus_{0 \leq q_s < 1} (\text{Ind}_W^W 1_{W_s}) \oplus L_W\).

We apply Lemma 6.1 for \(\pi = \pi(S)\), where the index set \(\{1, 2, \ldots, N\}\) is replaced by the set \(\{F; F \subseteq S\}\), and \(\mathbf{v} = \sum_{1 \leq i \leq N} \mathbf{v}_i\) is replaced by \(\mathbf{v}_{Q,S} = \sum_{F \subseteq S} \mathbf{v}_F\) above. The positive definite function associated with \(\mathbf{v}_F\) is given by \(f_F(g) := (d_{Q,S,F})^2 X_{W_F}(g) = \langle \Pi_F(g) \mathbf{v}_F, \mathbf{v}_F \rangle\) \((g \in W)\), and \(f_S = d_{Q,S,S} \delta_e\), \(d_{Q,S,S} \neq 0\). To guarantee that Lemma 6.1 is applicable, we have Lemma 5.2.

(ii) In this case, \(f_Q = \prod_{s \in S} f_s \sum_{s \in S \setminus F_0} f_{Q,S}\). We apply Proposition 5.1 for \(F := S \setminus F_0\). Note that \(W_F = \langle S \setminus F \rangle = (F_0)\). Then, using Lemmas 5.2 and 6.1 as for the assertion (i), we obtain (ii). □

The isomorphism between \(\pi_{f_Q}\) and the regular representation \(L_W\) is twisted in the sense given in the following theorem. This fact has an important meaning
when we consider a limiting process for a growing sequence of Coxeter groups as
\((W_n, S_n) \nearrow (W, S), S_n \nearrow S = \bigcup_{n \geq 1} S_n\), for instance, in the case of \(\mathfrak{S}_n \nearrow \mathfrak{S}_\infty\).

Here is the place where we ask the question formulated in our Problem 1, in connection with, e.g., [L2] and [L3]–[L4].

**Theorem 8.2.** Assume \(W\) is finite and \(0 \leq q_s < 1\ (s \in S)\). Put \(C_{fQ} := (f_Q(h^{-1}g))_{g, h \in W}\). Then the matrix \(C_{fQ}\) is Hermitian and strictly positive definite. A linear map \(\Psi_Q\) from \(V(\pi_{fQ})\) to \(\ell^2(W)\),
\[
\Psi_Q : \sum_{g \in W} c_g \pi_{fQ}(g) v_{fQ} \rightarrow \sum_{g \in W} c_g \delta_g,
\]
gives an algebraic isomorphism of cyclic representations \((\pi_{fQ}, v_{fQ})\) and \((\mathcal{L}_W, \delta_c)\). Moreover, for \(v = \sum_{g \in W} c_g \pi_{fQ}(g) v_{fQ} \in V(\pi_{fQ})\), express \(\Psi_Q(v) = \sum_{g \in W} c_g \delta_g \in \ell^2(W)\) as a column vector \(c = (c_g)_{g \in W}\). Then \(\|v\|_{V(\pi_{fQ})}^2 = \|\sqrt{C_{fQ}} c\|_{\ell^2(W)}^2\), where \(\sqrt{C_{fQ}}\) commutes with \(\mathcal{L}_W(g)\ (g \in W)\). In other words, \(\sqrt{C_{fQ}} \cdot \Psi_Q\) is a unitary \(W\)-map from \((\pi_{fQ}, V(\pi_{fQ}))\) onto \((\mathcal{L}_W, \ell^2(W))\), which maps the cyclic vector \(v_{fQ}\) for \(\pi_{fQ}\) to the one \(\delta_c\) for \(\mathcal{L}_W\).

9. The case of infinite Coxeter groups \((W, S)\)

In this section, we assume that \(W\) is infinite.

9.1. GR representations and induced representations.

**Lemma 9.1.** (i) Assume \(0 \leq q_s < 1\ (s \in S)\). For a finite subset \(F\) of \(S\), the GR representation \(\pi_{fQ}\) with cyclic vector \(v_{fQ}\) is isomorphic to the cyclic subrepresentation of
\[
\bigotimes_{s \in F} \Pi_s \otimes \bigotimes_{s \in F} (1_W \oplus \Pi_s) \otimes \pi_{fQ,S\setminus F},
\]
with a cyclic vector \(\bigotimes_{s \in F, \text{co-finite}} w_s \otimes \bigotimes_{s \in F, \text{co-infinite}} v_{1,s} \otimes v_{fQ,S\setminus F}\).

(ii) Let \(S\) be finite and \(0 < q_s < 1\ (s \in S)\). Then \((\pi_{fQ}, v_{fQ})\) is isomorphic to a cyclic subrepresentation of \(1_W \oplus \bigoplus_{F \supseteq S, \text{finite type}} \text{Ind}^{W_F}_{W_F} 1_{W_F} \oplus \mathcal{L}_W\) containing \(1_W \oplus \mathcal{L}_W\).

9.2. Intertwining operators among \(\Pi_F = \text{Ind}^W_{W_F} 1_{W_F}\). Taking into account Proposition 5.11 and Lemma 6.3, we study here intertwining operators among quasi-regular representations \(\Pi_F = \text{Ind}^W_{W_F} 1_{W_F}\) induced from parabolic subgroups \(W_F = \langle S \setminus F \rangle\).

9.2.1. Intertwining operators between \(\Pi_i = \text{Ind}^G_{H_i} 1_{H_i}\ (i = 1, 2)\). Let \(G\) be a discrete group, and \(H_i\ (i = 1, 2)\) its subgroups. The representation spaces \(V(\Pi_i)\)
consist of functions $\varphi_i$ on $G$ satisfying
\[
\varphi_i(g) = \varphi_i(gh_i) \quad (h_i \in H_i, g \in G), \quad \|\varphi_i\|^2 := \sum_{g \in G/H_i} |\varphi_i(g)|^2 < \infty.
\]

Any intertwining operator $T$ from $\Pi_1$ to $\Pi_2$ is given by a kernel function $K(g, g')$ as follows:
\[
T\varphi_1(g) = \sum_{g' \in G/H_i} K(g, g')\varphi_1(g') \quad (g \in G).
\]

Put $K'(g) := K(e, g) \quad (g \in G)$. Then $K(g, g') = K'(g^{-1}g')$, and $K'$ satisfies
\[
K'(h_2gh_1) = K'(g) \quad (h_i \in H_i, g \in G),
\]
\[
\sum_{g \in G/H_i} |K'(g)|^2 < \infty, \quad \sum_{g \in H \subseteq G} |K'(g)|^2 < \infty.
\]

Consider the restriction of $K'$ onto a double coset $H_2gH_1$. If the order of $H_2$-cosets $H_2 \setminus H_2gH_1$ is infinite, then $K'$ should be zero on $H_2gH_1$. Note that
\[
h_2gh_1 \in H_2(h_2'g_1') \Leftrightarrow h_1h_1'^{-1} \in g^{-1}H_2g \Leftrightarrow h_1h_1'^{-1} \in g^{-1}H_2g \cap H_1,
\]
etc. Then we have the following criterion:

\[
K'(g) \neq 0 \Rightarrow \begin{cases} 
|H_2gH_1/H_1| = |H_2/(gH_1g^{-1} \cap H_2)| < \infty, \\
|H_2 \setminus H_2gH_1| = |(g^{-1}H_2g \cap H_1) \setminus H_1| < \infty.
\end{cases}
\]

\section*{9.2.2. The case of a Coxeter group $W$ and $W_{F_i} = \langle S \setminus F_i \rangle$ (i = 1, 2).}

Put $\Pi_i := \Pi_{F_i} = \text{Ind}_{W_{F_i}}^W 1_{W_{F_i}}$ (i = 1, 2). An element $\varphi_i \in V(\Pi_i)$ can be considered as a function on $W$ which is $H_i$-invariant from the right. A $\varphi_1 \in V(\Pi_1)$ belongs also to $V(\Pi_2)$ only when it is also $H_2$-invariant from the right, and so invariant under $\langle H_1, H_2 \rangle = W_{F_1 \cap F_2}$ from the right.

\textbf{Lemma 9.2.} Let $\Pi_{F_i} = \text{Ind}_{W_{F_i}}^W 1_{W_{F_i}}$ (i = 1, 2). Then their spaces $V(\Pi_{F_1})$ and $V(\Pi_{F_2})$ have a non-trivial intersection (denoted by $V_{12}$) if and only if
\[
\begin{align*}
|\langle H_1, H_2 \rangle/H_1| &= |W_{F_1 \cap F_2}/W_{F_1}| < \infty, \\
|\langle H_1, H_2 \rangle/H_2| &= |W_{F_1 \cap F_2}/W_{F_2}| < \infty,
\end{align*}
\]
and in that case $\Pi_{F_1}$ and $\Pi_{F_2}$ have a common constituent realized on $V_{12} = l^2(W/W_{F_1 \cap F_2})$.

\textbf{Lemma 9.3.} There exists a non-zero intertwining operator from $\Pi_{F_1}$ to $\Pi_{F_2}$ if and only if there exists a $g \in W$ satisfying
\[
|W_{F_2}/(gW_{F_1}g^{-1} \cap W_{F_2})| < \infty, \\
|(g^{-1}W_{F_2}g \cap W_{F_1}) \setminus W_{F_1}| < \infty.
\]
Lemma 9.4. (i) The trivial representation $1_W$ is contained in the induced representation $\Pi_F = \text{Ind}_W^F 1_W$ if and only if $|W/W_F| < \infty$.
(ii) The quasi-regular representation $\Pi_F, F \subseteq S$, is not disjoint with the regular representation $L_W$ if and only if $W_F$ is finite. In that case, $\Pi_F$ is isomorphically imbedded into $L_W$.

10. THE CASE OF AFFINE WEYL GROUPS

Affine Weyl groups are a kind of infinite Coxeter groups $(W, S)$ generated by affine reflections in Euclidean spaces (cf. the definition in [21], Section 4.2). Irreducible affine Weyl groups are listed in [21], Section 4.7, and their Coxeter graphs are precisely the positive semidefinite ones which are not positive definite (cf. [21], Section 6.5). Note that, for any non-empty $F \subseteq S$, the parabolic subgroup $W_F = \langle S \setminus F \rangle$ is a finite Coxeter group. Then, by Lemmas 5.2, 6.1, 9.2, 9.3, and 9.4, we have the following.

Theorem 10.1. Let $(W, S)$ be an irreducible affine Weyl group, and $Q = (q_s)_{s \in S}$.
(i) Assume $0 < q_s < 1$ $(s \in S)$. Then $\pi_{f_Q} \cong 1_W \oplus L_W$.
(ii) Assume $q_s = 0$ $(s \in F_0 \neq \emptyset), 0 < q_s < 1 (s \not\in F_0)$. Then $\pi_{f_Q} \cong L_W$.
(iii) Assume $q_s = 1$ $(s \in F_1 \neq \emptyset), 0 < q_s < 1 (s \not\in F_1)$. Then $\pi_{f_Q} \cong 1_W \oplus \text{Ind}_W^{(F_1)} 1_{(F_1)}$.
(iv) Assume $q_s = 0$ $(s \in F_0 \neq \emptyset), q_s = 1$ $(s \in F_1 \neq \emptyset), 0 < q_s < 1$ otherwise. Then $\pi_{f_Q} \cong \text{Ind}_W^{(F_1)} 1_{(F_1)}$.

11. THE CASE OF HYPERBOLIC COXETER GROUPS

Consider the case where $(W, S)$ is irreducible, of rank $n$, and the bilinear form on $V$ is non-degenerate. Define a cone $C$ in $V$ by $C := \{ \lambda \in V; B(\lambda, \alpha_s) > 0 (s \in S) \}$. Such a Coxeter group $(W, S)$ is called hyperbolic if $B$ has signature $(n - 1, 1)$ and $B(\lambda, \lambda) < 0 (\lambda \in C)$; see [21], Section 6.8. An irreducible Coxeter group $(W, S)$ is hyperbolic if and only if the following conditions are satisfied:
(a) $B$ is non-degenerate, but not positive definite;
(b) for each $s \in S$, the Coxeter graph obtained by removing $s$ from $\Gamma$ is of positive type, or its bilinear form is positive semidefinite.

A hyperbolic Coxeter group $(W, S)$ is called compact if the quotient of $O(V)$ by $W$ is compact, where $O(V)$ is the orthogonal group for $B$. An irreducible Coxeter group $(W, S)$ is compact hyperbolic if and only if it satisfies (a) above and
(c) for each $s \in S$, the Coxeter graph obtained by removing $s$ from $\Gamma$ is positive definite or $W_s$ is finite.
Hyperbolic Coxeter groups exist only in ranks from 3 to 10, and their numbers are finite in each of the ranks from 4 to 10, as seen in Table 1 below.

<table>
<thead>
<tr>
<th>rank W</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>compact hyperbolic</td>
<td>∞</td>
<td>9</td>
<td>5</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td></td>
</tr>
<tr>
<td>non-compact hyperbolic</td>
<td>∞</td>
<td>23</td>
<td>9</td>
<td>12</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

11.1. The case of compact hyperbolic Coxeter groups. By [21], Section 6.8, in this case, for any $s \in S$, the sub-Coxeter group $(W_s, S \setminus \{s\})$ is a finite Coxeter group. Therefore, from Lemmas 6.1, 6.2, 7.5, and 9.1 we get the following.

**Theorem 11.1.** Assume that a Coxeter group $(W, S)$ is compact hyperbolic.

(i) Assume for $Q = (q_s)_{s \in S}, 0 < q_s < 1 (s \in S)$. Then $\pi_{f_Q} \cong 1_W \oplus \mathcal{L}_W$.

(ii) Assume $q_s = 0 (s \in F_0)$ and $0 < q_s < 1 (s \notin F_0)$ for an $F_0 \neq \emptyset$. Then $\pi_{f_Q} \cong \mathcal{L}_W$.

(iii) Assume $q_s = 1 (s \in F_1 \neq \emptyset), 0 < q_s < 1 (s \notin F_1)$. Then

$$\pi_{f_Q} \cong 1_W \oplus \text{Ind}_{W_{F_1}}^W 1_{(F_1)}.$$

(iv) Assume $q_s = 0 (s \in F_0 \neq \emptyset), q_s = 1 (s \in F_1 \neq \emptyset), 0 < q_s < 1$ otherwise. Then

$$\pi_{f_Q} \cong \text{Ind}_{W_{F_1}}^W 1_{(F_1)}.$$

11.2. The case of non-compact hyperbolic Coxeter groups. By [21], Section 6.8, in this case, for any $s \in S$, the sub-Coxeter group $(W_s, S \setminus \{s\})$ is a finite or affine Weyl group. We apply Lemmas 6.1, 6.2, 7.5, and 9.1 (ii).

**Theorem 11.2.** Assume that $(W, S)$ is irreducible and non-compact hyperbolic.

(i) For $Q = (q_s)_{s \in S}, 0 < q_s < 1 (s \in S)$, the GR representation $(\pi_{f_Q}, v_{f_Q})$ is isomorphic to a cyclic subrepresentation, containing $1_W \oplus \mathcal{L}_W$, of

$$1_W \oplus \bigoplus_{s \in S} \text{Ind}_{W_s}^W 1_{W_s} \oplus \mathcal{L}_W.$$

(ii) Assume that an infinite type $s \in S$ is unique and denote it by $s_0$. Then

$$\pi_{f_Q} \cong \begin{cases} 1_W \oplus \text{Ind}_{W_{s_0}}^W 1_{W_{s_0}} \oplus \mathcal{L}_W & \text{if } |W/W_{s_0}| = \infty, \\ \text{Ind}_{W_{s_0}}^W 1_{W_{s_0}} \oplus \mathcal{L}_W & \text{if } |W/W_{s_0}| < \infty. \end{cases}$$

Note that many of non-compact hyperbolic irreducible Coxeter groups, in the complete list in [21], Section 6.9, pp. 142–144, have unique $s \in S$ of infinite type.
Actually, for \( n = \text{rank } W \geq 7 \), except two cases for \( n = 7 \), one for \( n = 9 \), and two for \( n = 10 \), all such Coxeter groups have unique \( s \) of infinite type. However, it might not be easy to check the condition \(|W/W_s| = \infty|\).  

**Example 11.1.** Irreducible rank 3 Coxeter groups are divided into two cases.

**Case 1.** The Coxeter graphs are of the form \( \circ - m - n - \circ - 3 \leq m \leq n \leq \infty \). Assume \( n < \infty \). Then, except the following cases, the Coxeter group with this graph is compact hyperbolic, and its bilinear form \( B \) is of signature \((2,1)\):

<table>
<thead>
<tr>
<th>((m,n))</th>
<th>(A_3)</th>
<th>(B_3)</th>
<th>(H_3)</th>
<th>(\tilde{G}_2)</th>
<th>(\tilde{B}_2 = \tilde{C}_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((3,3))</td>
<td>((3,4))</td>
<td>((3,5))</td>
<td>((3,6))</td>
<td>((4,4))</td>
<td></td>
</tr>
</tbody>
</table>

**Case 2.** The Coxeter graphs are triangle with labels \( 3 \leq m \leq n \leq p \) on three edges. Assume \( p < \infty \). Then except only one case of \((m,n,p) = (3,3,3)\) for type \( A_2 \), all other graphs are for compact hyperbolic Coxeter groups, and Theorem 11.1 is applicable. In the case where only one component of labels \((m,n)\) or \((m,n,p)\) is \( \infty \), we can apply Theorem 11.2.

**Example 11.2 (cf. [21], Section 5.1).** An example of a non-compact hyperbolic Coxeter group of rank 3 is given as follows: \( S = \{s_1, s_2, s_3\}, m(s_1, s_2) = 3, m(s_2, s_3) = \infty, m(s_1, s_3) = 2 \). Then the Coxeter group \( W \) is isomorphic to \( PGL(2, \mathbb{Z}) = GL(2, \mathbb{Z})/\{\pm 1\} \) by sending the generators \( s_1, s_2, s_3 \), respectively, to

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\begin{pmatrix}
-1 & 1 \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix},
\]

where the canonical map \( GL(2, \mathbb{Z}) \rightarrow PGL(2, \mathbb{Z}) \) is denoted by

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \rightarrow \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}.
\]

In this case, \( WF \) is finite except for \( F = \{s_1\} \), where \( W_{s_1} = \langle s_2, s_3 \rangle \) is equal to the parabolic subgroup \( P \) of upper triangular matrices. Hence we have \(|W/W_{s_1}| = \infty|\), and for \( Q = \langle q_s \rangle_{s \in S}, 0 \leq q_s < 1 (s \in S) \), \( \pi_{f_Q} \cong 1_W \oplus \text{Ind}_P^W 1_P \oplus \mathcal{L}_W \), by Theorem 11.2(ii).

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