Abstract. Let $S_n$ be the total gain in $n$ repeated St. Petersburg games. It is known that $n^{-1}(S_n - n \log_2 n)$ converges in distribution along certain geometrically increasing subsequences and its possible limiting random variables can be parametrized as $Y(t)$ with $t \in [\frac{1}{2}, 1]$. We determine the Hausdorff and box-counting dimension of the range and the graph for almost all sample paths of the stochastic process $\{Y(t)\}_{t \in [\frac{1}{2}, 1]}$. The results are compared to the fractal dimension of the corresponding limiting objects when gains are given by a deterministic sequence initiated by Hugo Steinhaus.

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1. INTRODUCTION

The famous St. Petersburg game is easily formulated as a simple coin tossing game. The player’s gain $Y = 2^T$ in a single game can be expressed by means of the stopping time $T = \inf\{n \in \mathbb{N}: X_n = 1\}$ of repeated independent tosses $(X_n)_{n \in \mathbb{N}}$ of a fair coin until it first lands heads. For a sequence of gains $(Y_n)_{n \in \mathbb{N}}$ in independent St. Petersburg games the partial sum $S_n = \sum_{k=1}^{n} Y_k$ denotes the total gain in the first $n$ games. To find a fair entrance fee for playing the game is commonly called the \textit{St. Petersburg problem}, frequently raised to the status of a paradox. Since the expectation $\mathbb{E}[Y] = \infty$ is infinite, a fair premium cannot be constructed by the help of the usual law of large numbers. We refer to Jorland [24] and Dutka [12] for the history of the St. Petersburg game and for early solutions of the 300 year old problem.

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The first step towards a mathematically satisfactory solution has been achieved by Feller [19, 20] who showed that a premium depending on the number $n$ of games can fulfill a certain weak law of large numbers

$$\frac{S_n}{n \log_2 n} \rightarrow 1 \quad \text{in probability},$$

where $\log_2$ denotes logarithm to the base 2. However, Feller’s result does not tell if the game is disadvantageous or advantageous for the player, i.e., if $S_n - n \log_2 n$ is likely to be negative or positive. This question can only be answered by a weak limit theorem and the first theorem of this kind has been shown by Martin-Löf [28] for the subsequence $k(n) = 2^n$:

$$(1.1) \quad \frac{S_{k(n)} - k(n) \log_2 k(n)}{k(n)} \rightarrow X \quad \text{in distribution.}$$

The limit $X$ is infinitely divisible with characteristic function $\exp \left( \psi(y) \right)$, where

$$\psi(y) = \int_0^\infty e^{iyx} - 1 - iyx \cdot 1_{\{x \leq 1\}} \, d\phi(x)$$

and the Lévy measure $\phi$ is concentrated on $2\mathbb{Z}$ with $\phi(\{2^k\}) = 2^{-k}$ for $k \in \mathbb{Z}$. Hence $X$ is a semistable random variable and the corresponding Lévy process $\{X(t)\}_{t \geq 0}$ with $X(1) \overset{d}{=} X$ is a (non-strictly) semistable Lévy process fulfilling the semi-selfsimilarity condition

$$(1.2) \quad X(2^kt) \overset{d}{=} 2^k(X(t) + kt) \quad \text{for every } k \in \mathbb{Z} \text{ and } t \geq 0.$$ 

For details on semistable random variables and Lévy processes we refer to the monographs [30, 34]. For a semistable setup in general there exists a continuum of possible limit distributions by variation of the subsequence $k(n) \rightarrow \infty$ in (1.1). The possible limit distributions for the St. Petersburg game have been characterized by Csörgő and Dodunekova [7] as follows. For $n \in \mathbb{N}$ let us introduce the quantity

$$(1.3) \quad \gamma_n = n \cdot 2^{-[\log_2 n]} \in \left(\frac{1}{2}, 1\right],$$

which determines the relative position of $n \in \mathbb{N}$ between its nearest consecutive powers of 2. If $k(n) \rightarrow \infty$ is a subsequence such that $\gamma_{k(n)} \rightarrow t \in \left[\frac{1}{2}, 1\right]$, Csörgő and Dodunekova [7] have shown that

$$(1.4) \quad \frac{S_{k(n)} - k(n) \log_2 k(n)}{k(n)} \rightarrow Y(t) := t^{-1} \left( X(t) - t \log_2 t \right)$$

in distribution, where $Y\left(\frac{1}{2}\right) \overset{d}{=} Y(1) \overset{d}{=} X$ due to (1.2); cf. also [37]. In fact, in Theorem 2.2 of [7] the necessary and sufficient condition for convergence in distribution of the normalized and centralized sums $S_n$ along the subsequence $k(n) \rightarrow \infty$
should be stated in terms of the so-called circular convergence of $\gamma_{k(n)}$; for details we refer to page 301 in [5] or page 241 in [6]. It is also possible to interpret (1.4) as convergence in distribution of stochastic processes on the space $D_{[\frac{1}{2}, 1]}$ of càdlàg functions $\varphi : [\frac{1}{2}, 1] \to \mathbb{R}$ equipped with the Skorokhod $J_1$-topology as follows. Firstly, a direct application of Theorem 14.14 in [25] shows that

$$
\frac{S_{[2^n t]} - \lfloor 2^n t \rfloor n}{2^n} \xrightarrow{t \in [\frac{1}{2}, 1]} \{ X(t) \}_{t \in [\frac{1}{2}, 1]}
$$

in distribution on $D_{[\frac{1}{2}, 1]}$. Alternatively, one may deduce (1.5) from Theorem 2.1 of Fazekas [15]. Secondly, observe that

$$
\frac{S_{[2^n t]} - \lfloor 2^n t \rfloor \log_2 \lfloor 2^n t \rfloor}{2^n} = \frac{2^n}{2^n} \left( \frac{S_{[2^n t]} - \lfloor 2^n t \rfloor n}{2^n} - \frac{\lfloor 2^n t \rfloor \log_2 \lfloor 2^n t \rfloor}{2^n} \right),
$$

where convergence of the deterministic centerings

$$
x_n(t) = -\frac{\lfloor 2^n t \rfloor \log_2 \lfloor 2^n t \rfloor}{2^n} - t \log_2 t = x(t)
$$

and normalizations

$$
y_n(t) = \frac{2^n}{\lfloor 2^n t \rfloor} \to t^{-1} = y(t)
$$

can also be interpreted as convergence in $D_{[\frac{1}{2}, 1]}$ with continuous limit functions $x$ and $y$. Since addition and multiplication are continuous at every element of $D_{[\frac{1}{2}, 1]} \times D_{[\frac{1}{2}, 1]}$ for which the coordinate functions have no common discontinuities (see Theorem 4.1 in [39], respectively Theorem 13.3.2 in [30]), an application of the continuous mapping theorem yields

$$
\left\{ \frac{S_{[2^n t]} - \lfloor 2^n t \rfloor \log_2 \lfloor 2^n t \rfloor}{2^n} \right\}_{t \in [\frac{1}{2}, 1]} \to \left\{ t^{-1} (X(t) - t \log_2 t) = Y(t) \right\}_{t \in [\frac{1}{2}, 1]}
$$

in distribution on $D_{[\frac{1}{2}, 1]}$. Hence we have convergence in distribution of stochastic processes in (1.3) for $k(n) = \lfloor 2^n t \rfloor$ for which circular convergence of $\gamma_{k(n)}$ towards $t \in [\frac{1}{2}, 1]$ holds.

The object of our study are local fluctuations of the sample paths of the stochastic process $Y = \{ Y(t) \}_{t \in [\frac{1}{2}, 1]}$. Figure 11 shows typical (approximative) sample paths of $\{ Y(t) \}_{t \in [\frac{1}{2}, 1]}$ generated by $n = 2^{16}$ simulated St. Petersburg games. Note that the sample paths do only have upward jumps due to the fact that the Lévy measure $\phi$ is concentrated on $2^\mathbb{Z}$.

The main goal of our paper is to determine the Hausdorff dimension of the range $Y([\frac{1}{2}, 1]) = \{ Y(t) : t \in [\frac{1}{2}, 1] \}$ and the graph $G_Y([\frac{1}{2}, 1]) = \{ (t, Y(t)) : t \in [\frac{1}{2}, 1] \}$ of the stochastic process $Y$ encoding all the possible distributional
limits of St. Petersburg games. For an arbitrary subset $F \subseteq \mathbb{R}^d$ the $s$-dimensional Hausdorff measure is defined as

$$\mathcal{H}^s(F) = \liminf_{\delta \to 0} \left\{ \sum_{i=1}^{\infty} |F_i|^s : |F_i| \leq \delta \text{ and } F \subseteq \bigcup_{i=1}^{\infty} F_i \right\},$$

where $|F| = \sup\{\|x - y\| : x, y \in F\}$ denotes the diameter of a set $F \subseteq \mathbb{R}^d$ and $\| \cdot \|$ is the Euclidean norm. It can now be shown that there exists a unique value $\dim_H F \geq 0$ so that $\mathcal{H}^s(F) = \infty$ for all $0 \leq s < \dim_H F$ and $\mathcal{H}^s(F) = 0$ for all $s > \dim_H F$. This critical value is called the Hausdorff dimension of $F$. Specifically, we have

$$\dim_H F = \inf \left\{ s : \mathcal{H}^s(F) = 0 \right\} = \sup \left\{ s : \mathcal{H}^s(F) = \infty \right\}.$$

For details on the Hausdorff dimension we refer to [15] and [29].
Now let $F \subseteq \mathbb{R}^d$ be a Borel set and denote by $\mathcal{M}^1(F)$ the set of probability measures on $F$. For $s > 0$ the $s$-energy of $\mu \in \mathcal{M}^1(F)$ is defined by

$$I_s(\mu) = \int_F \int_F \mu(dx)\mu(dy) \|x - y\|^s.$$

By Frostman’s lemma (see, e.g., [25], [29]) there exists a probability measure $\mu \in \mathcal{M}^1(F)$ with $I_s(\mu) < \infty$ if $\dim_H F > s$. In this case $F$ is said to have positive $s$-capacity $C_s(F)$ given by

$$C_s(F) = \sup \{I_s(\mu)^{-1} : \mu \in \mathcal{M}^1(F)\},$$

and the capacitary dimension of $F$ is defined by

$$\dim C F = \sup \{s > 0 : C_s(F) > 0\} = \inf \{s > 0 : C_s(F) = 0\}.$$ 

A consequence of Frostman’s theorem (see, e.g., [25], [29]) is that for Borel sets $F \subseteq \mathbb{R}^d$ the Hausdorff and capacitary dimension coincide. Therefore, one can prove lower bounds for the Hausdorff dimension with a simple capacity argument: if $I_s(\mu) < \infty$ for some $\mu \in \mathcal{M}^1(F)$ then $\dim_H F = \dim_C F \geq s$.

An alternative fractal dimension is the so-called box-counting dimension (see, e.g., [15]). For this purpose let $N_\delta(F)$ be the smallest number of closed balls of radius $\delta$ that cover the set $F \subseteq \mathbb{R}^d$. The lower and the upper box-counting dimensions of an arbitrary set $F \subseteq \mathbb{R}^d$ are now defined as

$$\dim_B^\ell F = \liminf_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \quad \text{and} \quad \dim_B^\r F = \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

and the box-counting dimension of $F$ is given by

$$\dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

provided that this limit exists. The different fractal dimensions are related as follows:

$$\dim_H F \leq \dim_B^\ell F \leq \dim_B F \leq \dim_B^\r F \leq d.$$

Note that there are plenty of sets $F \subseteq \mathbb{R}^d$ where these inequalities are strict.

In Section 2 we will determine the Hausdorff and box-counting dimension of the range $Y\left([\frac{1}{2}, 1]\right)$ and the graph $G_Y\left([\frac{1}{2}, 1]\right)$ for almost all sample paths of the stochastic process $Y$. Additionally, in Section 3 we will also consider a deterministic sequence introduced by Steinhaus [35] which is called the Steinhaus sequence according to [8]. The Steinhaus sequence $(x_n)_{n \in \mathbb{N}}$ is defined by $x_n = 2^k$ if $n = 2^{k-1} + m \cdot 2^k$ for some $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$. Alternatively, as in Vardi [38],
one can define $x_n$ to be twice the highest power of 2 dividing $n$. The Steinhaus sequence is explicitly given by

$$2, 4, 2, 8, 2, 4, 2, 16, 2, 4, 2, 8, 2, 4, 2, 32, 2, 4, 2, 8, 2, 4, 2, 16, 2, 4, 2, 8, 2, 4, 2, 64, ...$$

and has relative frequencies $\lim_{n \to \infty} n^{-1} \text{card}\{1 \leq j \leq n : x_j = 2^k\} = 2^{-k}$ for $k \in \mathbb{N}$. The sequence $(x_n)_{n \in \mathbb{N}}$ has been considered as entrance fees for repeated St. Petersburg games in [35] and [9] and has been proven to be a sequence of nearly asymptotically fair premiums in a certain sense. For details we refer to [9]. In contrast to [35] and [9] we will consider the Steinhaus sequence as a sequence of possible gains in repeated St. Petersburg games. Again, we will determine the Hausdorff and box-counting dimension of the range and the graph of the corresponding limiting object of the centralized and normalized Steinhaus sequence. To do so, we will employ results for iterated function systems as presented in [17].

2. Hausdorff Dimension of the St. Petersburg Game

2.1. Hausdorff dimension of the range. In this section we evaluate the Hausdorff dimension of the range of the stochastic process $Y = \{Y(t)\}_{t \in [\frac{1}{2}, 1]}$. We employ common techniques used to calculate Hausdorff dimensions of selfsimilar Lévy processes (see [41], [31], [27]) and adapt them to our situation. Note that the given process $Y$ is neither a Lévy process nor does it have the selfsimilarity property of a semistable process. The result is stated in the theorem below.

**Theorem 2.1.** We have $\dim_H Y([\frac{1}{2}, 1]) = 1$ almost surely.

Note that Theorem 2.1 together with (1.7) yields

$$\dim_H Y([\frac{1}{2}, 1]) = \dim_B Y([\frac{1}{2}, 1]) = 1$$

almost surely. Since $Y$ is a process on $\mathbb{R}$ it is obvious that $\dim_H Y([\frac{1}{2}, 1]) \leq 1$ almost surely. Hence for the proof of Theorem 2.1 it is sufficient to prove the following lemma.

**Lemma 2.1.** We have $\dim_H Y([\frac{1}{2}, 1]) \geq 1$ almost surely.

**Proof.** As mentioned above we can write

$$Y(t) = t^{-1}(X(t) - t \log_2 t),$$

where $X = \{X(t)\}_{t \geq 0}$ is a semistable Lévy process. To prove the assertion we will apply Frostman’s theorem [25], [29] with the probability measure $\sigma = 2\lambda|[\frac{1}{2}, 1]|$, where $\lambda$ denotes Lebesgue measure. For this purpose let $0 < \gamma < 1$ and note that
\( \sigma \) is an admissible measure for Frostman’s lemma, i.e.,
\[
\int_{\frac{1}{2}}^{1} \frac{1}{|s-t|^{\gamma}} \sigma(ds) \sigma(dt) < \infty.
\]

By Frostman’s theorem it is now sufficient to show that for all \( \gamma \in (0, 1) \)
\[
\int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} \mathbb{E}|Y(s) - Y(t)|^{-\gamma} ds \, dt < \infty.
\]

For \( r \in (0, 1] \) let \( g_r \) be the Lebesgue density of \( X(r) \) chosen from the class \( C^\infty(\mathbb{R}) \) by Proposition 2.8.1 in [34]. Then we have \( M := \sup_{r \in (0,1]} \sup_{x \in \mathbb{R}} |g_r(x)| < \infty \) as in Lemma 3 of [34]; see also Lemma 2.2 in [27]. By symmetry of the integrand we get
\[
\int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} \mathbb{E}|Y(s) - Y(t)|^{-\gamma} ds \, dt = 2 \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} \mathbb{E}\left[ \left| s^{-1} X(s) - \log_2 s - t^{-1} \left( X(s) + (X(t) - X(s)) \right) \right| \right] \, ds \, dt
\]
\[
= 2 \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} \int_{\mathbb{R}^2} \left| s^{-1} x - \log_2 s - t^{-1}(x + y) + \log_2 t \right|^{-\gamma} g_s(x) g_{t-s}(y) \, d\lambda^2(x,y) \, ds \, dt
\]
\[
= 2 \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} \int_{\mathbb{R}^2} \left| \frac{t-s}{st} x + \log_2 \left( \frac{t}{s} \right) - \frac{y}{t} \right|^{-\gamma} g_s(x) g_{t-s}(y) \, d\lambda^2(x,y) \, ds \, dt
\]
\[
= 2 \int_{\frac{1}{2}}^{1} \int_{0}^{\frac{1}{2}} \int_{\mathbb{R}^2} \left| \frac{w}{t(t-w)} x + \log_2 \left( \frac{t}{t-w} \right) - \frac{y}{t} \right|^{-\gamma} g_{t-w}(x) g_w(y) \, d\lambda^2(x,y) \, dw \, dt,
\]

where in the last equality we substituted \( w = t-s \). Now we write \( w \in \left( 0, \frac{1}{2} \right] \) as \( w = 2^{-m} r \) with \( m = m(w) \in \mathbb{N} \) and \( r \in \left( \frac{1}{2}, 1 \right] \). This leads us to
\[
g_w(y) = \frac{d}{dy} \mathbb{P}(X(w) \leq y)
\]
\[
= \frac{d}{dy} \mathbb{P}(X(2^{-m} r) \leq y) = \frac{d}{dy} \mathbb{P}\left( 2^{-m}(X(r) - mr) \leq y \right)
\]
\[
= \frac{d}{dy} \mathbb{P}(X(r) \leq 2^m y + mr) = 2^m g_r(2^m y + mr).
\]

Using the substitutions
\[
v = 2^m y + mr \quad \text{and} \quad u = \frac{t}{2^{-m}} \left( \frac{w}{t(t-w)} x + \log_2 \left( \frac{t}{t-w} \right) + \frac{mw}{t} \right),
\]
\[
\frac{d}{dy} \mathbb{P}(X(r) \leq 2^m y + mr) = \frac{d}{du} \mathbb{P}(X(r) \leq 2^m y + mr) \frac{dv}{du} = 2^m g_r(2^m y + mr).
\]
we get

\[
\int_{\mathbb{R}^2} \frac{w}{t(t-w)} x + \log_2 \left( \frac{t}{t-w} \right) - \frac{y}{t} \left[ g_{t-w}(x) g_w(y) \right] d\lambda^2(x,y) \\
= 2m \int_{\mathbb{R}^2} \frac{w}{t(t-w)} x + \log_2 \left( \frac{t}{t-w} \right) - \frac{y}{t} \left[ g_{t-w}(x) g_r(2^m y + mr) \right] d\lambda^2(x,y) \\
= \int_{\mathbb{R}^2} \frac{w}{t(t-w)} x + \log_2 \left( \frac{t}{t-w} \right) - \frac{2^{-m}}{t} v + \frac{mw}{t} \left[ g_{t-w}(x) g_r(v) \right] d\lambda^2(x,v) \\
= \frac{t-w}{r} \int_{\mathbb{R}^2} \left| \frac{2^{-m}}{t} (u-v) \right|^{-\gamma} g_{t-w}(x(u)) g_r(v) d\lambda^2(u,v) \\
= \frac{t^\gamma (t-w)^{2m\gamma}}{r} \left( \int_A + \int_{A^c} \right) |u-v|^{-\gamma} g_{t-w}(x(u)) g_r(v) d\lambda^2(u,v),
\]

where \( A = \{(u,v) \in \mathbb{R}^2 : |u-v| \leq 1\} \). We now estimate the two integrals separately. Firstly,

\[
\int_A |u-v|^{-\gamma} g_{t-w}(x(u)) g_r(v) d\lambda^2(u,v) \\
\leq M \int_{\mathbb{R}} \left( \int_{v-1}^v (v-u)^{-\gamma} du + \int_u^{v+1} (u-v)^{-\gamma} du \right) g_r(v) dv \\
= M \int_{\mathbb{R}} \frac{2}{1-\gamma} g_r(v) dv = \frac{2M}{1-\gamma}
\]

and secondly,

\[
\int_{A^c} |u-v|^{-\gamma} g_{t-w}(x(u)) g_r(v) d\lambda^2(u,v) \leq \int_{A^c} g_{t-w}(x(u)) g_r(v) d\lambda^2(u,v) \\
\leq \frac{r}{t-w} \int_{\mathbb{R}^2} g_{t-w}(x) g_r(v) d\lambda^2(x,v) = \frac{r}{t-w}.
\]

This leads us to

\[
\int_{\mathbb{R}^2} \frac{w}{t(t-w)} x + \log_2 \left( \frac{t}{t-w} \right) - \frac{y}{t} \left[ g_{t-w}(x) g_w(y) \right] d\lambda^2(x,y) \\
\leq t^\gamma 2^{m\gamma} \left( \frac{2M(t-w)}{r(1-\gamma)} + 1 \right) \leq t^\gamma 2^{m\gamma} \left( \frac{4M}{1-\gamma} + 1 \right) =: K t^\gamma 2^{m\gamma}.
\]
Taken all together, we obtain

\[
\int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} \mathbb{E}[|Y(s) - Y(t)|^{-\gamma}] ds \, dt \leq 2K \int_{0}^{1} \int_{\frac{1}{2}}^{t} t^{\gamma}2m(w)^{-\gamma} \, dw \, dt
\]

\[
= 2K \int_{0}^{1} \int_{\frac{1}{2}}^{1} t^{\gamma}2m(w)^{-\gamma} \, dw \, dt = 2K \sum_{m \in \mathbb{N}} 2^{-m} \int_{\frac{1}{2}}^{1} t^{\gamma}2m^{-\gamma} \, dt \, dw
\]

\[
\leq 2K \sum_{m \in \mathbb{N}} \int_{\frac{1}{2}}^{1} t^{\gamma}2m^{-\gamma} \, dt \, 2^{-m} \, dr = K \sum_{m \in \mathbb{N}} (2^{\gamma-1})^{-m} \, t^{\gamma} \, dt < \infty,
\]

since \(\gamma - 1 < 0\). This concludes our proof. \(\blacksquare\)

### 2.2. Hausdorff dimension of the graph.

In this section we show that the dimension result for the range of the stochastic process \(Y\) also holds for its graph \(G_Y([\frac{1}{2}, 1])\). We will split the proof into two parts, firstly verifying \(\alpha = 1\) as an upper bound and secondly as a lower bound for the Hausdorff dimension of the graph.

We first calculate the upper bound for the Hausdorff dimension of the graph of the semistable Lévy process \(X\) and later on transfer the result to the process \(Y\).

As \(X\) is not strictly semistable, we cannot use the dimension results of [27] without modifying it according to our situation. This parallels investigations for stable Lévy processes, where dimension results for the strictly stable (symmetric) Cauchy process and for an asymmetric (non-strictly) stable Cauchy process have been addressed separately. Nevertheless, the Hausdorff dimension of the range, respectively the graph, coincides for both Cauchy processes; see [2], [22], [36].

**Proposition 2.1.** Let \(\{Z(t) := (t, X(t))\}_{t \geq 0}\). Then almost surely

\[
\dim H Z([\frac{1}{2}, 1]) \leq 1.
\]

Let \(T(a, s) = \int_{0}^{s} 1_{B(0, a)}(Z(t)) \, dt\) denote the sojourn time of the Lévy process \(Z\) up to time \(s\) in the closed ball \(B(0, a) \subseteq \mathbb{R}^2\) with radius \(a\) centered at the origin. To prove Proposition 2.1 we need the following lemma.

**Lemma 2.2.** Let \(Z\) be the stochastic process defined in Proposition 2.1. There exists a positive and finite constant \(K\) such that for all \(0 < a \leq 1\) and \(\frac{a}{\sqrt{2}} \leq s \leq 1\) we have

\[
\mathbb{E}[T(a, s)] \geq Ka.
\]

**Proof.** Fix \(0 < a \leq 1\) and let \(0 < \delta \leq \frac{1}{\sqrt{2}}\), so that

\[
2^{-i_0+1} < a\delta \leq 2^{-i_0} < \frac{a}{\sqrt{2}} < s
\]
for some \( i_0 \in \mathbb{N}_0 \). Furthermore, we choose \( 0 < \delta \leq \frac{1}{\sqrt{2}} \) small enough (i.e. \( i_0 \in \mathbb{N}_0 \) big enough) so that

\[
2.3 \quad 2i \leq \frac{a}{2\sqrt{2}}2^i \quad \text{for all } i > i_0
\]

and that additionally

\[
2.2 \quad P \left( \sup_{r \in [1,2]} X(r) < \frac{1}{\delta 2\sqrt{2}} \right) - P \left( \inf_{r \in [1,2]} X(r) \leq -\frac{1}{\delta 2\sqrt{2}} \right) \geq \frac{1}{2}.
\]

Inequality (2.3) holds for \( \delta > 0 \) small enough since \( X \) is a Lévy process and it can be assumed that it has càdlàg paths. Thus both \( \sup_{r \in [1,2]} X(r) \) and \( \inf_{r \in [1,2]} X(r) \) are random variables. We have

\[
E \left[ T(a, s) \right] = \int_0^s P \left( \| Z(t) \| < a \right) dt \geq \int_0^s P \left( |X(t)| < \frac{a}{\sqrt{2}}, t < \frac{a}{\sqrt{2}} \right) dt
\]

\[
= \int_0^{a/\sqrt{2}} P \left( |X(t)| < \frac{a}{\sqrt{2}} \right) dt \geq \int_0^{a/\sqrt{2}} P \left( |X(t)| < \frac{a}{\sqrt{2}} \right) dt
\]

\[
= \sum_{i=i_0+1}^{2^{i+1}} \int_{2^{-i}}^{2^{i+1}} P \left( |X(t)| < \frac{a}{\sqrt{2}} \right) dt = \sum_{i=i_0+1}^{\infty} 2^{-i} \int_1^{2} P \left( |X(2^{-i}r)| < \frac{a}{\sqrt{2}} \right) dr
\]

\[
= \sum_{i=i_0+1}^{\infty} 2^{-i} \int_1^{2} P \left( |X(r) - ir| < \frac{2^i a}{\sqrt{2}} \right) dr.
\]

By (2.2) and (2.3) the probability above can be estimated from below by

\[
P \left( |X(r) - ir| < \frac{2^i a}{\sqrt{2}} \right) = P \left( -\frac{a}{\sqrt{2}}2^i + ir < X(r) < \frac{a}{\sqrt{2}}2^i + ir \right)
\]

\[
\geq P \left( -\frac{a}{\sqrt{2}}2^i + 2i \leq X(r) < \frac{a}{\sqrt{2}}2^i \right)
\]

\[
= P \left( X(r) < \frac{a}{\sqrt{2}}2^i \right) - P \left( X(r) \leq -\frac{a}{\sqrt{2}}2^i + 2i \right)
\]

\[
\geq P \left( \sup_{r \in [1,2]} X(r) < \frac{a}{\sqrt{2}}2^i \right) - P \left( \inf_{r \in [1,2]} X(r) \leq -\frac{a}{\sqrt{2}}2^i + 2i \right)
\]

\[
\geq P \left( \sup_{r \in [1,2]} X(r) < \frac{1}{\delta 2\sqrt{2}} \right) - P \left( \inf_{r \in [1,2]} X(r) \leq -\frac{1}{\delta 2\sqrt{2}} \right) \geq \frac{1}{2}.
\]
Note that $\delta$ does not depend on $a$. It follows that
\[
\mathbb{E}[T(a,s)] \geq \sum_{i=i_0+1}^{\infty} 2^{-i} \int_0^2 \frac{1}{2} \, dr = \frac{1}{2} \sum_{i=i_0+1}^{\infty} 2^{-i} = \frac{1}{2} - i_0 \geq \frac{1}{2} \delta a =: Ka,
\]
which concludes the proof. ■

Proof of Proposition 2.1. Let $K_1 > 0$ be a fixed constant. A family $\Lambda(a)$ of cubes of side $a \in (0,1]$ in $\mathbb{R}^2$ is called $K_1$-nested if no ball of radius $a$ in $\mathbb{R}^2$ can intersect more than $K_1$ cubes of $\Lambda(a)$. For any $u \geq 0$ let $M_u(a,s)$ be the number of these cubes hit by the Lévy process $Z$ at some time $t \in [u,u+s]$. Then a famous covering lemma of Pruitt and Taylor ([33], Lemma 6.1) states that
\[
\mathbb{E}[M_u(a,s)] \leq 2K_1s \cdot (\mathbb{E}[T(a/3,s)])^{-1}.
\]
Lemma now enables us to construct a covering of $Z([\frac{1}{2},1])$ whose expected $s$-dimensional Hausdorff measure is finite for every $s > 1$. The arguments are analogous to the proof of part (i) of Lemma 3.4 in [27] and thus omitted. ■

In order to transfer the result of Proposition 2.1 to the process $Y$ we introduce the continuous function $\tau : [\frac{1}{2},1] \times K_1 \rightarrow [\frac{1}{2},1] \times K_2$ with
\[
\tau(t,x) = \left( t^{-1}x - \log_2 t \right),
\]
where $K_1, K_2 \subseteq \mathbb{R}$ are not further specified compact intervals that can vary throughout the paper. It can easily be shown that for a fixed compact interval $K_1 \subseteq \mathbb{R}$ the function $\tau$ is bi-Lipschitz when choosing $K_2$ such that $[\frac{1}{2},1] \times K_2 = \text{Im}(\tau)$. We can now write all elements $(t,Y(t))^\top \in G_Y([\frac{1}{2},1])$ as
\[
\left( \begin{array}{c} t \\ Y(t) \end{array} \right) = \left( \begin{array}{c} t \\ t^{-1}X(t) - \log_2 t \end{array} \right) = \tau(t,X(t)).
\]
Since $X$ is a Lévy process, it can be assumed that all paths are càdlàg, and hence that for all fixed $\omega \in \Omega$ there exists a compact interval $K_1 \subseteq \mathbb{R}$ such that $X(t)(\omega) \in K_1$ for all $t \in [\frac{1}{2},1]$. This means that for $Z = (Z(t) = (t, X(t)))_{t \in [\frac{1}{2},1]}$ and all $\omega \in \Omega$ we have
\[
\dim_H Z([\frac{1}{2},1])(\omega) = \dim_H \tau(Z([\frac{1}{2},1]))(\omega) = \dim_H G_Y([\frac{1}{2},1])(\omega)
\]
by Lemma 1.8 in [13]. Since we have shown in Proposition 2.1 that $\dim_H Z([\frac{1}{2},1]) \leq 1$ almost surely, we have thus proven the following upper bound.
THEOREM 2.2. We have $\dim H G Y \left( \left[ \frac{1}{2}, 1 \right] \right) \leq 1$ almost surely.

To prove the lower bound for the Hausdorff dimension of the graph we can use the same technique as for the lower bound in case of the range of $Y$.

THEOREM 2.3. We have $\dim H G Y \left( \left[ \frac{1}{2}, 1 \right] \right) \geq 1$ almost surely.

Proof. Let $0 < \gamma < 1$. By (2.1) we get

\[
\frac{1}{2} \int_1^1 \mathbb{E} \left[ \left\| (s, Y(s)) - (t, Y(t)) \right\|^{-\gamma} \right] ds dt
= \frac{1}{2} \int_1^1 \mathbb{E} \left[ (s - t)^2 + (Y(s) - Y(t))^2 \right]^{\gamma/2} ds dt
\leq \frac{1}{2} \int_1^1 \mathbb{E} |Y(s) - Y(t)|^{-\gamma} ds dt < \infty.
\]

As in Lemma 2.1 the assertion follows by Frostman’s theorem. ■

With similar techniques it is also possible to prove the following dimension result for the box-counting dimension of the graph of the St. Petersburg process $Y$.

THEOREM 2.4. We have $\dim B G Y \left( \left[ \frac{1}{2}, 1 \right] \right) = 1$ almost surely.

Proof. The lower bound follows directly from the almost sure inequalities

\[1 \leq \dim H G Y \left( \left[ \frac{1}{2}, 1 \right] \right) \leq \dim B G Y \left( \left[ \frac{1}{2}, 1 \right] \right) \leq \overline{\dim B} G Y \left( \left[ \frac{1}{2}, 1 \right] \right)\]

For the upper bound it is now sufficient to verify $\overline{\dim B} G Y \left( \left[ \frac{1}{2}, 1 \right] \right) \leq 1$ almost surely. Due to the nature of the upper box-counting dimension (see (1.6)) we can again calculate the upper bound for $\overline{\dim B} Z \left( \left[ \frac{1}{2}, 1 \right] \right) \leq 1$ with the same covering arguments as in the proof of part (i) of Lemma 3.4 in [22]. With the bi-Lipschitz invariance of the upper box-counting dimension (see Section 3.2 in [15]) the proof concludes. ■

REMARK 2.1. If one prefers to flip an unfair coin, this naturally leads to so-called generalized St. Petersburg games as treated in [8], [21], [32]. Let $p \in (0, 1)$ be the probability of the coin falling heads and let $q = 1 - p$. The single gain in a generalized St. Petersburg game is given by $q^{T/\alpha}$ for some $\alpha > 0$. We focus on the classical situation $\alpha = 1$ and slightly modify the gain to $q^{1-T} p^{-1}$ for ease of the notation, which results in the limit theorem

\[
\frac{S_{k(n)} - k(n) \log_{1/q} k(n)}{k(n)} \to Y(t) = t^{-1} (X(t) - t \log_{1/q} t)
\]
in distribution, whenever
\[
q^{[\log_{1/q} k(n)]} k(n) \to t \in [q, 1],
\]
where \( \{ X(t) \}_{t \geq 0} \) is a semistable Lévy process with the semi-selfsimilarity property
\[
X(q^{-k} t) \overset{d}{=} q^{-k} (X(t) + k t) \quad \text{for every } k \in \mathbb{Z} \text{ and } t \geq 0.
\]

We emphasize that with the above techniques our Theorems 2.1–2.4 also hold for the process \( \{ Y(t) \}_{t \in [q, 1]} \) in this generalized situation when replacing the interval by \([q, 1]\). Presumably, similar results can be shown for general \( \alpha > 0 \).

3. Hausdorff Dimension of the Steinhaus Sequence

Recall the definition of the Steinhaus sequence \((x_n)_{n \in \mathbb{N}}\) given in the Introduction. The asymptotic properties of \((x_n)_{n \in \mathbb{N}}\) have been analyzed in full detail by Csörgő and Simons [9]. Let \( s(n) = x_1 + \ldots + x_n \) and \( \gamma_n = n \cdot 2^{-[\log_2 n]} \in \left( \frac{1}{2}, 1 \right) \) as in (3.1). Then by Theorem 3.3 in [9] we have for any \( n \in \mathbb{N} \)
\[
(3.1) \quad \frac{s(n) - n \log_2 n}{n} = \xi(\gamma_n),
\]
where the function \( \xi : \left[ \frac{1}{2}, 1 \right] \to [0, 2] \) is defined by
\[
\xi(\gamma) = 2 - \log_2 \gamma - \frac{1}{\gamma} \sum_{k=1}^{\infty} k \varepsilon_k 2^{-k},
\]
and the sequence \( (\varepsilon_k)_{k \in \mathbb{N}} \subseteq \{0, 1\} \) is given by the dyadic expansion \( \gamma = \sum_{k=0}^{\infty} \varepsilon_k \gamma_k \) of \( \gamma \in \left[ \frac{1}{2}, 1 \right] \) with the convention that \( \varepsilon_k = 0 \) for infinitely many \( k \in \mathbb{N} \). By Theorem 3.1 in [9] the function \( \xi \) is càdlàg with \( \xi(\frac{1}{2}) = 2 = \xi(1) \) and has jumps precisely at the dyadic rationals in \( \left( \frac{1}{2}, 1 \right) \). All these jumps are upward and the largest jump occurs from \( \xi(1^{-}) = 0 \) to \( \xi(1) = 2 \). The graph of \( \xi \) seems to inhere fractal properties as can be seen in Figure 2 below, a replication of Figure 1 in [9]. It follows directly from (3.1) that the sequence \( (s(n))_{n \in \mathbb{N}} \) of total gains satisfies the asymptotic property of Feller
\[
(3.2) \quad \frac{s(n)}{n \log_2 n} \to 1 \quad \text{as } n \to \infty; \quad \text{see [9].}
\]
Note that Feller’s law of large numbers does not hold in an almost sure sense. According to classical results in [3], [1], [10] it is known that
\[
(3.3) \quad \lim_{n \to \infty} \sup_{n \in \mathbb{N}} \frac{S_n}{n \log_2 n} = \infty \quad \text{and} \quad \lim_{n \to \infty} \inf_{n \in \mathbb{N}} \frac{S_n}{n \log_2 n} = 1 \quad \text{almost surely.}
\]
More precisely, we have \( \text{LIM}\{S_n/(n \log_2 n) : n \in \mathbb{N}\} = [1, \infty] \) almost surely by Corollary 1 in [38], where LIM denotes the set of accumulation points. But there is a version of the strong law of large numbers by Csörgő and Simons [11] when neglecting the largest gain
\[
\frac{S_n - \max_{1 \leq k \leq n} X_k}{n \log_2 n} \to 1 \quad \text{almost surely.}
\]

A comparison of (3.2) and (3.3) shows that the Steinhaus sequence belongs to an exceptional null set with respect to (3.3) concerning the almost sure limit behavior of the total gain in repeated St. Petersburg games. Moreover, for any sequence \( k_n \to \infty \) with \( \gamma_{k_n} = k_n \cdot 2^{-\lceil \log_2 k_n \rceil} \to \gamma \in \left[\frac{1}{2}, 1\right] \) we get from (3.1)
\[
\emptyset \neq \text{LIM}\left\{ \frac{s(k_n) - k_n \log_2 k_n}{k_n} : n \in \mathbb{N}\right\} \subseteq \{\xi(\gamma), \xi(\gamma^-)\}.
\]

Hence we may consider the function \( \xi \) as the corresponding limiting object when using the same centering and normalization sequences as in (1.4). We will now show that the Steinhaus sequence is not exceptional concerning the local fluctuations of the limit measured by the Hausdorff or box-counting dimension.

It follows directly from the above-stated properties of \( \xi \) given in Theorem 3.1 of [29] that the range \( \xi\left(\left[\frac{1}{2}, 1\right]\right) \) equals the interval \( (0, 2] \) and hence \( \dim_H \xi\left(\left[\frac{1}{2}, 1\right]\right) \)
Dimension results related to the St. Petersburg game

= 1 by Theorem 1.12 in [13]. This shows that \( \dim_H \xi([1/2,1]) \) coincides with the Hausdorff dimension of the range of a typical sample path of \( \{Y(t)\}_{t \in [1/2,1]} \).

Clearly, by (3.4) we also have \( \dim_H \xi([1/2,1]) = \dim_H \xi([1,1]) = 1 \). A look at Figure 4 suggests that it is merely the graph and not the range of \( \xi \) that should here inherit fractal properties. In the sequel we will argue that also the graph \( G_\xi([1/2,1]) \) is typical concerning the almost sure dimension properties of the sample graph of \( \{Y(t)\}_{t \in [1/2,1]} \). To this aim we will apply the inverse \( \tau^{-1} \) of the bi-Lipschitz function \( \tau \) defined in (3.4) with \( K_2 = [0,2] \subseteq \mathbb{R} \). Namely, we now consider the function \( \tau^{-1} : [1/2,1] \times [0,2] \to [1,1] \times K_1 \) with \( \tau^{-1}(t,x) = (t,t(x + \log_2 t)) \), where the compact interval \( K_1 \subseteq \mathbb{R} \) is chosen such that \( [1/2,1] \times K_1 = \text{Im}(\tau^{-1}) \).

Applied to the graph of \( \xi \) we get for any \( \gamma \in [1/2,1] \)

\[
\tau^{-1}(\gamma, \xi(\gamma)) = \left( \frac{\gamma}{\gamma + \log_2 \gamma} \right) = 2\gamma - \sum_{k=1}^{\infty} \frac{\gamma}{(k \varepsilon_k)/2^k}
\]

and by bi-Lipschitz invariance we have

(3.4) \( \dim_H G_\xi([1/2,1]) = \dim_H \tau^{-1}(G_\xi([1/2,1])) \).

The same equality holds for upper and lower box-counting dimensions; see, e.g., [13]. The image \( \tau^{-1}(G_\xi([1/2,1])) \) is illustrated in Figure 4 and shows perfect self-similarity. To see this, we may write \( \tau^{-1}(\gamma, \xi(\gamma)) = (\gamma, f(\gamma)) \) with

\[
f(\gamma) = 2\gamma - \sum_{k=1}^{\infty} \frac{k \varepsilon_k}{2^k}.
\]

**Lemma 3.1.** For any \( \gamma \in [1/2,1] \) we have

\[
f(\gamma + 1/2) = \frac{1}{2}(1 - \gamma + f(\gamma)) = f(\gamma + 1/4).
\]

**Proof.** For the dyadic expansion \( \gamma = \sum_{k=1}^{\infty} \varepsilon_k/2^k \) of \( \gamma \in [1/2,1] \) we necessarily have \( \varepsilon_1 = 1 \). Consequently,

\[
\frac{1}{2}\gamma + \frac{1}{2} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{\varepsilon_k}{2^k + 1} = \sum_{k=1}^{\infty} \varepsilon'_k \quad \text{with} \quad \varepsilon'_k = \begin{cases} 1, & k = 1, \\ \varepsilon_{k-1}, & k \geq 2, \end{cases}
\]

and

\[
\frac{1}{2}\gamma + \frac{1}{4} = \frac{1}{4} + \sum_{k=1}^{\infty} \frac{\varepsilon_k}{2^k + 1} = \sum_{k=1}^{\infty} \varepsilon''_k \quad \text{with} \quad \varepsilon''_k = \begin{cases} 1, & k = 1, \\ 0, & k = 2, \\ \varepsilon_{k-1}, & k \geq 3. \end{cases}
\]
It follows that
\[
    f\left(\frac{1}{2} \gamma + \frac{1}{2}\right) = 2\left(\frac{1}{2} \gamma + \frac{1}{2}\right) - \sum_{k=1}^{\infty} \frac{k \varepsilon'_k}{2^k} = \gamma + \frac{1}{2} - \sum_{k=2}^{\infty} \frac{k \varepsilon_{k-1}}{2^k},
\]
and
\[
    f\left(\frac{1}{2} \gamma + \frac{1}{4}\right) = 2\left(\frac{1}{2} \gamma + \frac{1}{4}\right) - \sum_{k=1}^{\infty} \frac{k \varepsilon''_k}{2^k} = \gamma - \sum_{k=3}^{\infty} \frac{k \varepsilon_{k-1}}{2^k} = \gamma + \frac{1}{2} - \sum_{k=2}^{\infty} \frac{k \varepsilon_{k-1}}{2^k}.
\]

This shows that \( f\left(\frac{1}{2} \gamma + \frac{1}{2}\right) = f\left(\frac{1}{2} \gamma + \frac{1}{4}\right) = \gamma + \frac{1}{2} - \sum_{k=2}^{\infty} (k \varepsilon_{k-1})/2^k \), and furthermore we get
\[
    \gamma + \frac{1}{2} - \sum_{k=2}^{\infty} \frac{k \varepsilon_{k-1}}{2^k} = \gamma + \frac{1}{2} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{(k+1) \varepsilon_k}{2^k}
    = \gamma + \frac{1}{2} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{k \varepsilon_k}{2^k} - \frac{1}{2} \gamma = \frac{1}{2} \left(1 - \gamma + f(\gamma)\right),
\]
concluding the proof. \( \blacksquare \)
Let \( \tau_0, \tau_1 : \left[ \frac{1}{2}, 1 \right] \times [0, \frac{1}{2}] \to \left[ \frac{1}{2}, 1 \right] \times [0, \frac{1}{2}] \) be the affine contractions given by

\[
\tau_0(x, y) = \left( \frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{1}{4} \right) = \left( \frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{1}{4} \right),
\][2x37]0.1
\[
\tau_1(x, y) = \left( \frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y + \frac{1}{2} \right) = \left( \frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y + \frac{1}{2} \right).
\]

Then \( \tau_0 \) maps \( \tau^{-1}(G_\xi([\frac{1}{2}, 1])) \) onto its left half and \( \tau_1 \) onto its right half, i.e., for any \( \gamma \in \left[ \frac{1}{2}, 1 \right) \) we have

\[
\tau^{-1}\left( \frac{1}{2} \gamma + \frac{1}{2} \xi\left( \frac{1}{2} \gamma + \frac{1}{2} \right) \right) = \tau_0(\gamma, f(\gamma)) = \tau_0(\tau^{-1}(\gamma, \xi(\gamma)))
\]
and

\[
\tau^{-1}\left( \frac{1}{2} \gamma + \frac{1}{2} \xi\left( \frac{1}{2} \gamma + \frac{1}{2} \right) \right) = \tau_1(\gamma, f(\gamma)) = \tau_1(\tau^{-1}(\gamma, \xi(\gamma))),
\]

which follows directly from Lemma 5.1. These contraction properties are illustrated in Figure 4 and show that the image \( \tau^{-1}(G_\xi([\frac{1}{2}, 1])) \) can be generated by an iterated function system. By Hutchinson [23] there exists a unique non-empty com-

\[
\text{Figure 4. Contractions generating the image.}
\]

The highlighted parallelograms are \( \tau_0(D), \tau_1(D) \) with \( D = \left[ \frac{1}{2}, 1 \right] \times [0, \frac{1}{2}] \) (left) and their first iterates \( \tau_0(\tau_0(D)), \tau_1(\tau_0(D)), \tau_0(\tau_1(D)), \tau_1(\tau_1(D)) \) (right)

pact set \( F \subseteq \left[ \frac{1}{2}, 1 \right] \times [0, \frac{1}{2}] \), called the attractor, such that \( F = \tau_0(F) \cup \tau_1(F) \), which fulfills

\[
F = \bigcap_{r=1}^{\infty} \bigcup_{(i_1, \ldots, i_r) \in \{0, 1\}^r} \tau_{i_r} \circ \ldots \circ \tau_{i_1}\left( \left[ \frac{1}{2}, 1 \right] \times [0, \frac{1}{2}] \right).
\]

In fact, for any \( (i_n)_{n \in \mathbb{N}} \in \{0, 1\}^\mathbb{N} \) the iterated contractions \( \tau_{i_r} \circ \ldots \circ \tau_{i_1} \) applied to the square \( \left[ \frac{1}{2}, 1 \right] \times [0, \frac{1}{2}] \) converge to a single point of \( F \) as \( r \to \infty \) and every element of \( F \) can be obtained in this way. More precisely, our construction shows that for \( \gamma \in \left[ \frac{1}{2}, 1 \right] \) with dyadic expansion \( \gamma = \sum_{k=1}^{\infty} \varepsilon_k / 2^k \) we have \( \varepsilon_1 = 1 \) and

\[
d(\tau_{i_r} \circ \ldots \circ \tau_{i_1}\left( \left[ \frac{1}{2}, 1 \right] \times [0, \frac{1}{2}] \right), \tau^{-1}(\gamma, \xi(\gamma))) \to 0
\]
as \( r \to \infty \), where \( d(A, x) = \inf \{ \| y - x \| : y \in A \} \) for \( A \subseteq \mathbb{R}^2 \) and \( x \in \mathbb{R}^2 \). Since we required \( \epsilon_k = 0 \) for infinitely many \( k \in \mathbb{N} \), the only possible limit points of \( F \) missing are those with \( \tau_{\epsilon_k} = \tau_1 \) for all but finitely many \( k \geq 2 \). For these necessarily \( \gamma = \frac{1}{2} + \sum_{k=2}^{\infty} \frac{\epsilon_k}{2^k} \in \left( \frac{1}{2}, 1 \right] \) is a dyadic rational and we have
\[
d\left( \tau_{\epsilon_r} \circ \ldots \circ \tau_{\epsilon_2} \left( \left[ \frac{1}{2}, 1 \right] \times \left[ 0, \frac{1}{2} \right] \right), \tau^{-1}(\gamma, \xi(\gamma-)) \right) \to 0
\]
as \( r \to \infty \). The above arguments show that \( F \) is the closure of \( \tau^{-1}(G_\xi \left( \left[ \frac{1}{2}, 1 \right] \right)) \) and since the dyadic rationals are countable, by elementary properties of the Hausdorff dimension and (3.4) we get
\[
(3.5) \quad \dim H F = \dim H \tau^{-1}(G_\xi \left( \left[ \frac{1}{2}, 1 \right] \right)) = \dim H G_\xi \left( \left[ \frac{1}{2}, 1 \right] \right).
\]
The same equality holds for upper and lower box-counting dimensions; see, e.g., page 44 in \( [13] \).

A common way to calculate the fractal dimension of the self-affine invariant set \( F \) is by means of the singular value function. For an overview of such methods we refer to \( [17] \). The linear part of both affine mappings \( \tau_0 \) and \( \tau_1 \) is equal to the linear contraction with associated matrix
\[
L = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1/2 \end{pmatrix}.
\]
By induction one easily calculates for \( r \in \mathbb{N} \)
\[
L^r = \begin{pmatrix} 1/2^r & 0 \\ -r/2^r & 1/2^r \end{pmatrix}
\]
and the singular values of \( L^r \) are the positive roots of the eigenvalues of \( (L^r)^T L^r \) which can be calculated as
\[
\alpha_1^{(r)} = \frac{1}{2^r} \sqrt{\frac{r^2 + 2 + \sqrt{r^4 + 4r^2}}{2}} \quad \text{and} \quad \alpha_2^{(r)} = \frac{1}{2^r} \sqrt{\frac{r^2 + 2 - \sqrt{r^4 + 4r^2}}{2}}.
\]
These determine the singular value function of \( L^r \) for \( r \in \mathbb{N} \) given by
\[
(3.6) \quad \varphi^s(L^r) = \begin{cases} 
(\alpha_1^{(r)})^s & \text{for } 0 < s \leq 1, \\
(\alpha_1^{(r)})(\alpha_2^{(r)})^{s-1} & \text{for } 1 < s \leq 2.
\end{cases}
\]
Now the affinity dimension of \( F \) is defined by
\[
\dim_A F = \inf \left\{ s > 0 : \sum_{r=1}^{\infty} \sum_{(i_1, \ldots, i_r) \in \{0, 1\}^r} \varphi^s(L_{i_r} \circ \ldots \circ L_{i_1}) < \infty \right\} = \inf \left\{ s > 0 : \sum_{r=1}^{\infty} 2^r \varphi^s(L^r) < \infty \right\},
\]
where $L_0$, $L_1$ are the linear parts of the affine contractions $\tau_0$, $\tau_1$, respectively, and the last equality holds since $L_0 = L = L_1$ in our situation. The special form of the singular values $\alpha_1^{(r)}$, $\alpha_2^{(r)}$ of $L^r$ together with (5.7) shows that $\dim_A F = 1$.

Since the union $F = \tau_0(F) \cup \tau_1(F)$ is disjoint, by Proposition 2 in [16] we get a lower bound for the Hausdorff dimension of $F$:

$$\dim_H F \geq \inf \left\{ s > 0 : \sum_{r=1}^{\infty} \sum_{(i_1, \ldots, i_r) \in \{0,1\}^r} (\varphi^s((L_{i_r} \circ \cdots \circ L_{i_1})^{-1}))^{-1} < \infty \right\} = \inf \left\{ s > 0 : \sum_{r=1}^{\infty} 2^r (\varphi^s(L^{-r}))^{-1} < \infty \right\}.$$

Again, by induction one easily calculates for $r \in \mathbb{N}$

$$L^{-r} = \begin{pmatrix} 2^r & 0 \\ \cdot & 2^r & 2^r \end{pmatrix}$$

and the singular values of $L^{-r}$ are

$$\beta_1^{(r)} = 2^r \sqrt{r^2 + 2 + \sqrt{r^4 + 4r^2}} \quad \text{and} \quad \beta_2^{(r)} = 2^r \sqrt{r^2 + 2 - \sqrt{r^4 + 4r^2}},$$

which shows that $\dim_H F \geq 1$. Since by Proposition 1 in [17] we have

$$\dim_H F \leq \dim_B F \leq \overline{\dim_B} F \leq \dim_A F,$$

the above calculations altogether show:

**THEOREM 3.1.** We have $\dim_H G_{\xi}([\frac{1}{2}, 1]) = 1 = \dim_B G_{\xi}([\frac{1}{2}, 1]).$

This shows that the graph of $\xi$, being the limiting object of the Steinhaus sequence (considered as a possible sequence of total gains in repeated St. Petersburg games), is not exceptional concerning the Hausdorff or box-counting dimension of the sample graph $G_Y([\frac{1}{2}, 1])$ calculated in Section 2.

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Peter Kern
Institute of Mathematics
Heinrich-Heine-Universität Düsseldorf
Universitätsstr. 1
40225 Düsseldorf, Germany
E-mail: kern@math.uni-duesseldorf.de

Lina Wedrich
Institute of Mathematics
Heinrich-Heine-Universität Düsseldorf
Universitätsstr. 1
40225 Düsseldorf, Germany
E-mail: wedrich@math.uni-duesseldorf.de

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