ALMOST SURE CENTRAL LIMIT THEOREMS FOR RANDOM RATIOS
AND APPLICATIONS TO LSE FOR FRACTIONAL
ORNSTEIN–UHLENBECK PROCESSES

BY

PEGGY CÉNAC (DIJON) AND KHALIFA ES-SEBAIY (MARRAKESH)

Abstract. We will investigate an almost sure central limit theorem
(ASCLT) for sequences of random variables having the form of a ratio
of two terms such that the numerator satisfies the ASCLT and the denominator
is a positive term which converges almost surely to one. This result leads
to the ASCLT for least squares estimators for Ornstein–Uhlenbeck process
driven by fractional Brownian motion.

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tic integrals.

1. INTRODUCTION

The almost sure central limit theorem (ASCLT) was simultaneously proved by
Brosamler [5] and Schatte [16]. The simplest form of the ASCLT (see Lacey and
Philipp [10]) states that if \( \{X_n, n \geq 1\} \) is a sequence of real-valued independent
identically distributed random variables with \( \mathbb{E}(X_1) = 0, \mathbb{E}(X_1^2) = 1 \), and if we
denote by \( S_n = \frac{1}{\sqrt{n}} (X_1 + \ldots + X_n) \) the normalized partial sums, then, almost
surely, for all \( z \in \mathbb{R} \),

\[
\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \mathbb{I}\{S_k \leq z\} \xrightarrow{a.s.} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} \, dx,
\]

where \( N \) is a \( \mathcal{N}(0, 1) \) random variable and \( \mathbb{I}\{A\} \) denotes the indicator of the
set \( A \). Equivalently, for any bounded and continuous function \( \varphi : \mathbb{R} \to \mathbb{R} \), one has,
almost surely,

\[
\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \varphi(S_k) \xrightarrow{a.s.} \mathbb{E}(\varphi(N)).
\]

The ASCLT was first stated, without proof, by Lévy [11]. For more discussion
about ASCLT see, e.g., Berkes and Csáki [4] and the references in the survey paper
by Berkes [3].
Ibragimov and Lifshits [8], [9] give a criterion (see Theorem 2.1) for the ASCLT based on the rate of convergence of the empirical characteristic function. Using this criterion and Malliavin calculus, Bercu et al. [2] provide a criterion for ASCLT for functionals of general Gaussian fields.

Our first aim is to prove an almost sure central limit theorem for a sequence of the form \( \{G_n/R_n\}_{n \geq 1} \), where \( \{G_n\}_{n \geq 1} \) satisfies the ASCLT and \( \{R_n\}_{n \geq 1} \) is a sequence of positive random variables not necessarily independent of \( \{G_n\} \) and converging almost surely to one (see Theorem 3.1). We apply our ASCLT to a fractional Ornstein–Uhlenbeck process \( X = \{X_t; t \geq 0\} \) defined as

\[
X_0 = 0, \quad dX_t = -\theta X_t dt + dB_t, \quad t \geq 0,
\]

where \( B = \{B_t; t \geq 0\} \) is a fractional Brownian motion with Hurst parameter \( H \in (\frac{1}{2}, 1) \), and \( \theta \) is a real parameter. \( \theta \) is unknown and estimated with least squares estimators (LSE). Theorem 3.1 leads to the ASCLT for the LSE in this model.

**Continuous observations.** Recently, the parametric estimation of the continuously observed fractional Ornstein–Uhlenbeck process defined in (1.1) was studied by using the least squares estimator (LSE) defined by

\[
\widehat{\theta}_T = \frac{\int_0^T X_t \delta X_t}{\int_0^T X_t^2 dt}.
\]

In the ergodic case, that is, when \( \theta > 0 \), Hu and Nualart [6] proved that the LSE \( \widehat{\theta}_T \) of \( \theta \) is strongly consistent and asymptotically normal. In addition, they also proved that the estimator

\[
\overline{\theta}_T = \left( \frac{1}{HT(2H)} \int_0^T X_t^2 dt \right)^{-1/(2H)}
\]

is strongly consistent and asymptotically normal. In the non-ergodic case \( \theta < 0 \), Belfadli et al. [11] established that the LSE \( \theta_T \) of \( \theta \) is strongly consistent and asymptotically Cauchy.

In this paper, we focus our discussion on the ergodic case \( \theta > 0 \). We shall prove that when \( H \in (1/2, 3/4) \), the sequence \( \{\sqrt{n}(\theta - \hat{\theta}_n)\}_{n \geq 1} \) satisfies the ASCLT (see Theorem 4.2).

**Discrete observations.** Assume that the process \( X \) is observed equidistantly in time with the step size \( h > 0 \), that is, for any \( i \in \{0, \ldots, n\} \), \( t_i = ih \). Hu and Song [7], motivated by the estimator \( \overline{\theta}_T \), proved that the estimator

\[
\tilde{\theta}_n = \left( \frac{1}{HT(2H)n} \sum_{i=1}^n X_{t_i}^2 \right)^{-1/(2H)}
\]

is strongly consistent and asymptotically normal.
In the present work, we shall also prove that, in the case when $H \in (1/2, 3/4)$, the sequence

$$\left\{ \frac{\sqrt{n}}{\sigma(H, \theta)} (\theta - \tilde{\theta}_n) \right\}_{n \geq 1}$$

satisfies the ASCLT (see Theorem 4.4).

The paper is organized as follows. Section 2 contains the basic tools of Malliavin calculus for the fractional Brownian motion needed throughout the paper. In Section 3 we prove the ASCLT for a sequence of random variables having the form of a ratio of two terms such that the numerator satisfies the ASCLT and the denominator is a positive term which converges almost surely to one. In Section 4, we use our ASCLT to study the ASCLT for the estimators $\tilde{\theta}_n$ and $\hat{\theta}_n$.

2. PRELIMINARIES

In this section we describe some basic facts on the stochastic calculus with respect to a fractional Brownian motion. For more complete presentation on the subject, see Nualart [14].

The fractional Brownian motion $\{B_t, t \geq 0\}$ with Hurst parameter $H \in (0, 1)$ is defined as a centered Gaussian process starting from zero with covariance

$$R_H(t, s) := \mathbb{E}(B_t B_s) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}).$$

Assume that $B$ is defined on a complete probability space $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F}$ is the sigma-field generated by $B$. By Kolmogorov’s continuity criterion and the equality

$$\mathbb{E} (B_t - B_s)^2 = |t - s|^{2H}, \quad s, t \geq 0,$$

$B$ has Hölder continuous paths of order $H - \varepsilon$ for all $\varepsilon \in (0, H)$.

Fix a time interval $[0, T]$. We denote by $\mathcal{H}$ the canonical Hilbert space associated with the fractional Brownian motion $B$. That is, $\mathcal{H}$ is the closure of the linear span $\mathcal{E}$ generated by the indicator functions $\mathbbm{1}_{(0, t]}$, $t \in [0, T]$, with respect to the scalar product

$$\langle \mathbbm{1}_{(0, t]}, \mathbbm{1}_{(0, s]} \rangle = R_H(t, s).$$

We denote by $| \cdot |_{\mathcal{H}}$ the associated norm. The mapping $\mathbbm{1}_{(0, t]} \mapsto B_t$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space associated with $B$. We denote this isometry by

$$\varphi \mapsto B(\varphi) = \int_0^T \varphi(s) dB_s.$$

When $H > \frac{1}{2}$, the elements of $\mathcal{H}$ may be not functions but distributions of negative order (see Pipiras and Taqqu [15]). Therefore, it is of interest to know significant subspaces of functions contained in it.
Let $|\mathcal{H}|$ be the set of measurable functions $\varphi$ on $[0, T]$ such that
\[
\|\varphi\|_{|\mathcal{H}|}^2 := H(2H - 1) \int_0^T \int_0^T |\varphi(u)||\varphi(v)||u - v|^{2H-2} du dv < \infty.
\]
Note that if $\varphi, \psi \in |\mathcal{H}|$, then
\[
\mathbb{E}(B(\varphi)B(\psi)) = H(2H - 1) \int_0^T \int_0^T \varphi(u)\psi(v)|u - v|^{2H-2} du dv.
\]
It follows actually from Pipiras and Taqqu [15] that the space $|\mathcal{H}|$ is a Banach space for the norm $\| \cdot \|_{|\mathcal{H}|}$ and it is included in $\mathcal{H}$. Moreover, one has
\[
L^2([0, T]) \subset L^1([0, T]) \subset |\mathcal{H}| \subset H.
\]
Let $C^\infty_b(\mathbb{R}^n, \mathbb{R})$ be the class of infinitely differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $f$ and all its partial derivatives are bounded. We denote by $\mathcal{S}$ the class of cylindrical random variables $F$ of the form
\[
F = f(B(\varphi_1), \ldots, B(\varphi_n)),
\]
where $n \geq 1$, $f \in C^\infty_b(\mathbb{R}^n, \mathbb{R})$ and $\varphi_1, \ldots, \varphi_n \in \mathcal{H}$. The derivative operator $D$ of a cylindrical random variable $F$ of the form (2.2) is defined as the $\mathcal{H}$-valued random variable
\[
D_t F = \sum_{i=1}^N \frac{\partial f}{\partial x_i}(B(\varphi_1), \ldots, B(\varphi_n))\varphi_i(t).
\]
In this way the derivative $DF$ is an element of $L^2(\Omega; \mathcal{H})$. For $p \geq 1$, let $D^{1,p}$ be the closure of $\mathcal{S}$ with respect to the norm defined by
\[
\|F\|_{1,p}^p := \mathbb{E}(\|F\|^p) + \mathbb{E}(\|DF\|_{\mathcal{H}}^p).
\]
The divergence operator $\delta$ is the adjoint of the derivative operator $D$. Concretely, a random variable $u \in L^2(\Omega; \mathcal{H})$ belongs to the domain of the divergence operator $\text{Dom}(\delta)$ if, for every $F \in \mathcal{S},$
\[
\mathbb{E} |\langle DF, u \rangle_{\mathcal{H}}| \leq c\|F\|_{L^2(\Omega)}.
\]
In this case $\delta(u)$ is given by the duality relationship
\[
\mathbb{E}(F\delta(u)) = \mathbb{E} \langle DF, u \rangle_{\mathcal{H}}
\]
for any $F \in D^{1,2}$. We will make use of the notation
\[
\delta(u) = \int_0^T u_s dB_s, \quad u \in \text{Dom}(\delta).
\]
In particular, for $h \in \mathcal{H}$, $B(h) = \delta(h) = \int_0^T h_s dB_s$. 
For every $n \geq 1$, let $\mathcal{H}_n$ be the $n$-th Wiener chaos of $B$, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables \( \{ H_n(B(h)), h \in \mathcal{H}, \|h\|_p = 1 \} \), where $H_n$ is the $n$-th Hermite polynomial. The mapping \( I_n(h \otimes^n) = n!H_n(B(h)) \) provides a linear isometry between the symmetric tensor product $\mathcal{H} \otimes^n$ (equipped with the modified norm $\| \cdot \|_{\mathcal{H} \otimes^n} = \frac{1}{\sqrt{n!}} \| \cdot \|_{\mathcal{H}^\otimes n}$) and $\mathcal{H}_n$. For every $f, g \in \mathcal{H} \otimes^n$ the following multiplication formula holds:

\[
\mathbb{E}(I_n(f)I_n(g)) = n!\langle f, g \rangle_{\mathcal{H}^\otimes n}.
\]

On the other hand, it is well known that $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathcal{H}_n$. That is, any square integrable random variable $F \in \mathbb{L}^2(\Omega)$ admits the following chaotic expansion:

\[
F = \mathbb{E}(F) + \sum_{n=1}^{\infty} I_n(f_n),
\]

where $f_n \in \mathcal{H} \otimes^n$ are uniquely determined by $F$.

Let \( \{ e_n, n \geq 1 \} \) be a complete orthonormal system in $\mathcal{H}$. Given $f \in \mathcal{H} \otimes^p$ and $g \in \mathcal{H} \otimes^q$, for every $r = 0, \ldots, p \wedge q$, the $r$-th contraction of $f$ and $g$ is the element of $\mathcal{H} \otimes (p+q-2r)$ defined as

\[
f \otimes_r g = \sum_{i_1, \ldots, i_r=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathcal{H} \otimes^r} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathcal{H} \otimes^r}.
\]

In particular, note that $f \otimes_0 g = f \otimes g$ and, when $p = q$, $f \otimes_p g = \langle f, g \rangle_{\mathcal{H} \otimes^p}$. Since, in general, the contraction $f \otimes_r g$ is not necessarily symmetric, we denote its symmetrization by $f \tilde{\otimes}_r g \in \mathcal{H} \otimes (p+q-2r)$. When $f \in \mathcal{H} \otimes^q$, we write $I_q(f)$ to indicate its $q$-th multiple integral with respect to $X$. The following formula is useful to compute the product of such multiple integrals: if $f \in \mathcal{H} \otimes^p$ and $g \in \mathcal{H} \otimes^q$, then

\[
I_p(f)I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \tilde{\otimes}_r g).
\]

Let us now recall the criterion of Ibragimov and Lifshits [9], which plays a crucial role in Bercu et al. [2] to study ASCLTs for sequences of functionals of general Gaussian fields.

**Theorem 2.1 (Ibragimov and Lifshits [9])**. Let \( \{ G_n \} \) be a sequence of random variables converging in distribution towards a random variable $G_\infty$, and set

\[
\Delta_n(t) = \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} (e^{itG_k} - \mathbb{E}(e^{itG_\infty})).
\]
Assume that, for all \( r > 0 \),
\[
\sup_{|t| \leq r} n \sum_{n} \frac{E|\Delta_n(t)|^2}{n \log n} < \infty.
\]
Then, almost surely, for all continuous and bounded functions \( \varphi : \mathbb{R} \to \mathbb{R} \), one has
\[
\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \varphi(G_k) \xrightarrow{a.s. \ n \to \infty} \mathbb{E}(\varphi(G_\infty)).
\]

For the rest of the paper, we will use the standard notation \( \phi(z) := P(N \leq z) \), where \( N \) is an \( \mathcal{N}(0, 1) \) random variable. We will denote by \( C(\theta, H) \) a generic positive constant which depends only on \( \theta \) and \( H \).

3. ALMOST SURE CENTRAL LIMIT THEOREMS

In this section we shall state and prove our results concerning the ASCLT for the sequences of \( \mathbb{R} \)-valued random variables of the form \( \{G_n/R_n\}_{n \geq 1} \) and \( \{G_n + R_n\}_{n \geq 1} \).

**Theorem 3.1.** Let \( \{G_n\}_{n \geq 1} \) be a sequence of \( \mathbb{R} \)-valued random variables satisfying the ASCLT. Let \( \{R_n\}_{n \geq 1} \) be a sequence of positive random variables converging almost surely to one. Then \( \{G_n/R_n\}_{n \geq 1} \) satisfies the ASCLT. In other words, if \( N \) is an \( \mathcal{N}(0, 1) \) random variable, then, almost surely, for all \( z \in \mathbb{R} \),
\[
\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}_{\{G_k \leq zR_k\}} \xrightarrow{a.s. \ n \to \infty} \phi(z).
\]

**Theorem 3.2.** Let \( \{G_n\}_{n \geq 1} \) be a sequence of \( \mathbb{R} \)-valued random variables satisfying the ASCLT. Let \( \{R_n\}_{n \geq 1} \) be a sequence of \( \mathbb{R} \)-valued random variables converging almost surely to zero. Then \( \{G_n + R_n\}_{n \geq 1} \) satisfies the ASCLT. In other words, almost surely, for all \( z \in \mathbb{R} \),
\[
\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}_{\{G_k + R_k \leq z\}} \xrightarrow{a.s. \ n \to \infty} \phi(z).
\]

**Remark 3.1.** A similar result to Theorem 3.2 for the ASCLT of the sequence \( \{G_n + R_n\}_{n \geq 1} \), where \( \{R_n\}_{n \geq 1} \) converges in \( L^2(\Omega) \) to zero, and such that
\[
\sum_{n=2}^{\infty} \frac{1}{n \log^2 n} \sum_{k=1}^{n} \mathbb{E}|R_k|^2 < \infty
\]
was established by Nourdin and Peccati in [13].

The proofs of Theorems 3.1 and 3.2 are respectively direct consequences of the following two lemmas:
LEMMA 3.1. Let \( \{G_n\}_{n \geq 1} \) and \( \{R_n\}_{n \geq 1} \) be two sequences of real-valued random variables. Define

\[
U_{n, \varepsilon} := \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}_{\{G_k \leq z(1-\varepsilon)\}} - \phi(z(1-\varepsilon)),
\]

\[
V_{n, \varepsilon} := \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}_{\{G_k \leq z(1+\varepsilon)\}} - \phi(z(1+\varepsilon)).
\]

Then, for all \( z \in \mathbb{R} \) and \( \varepsilon > 0 \),

\[
\left| \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}_{\{G_k \leq zR_k\}} - \phi(z) \right| \leq \max(U_{n, \varepsilon}, V_{n, \varepsilon}) + \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}_{\{|R_k-1|\geq \varepsilon\}} + \varepsilon.
\]

LEMMA 3.2. Let \( \{S_n\}_{n \geq 1} \) and \( \{R_n\}_{n \geq 1} \) be two sequences of real-valued random variables. Define

\[
T_{n, \eta} := \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}_{\{G_k \leq z+\eta\}} - \phi(z + \eta),
\]

\[
W_{n, \eta} := \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}_{\{G_k \leq z-\eta\}} - \phi(z - \eta).
\]

Then, for all \( z \in \mathbb{R} \) and \( \eta > 0 \),

\[
\left| \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}_{\{G_k + R_k \leq z\}} - \phi(z) \right| \leq \max(T_{n, \eta}, W_{n, \eta}) + \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}_{\{|R_k| \geq \eta\}} + \frac{\eta}{\sqrt{2\pi}}.
\]

Proof of Lemma 3.1. It is inspired by Lemma 1 from Michael and Pfanzagl [12], p. 78. The case \( \varepsilon \geq 1 \) is easy. We now assume that \( \varepsilon \in (0, 1) \). When \( z \geq 0 \), using the inclusion

\[
\{G_k \leq (1-\varepsilon)z\} \subset \{G_k \leq zR_k\} \cup \{R_k \leq 1 - \varepsilon\},
\]

we have

\[
\mathbb{I}_{\{G_k \leq z(1-\varepsilon)\}} \leq \mathbb{I}_{\{G_k \leq zR_k\}} + \mathbb{I}_{\{|R_k-1| \geq \varepsilon\}}.
\]

Since, for every \( x \geq 0 \), \( xe^{-x^2/2} \leq e^{-1/2} \), we get

\[
\left| \phi(z) - \phi(z(1-\varepsilon)) \right| \leq \min \left( \frac{1}{2}, \frac{z\varepsilon}{\sqrt{2\pi}} \exp \left( -\frac{z^2(1-\varepsilon)^2}{2} \right) \right) \leq \varepsilon.
\]
Combining (3.3) and (3.6), we obtain
\[ \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}_{\{G_k \leq \varepsilon R_k\}} - \phi(z) \geq -U_{n,\varepsilon} - \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}_{\{|R_k-1|\geq \varepsilon\}} - \varepsilon. \]

Now, when \( z \leq 0 \), the inclusion \( \{G_k \leq (1 + \varepsilon)z\} \subset \{G_k \leq zR_k\} \cup \{R_k \geq 1 + \varepsilon\} \) leads to
\[ \mathbb{I}_{\{G_k \leq z(1+\varepsilon)\}} \leq \mathbb{I}_{\{G_k \leq zR_k\}} + \mathbb{I}_{\{|R_k-1|\geq \varepsilon\}}. \]

Moreover, since
\[ |\phi(z) - \phi(z(1+\varepsilon))| \leq \frac{|z|\varepsilon}{\sqrt{2\pi}} \exp\left(\frac{-z^2(1+\varepsilon)^2}{2}\right) \leq \varepsilon, \]
we have
\[ \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}_{\{G_k \leq zR_k\}} - \phi(z) \geq -V_{n,\varepsilon} - \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}_{\{|R_k-1|\geq \varepsilon\}} - \varepsilon. \]

Thus, for every \( z \in \mathbb{R} \),
\[ \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}_{\{G_k \leq zR_k\}} - \phi(z) \geq -\max(U_{n,\varepsilon}, V_{n,\varepsilon}) - \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}_{\{|R_k-1|\geq \varepsilon\}} - \varepsilon. \]

Following the same guidelines as above and using
\[
\begin{align*}
\{G_k \leq zR_k\} &\subset \{G_k \leq (1 + \varepsilon)z\} \cup \{R_k \geq 1 + \varepsilon\} \quad \text{for } z \geq 0, \\
\{G_k \leq zR_k\} &\subset \{G_k \leq (1 - \varepsilon)z\} \cup \{R_k \leq 1 - \varepsilon\} \quad \text{for } z \leq 0
\end{align*}
\]
we get, for every \( z \in \mathbb{R} \),
\[ \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}_{\{G_k \leq zR_k\}} - \phi(z) \leq \max(U_{n,\varepsilon}, V_{n,\varepsilon}) + \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}_{\{|R_k-1|\geq \varepsilon\}} + \varepsilon. \]

This completes the proof of Lemma 3.2. ■

**Proof of Lemma 3.2.** Fix \( z \in \mathbb{R} \) and \( \eta > 0 \). Remark that
\[
\{G_k + R_k \leq z\} \subset \{G_k \leq z + \eta\} \cup \{|R_k| > \eta\}.
\]

Thus we obtain
\[
\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}_{\{G_k + R_k \leq z\}} - \phi(z)
\leq \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}_{\{G_k \leq z + \eta\}} - \phi(z + \eta)
+ \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}_{\{|R_k| > \eta\}} + \phi(z + \eta) - \phi(z)
\leq T_{n,\eta} + \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}_{\{|R_k| > \eta\}} + \frac{\eta}{\sqrt{2\pi}}.
\]
On the other hand, it follows from the inclusion
\[ \{G_k \leq z - \eta\} \subset \{G_k + R_k \leq z\} \cup \{|R_k| > \eta\} \]
that
\[
\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{1}_{\{G_k + R_k \leq z\}} - \phi(z) \\
\geq \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{1}_{\{G_k \leq z + \eta\}} - \phi(z - \eta) - \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{1}_{\{|R_k| > \eta\}} + \phi(z - \eta) - \phi(z) \\
\geq -W_{n,\eta} - \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{1}_{\{|R_k| > \eta\}} - \frac{\eta}{\sqrt{2\pi}}.
\]

The desired conclusion follows. \( \blacksquare \)

## 4. APPLICATION TO LSE FOR FRACTIONAL ORNSTEIN–UHLENBECK PROCESS

First we recall a result of \cite{2} concerning the ASCLT for multiple stochastic integrals.

**Theorem 4.1 (Bercu et al. \cite{2}).** Let \( q \geq 2 \) be an integer and let \( \{G_n\}_{n \geq 1} \) be a sequence of the form \( G_n = I_q(f_n) \) with \( f_n \in \mathcal{H}^{\otimes q} \). Assume that \( \mathbb{E}(G_n^2) = q! \|f_n\|_{\mathcal{H}^{\otimes q}}^2 = 1 \) for all \( n \) and that \( G_n \) converges in distribution towards a standard Gaussian. Moreover, assume that

\[
\sum_{n=2}^{\infty} \frac{1}{n \log n} \sum_{k=1}^{n} \frac{1}{k} \|f_k \otimes_r f_k\|_{\mathcal{H}^{\otimes (q-r)}} < \infty \quad \text{for every} \quad 1 \leq r \leq q - 1, \tag{4.1}
\]

\[
\sum_{n=2}^{\infty} \frac{1}{n \log n} \sum_{k,l=1}^{n} \frac{|\langle f_k, f_l \rangle_{\mathcal{H}^{\otimes q}}|}{kl} < \infty. \tag{4.2}
\]

Then \( \{G_n\}_{n \geq 1} \) satisfies an ASCLT. In other words, almost surely, for all \( z \in \mathbb{R} \),

\[
\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{1}_{\{G_k \leq z\}} \xrightarrow{\text{a.s.}} \frac{\phi(z)}{n \to \infty}
\]

or, equivalently, almost surely, for any bounded and continuous function \( \varphi : \mathbb{R} \to \mathbb{R} \), we have

\[
\frac{1}{\log(n)} \sum_{k=1}^{n} \frac{1}{k} \varphi(G_k) \xrightarrow{\text{a.s.}} \mathbb{E}\varphi(N).
\]

### 4.1. Continuous case

In this section we apply Theorem 4.1 to a least squares estimator for fractional Ornstein–Uhlenbeck processes based on continuous-time observations.
Let us consider the fractional Ornstein–Uhlenbeck process $X = \{X_t, t \geq 0\}$ given by the linear stochastic differential equation

\begin{equation}
X_0 = 0 \quad \text{and} \quad dX_t = -\theta X_t dt + dB_t, \quad t \geq 0,
\end{equation}

where $B = \{B_t, t \geq 0\}$ is a fractional Brownian motion of Hurst index $H \in \left(\frac{1}{2}, 1\right)$ and $\theta$ is a real unknown parameter. Let $\hat{\theta}_t$ be a least squares estimator (LSE) of $\theta$ given by

\begin{equation}
\hat{\theta}_t = -\frac{\int_0^t X_s \delta X_s}{\int_0^t X_s^2 ds}, \quad t > 0.
\end{equation}

This LSE is obtained by the least squares technique, that is, $\hat{\theta}_t$ (formally) minimizes

$\theta \mapsto \int_0^t \left(\hat{X}_s + \theta X_s\right)^2 ds$.

The linear equation (4.3) has the following explicit solution:

\begin{equation}
X_t = e^{-\theta t} \int_0^t e^{\theta s} dB_s, \quad t > 0.
\end{equation}

Using the equations (4.3) and (4.5) we can write the LSE $\{\hat{\theta}_t\}$ defined in (4.4) as follows:

\begin{equation}
\hat{\theta}_t - \theta = -\frac{\int_0^t X_s \delta B_s}{\int_0^t X_s^2 ds} = -\frac{\int_0^t \delta B_s e^{\theta s} \int_0^s \delta B_r e^{-\theta r} ds}{\int_0^t X_s^2 ds}.
\end{equation}

Thus, we have

\begin{equation}
\sqrt{t}(\theta - \hat{\theta}_t) = \frac{F_t}{t^{-1} \int_0^t X_s^2 ds}, \quad t > 0,
\end{equation}

where

$F_t := I_2(f_t)$

is a multiple integral of $f_t$ with

$f_t(u, v) = \frac{1}{2\sqrt{t}} e^{-\theta |u - v|} \mathbb{I}_{\{[0, t]\}}(u, v)$. 


Until the end of this paper we will use the following notation for all \( t > 0 \):

\[
\sigma_t = \lambda(\theta, H) \sqrt{\mathbb{E}(F_t^2)} \quad \text{with} \quad \lambda(\theta, H) := \theta^{-2H} H \Gamma(2H). \tag{4.8}
\]

We are now ready to state the main result of this subsection. First we recall some results of Hu and Nualart \([6]\) needed throughout the paper:

\[
\mathbb{E}(F_t^2) \xrightarrow{t \to \infty} A(\theta, H), \tag{4.9}
\]

where

\[
A(\theta, H) = \theta^{1-4H} \left( H^2(4H - 1) \left[ \Gamma(2H)^2 + \frac{\Gamma(2H)\Gamma(3 - 4H)\Gamma(4H - 1)}{\Gamma(2 - 2H)} \right] \right).
\]

Moreover, for every \( t \geq 0 \)

\[
\mathbb{E}[(||DF_t||_{\mathcal{H}}^2 - \mathbb{E}[||DF_t||_{\mathcal{H}}^2])^2] \leq C(\theta, H)t^{8H-6}, \tag{4.10}
\]

and as \( t \to \infty \)

\[
F_t \xrightarrow{d} N \sim N(0, A(\theta, H)) \tag{4.11}
\]

(where \( d \to \) means convergence in distribution). At last, we have the convergence

\[
\frac{1}{t} \int_0^t X_s^2 ds \xrightarrow{a.s. \quad t \to \infty} \lambda(\theta, H) \tag{4.12}
\]

as \( t \to \infty \).

**Theorem 4.2.** Assume \( H \in (1/2, 3/4) \). Then, almost surely, for all \( z \in \mathbb{R} \),

\[
\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \mathbb{1}_{\{((\sqrt{\mathbb{E}/\sigma_k})/(\theta - \tilde{\theta}_k)) \leq z\}} \xrightarrow{n \to \infty} \phi(z)
\]

or, equivalently, almost surely, for any bounded and continuous function \( \varphi \)

\[
\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \varphi \left( \frac{\sqrt{\mathbb{E}}}{\sigma_k} (\theta - \tilde{\theta}_k) \right) \xrightarrow{n \to \infty} \mathbb{E}(\varphi(N))
\]

**Proof.** Let us consider, for each \( t > 0 \),

\[
G_t = \frac{1}{\sqrt{\mathbb{E}(F_t^2)}} F_t = \frac{1}{\sqrt{\mathbb{E}(F_t^2)}} I_2(f_t)
\]

and

\[
R_t = \frac{1}{\lambda(\theta, H)t} \int_0^t X_s^2 ds.
\]
Thus, (4.7) leads to
\[ \sqrt{n} \frac{\theta - \hat{\theta}_n}{\sigma_n} = G_n/R_n, \quad n \geq 1. \]

It follows from (4.12) that \( R_n \) converges almost surely to one as \( n \) tends to infinity. Then, using Theorem 3.1 it suffices to show that \( \{G_n\}_{n \geq 1} \) satisfies the ASCLT. To do that, it is sufficient to prove that \( \{G_n\}_{n \geq 1} \) satisfies the conditions of Theorem 4.1.

We have \( E(G_n^2) = 1 \). In addition, the convergence of \( G_n \) towards the standard Gaussian is a straightforward consequence of (4.9) and (4.11). It remains to fulfill the conditions (4.1) and (4.2). Hence, we shall prove that
\[
I = \sum_{n \geq 2} \frac{1}{n \log^2(n)} \sum_{k=1}^{n} \frac{1}{k} \|f_k \otimes_1 f_k\|_{\mathcal{H}^\otimes 2} < \infty,
\]
and
\[
J = \sum_{n \geq 2} \frac{1}{n \log^2(n)} \sum_{k,l=1}^{n} \frac{|\langle f_k, f_l \rangle_{\mathcal{H}^\otimes 2}|}{kl} < \infty.
\]

Let us deal with the first convergence (4.13). For every \( t > 0 \), we have
\[
E[(\|DF_t\|_{\mathcal{H}}^2 - E\|DF_t\|_{\mathcal{H}}^2)^2] = 16 \|f_t \otimes_1 f_t\|_{\mathcal{H}^\otimes 2}^2,
\]
Combining (4.10) and (4.15) we obtain
\[
I \leq C(\theta, H) \sum_{n \geq 2} \frac{1}{n \log^2(n)} \sum_{k=1}^{n} \frac{1}{k^{4-3H}},
\]
and, consequently,
\[
I \leq C(\theta, H) \sum_{n \geq 2} \frac{1}{n^{4-3H}} < \infty,
\]
since \( H < 3/4 \), where \( C(\theta, H) \) is a generic constant depending only on \( \theta, H \).

Now, we prove (4.14). Let \( k < l \). Then for some \( k^* \in [0, k] \) we have
\[
|\langle f_k, f_l \rangle_{\mathcal{H}}| = H^2 (2H - 1)^2 \frac{1}{\sqrt{kl}}
\times \int_{[0,k]^2} dx du e^{-\theta|\mathbf{x} - \mathbf{u}|} \int_{[0,l]^2} dy dv e^{-\theta|y - v|} \cdot |x - y|^{2H-2} |u - v|^{2H-2}
\]
\[
= 2H^2 (2H - 1)^2 \sqrt{\frac{k}{l}} \int_{[0,k^*]} du e^{-\theta|k^* - u|}
\times \int_{[0,l]^2} dy dv e^{-\theta|y - v|} \cdot |k^* - y|^{2H-2} |u - v|^{2H-2}
\]
\[
:= 2H^2 (2H - 1)^2 \sqrt{\frac{k}{l}} (D^{(1)} + D^{(2)} + D^{(3)} + D^{(4)}),
\]
Moreover, the first term can be bounded above by

\[
D^{(1)} = \int_{[0,k^*]} du e^{-\theta(k^*-u)} \int_{[0,k^*]^2} dydv e^{-\theta|y-v|} (k^* - y)^{2H-2} |u - v|^{2H-2} \\
= \int_{[0,k^*]^3} e^{-\theta u} e^{-\theta|y-v|} y^{2H-2} |u - v|^{2H-2} du dv dy \\
\leq \int_{[0,\infty)^3} e^{-\theta u} e^{-\theta|y-v|} y^{2H-2} |u - v|^{2H-2} du dv dy < \infty.
\]

The last inequality is a consequence of the proof of Lemma 5.3 (see only web Appendix) in [3]. Following the same guidelines, we get for the other terms:

\[
D^{(2)} = \int_{[0,k^*]} du e^{-\theta(k^*-u)} \int_{[k^*,l]^2} dy dv e^{-\theta|y-v|} (y - k^*)^{2H-2} |u - v|^{2H-2} \\
= \int_{[0,k^*,l]^3} e^{-\theta u} e^{-\theta|y-v|} y^{2H-2} (u + v)^{2H-2} du dv dy \\
\leq \int_{[0,\infty)^3} e^{-\theta u} e^{-\theta|y-v|} y^{2H-2} |u - v|^{2H-2} du dv dy < \infty,
\]

\[
D^{(3)} = \int_{[0,k^*]} du e^{-\theta(k^*-u)} \int_{[0,k^*,l]} dy dv e^{-\theta|y-v|} (k^* - y)^{2H-2} |u - v|^{2H-2} \\
= \int_{[0,k^*,l]^3} e^{-\theta u} e^{-\theta|y+v|} y^{2H-2} (u + v)^{2H-2} du dv dy \\
\leq \int_{[0,\infty)^3} e^{-\theta u} e^{-\theta|y-v|} y^{2H-2} |u - v|^{2H-2} du dv dy < \infty,
\]

and

\[
D^{(4)} = \int_{[0,k^*]} du e^{-\theta(k^*-u)} \int_{[k^*,l]} dy dv e^{-\theta|y-v|} (y - k^*)^{2H-2} |u - v|^{2H-2} \\
= \int_{[0,k^*,l]^3} e^{-\theta u} e^{-\theta|y+v|} y^{2H-2} |u - v|^{2H-2} du dv dy \\
\leq \int_{[0,\infty)^3} e^{-\theta u} e^{-\theta|y-v|} y^{2H-2} |u - v|^{2H-2} du dv dy < \infty.
\]

Thus, we deduce that, for every \(k < l\),

\[
|\langle f_k, f_l \rangle| = C(\theta, H) \sqrt{\frac{k}{l}}.
\]
Consequently, we obtain

\begin{align}
J & \leq C(\theta, H) \sum_{n \geq 2} \frac{1}{n} \log^3(n) \sum_{l=1}^{n} \frac{1}{l^{3/2}} \sum_{k=1}^{l} \frac{1}{\sqrt{k}} \\
& \leq C(\theta, H) \sum_{n \geq 2} \frac{1}{n} \log^3(n) \sum_{l=1}^{n} \frac{1}{l} \\
& \leq C(\theta, H) \sum_{n \geq 2} \frac{1}{n} \log^2(n) < \infty,
\end{align}

which concludes the proof. ■

4.2. Discrete case. Consider the fractional Ornstein–Uhlenbeck process \( X = \{X_t, t \geq 0\} \) defined in (4.3). Assume that the process \( X \) is observed equidistantly in time with the step size \( h > 0 \): \( t_i = ih, i = 0, \ldots, n \).

**Theorem 4.3.** Assume \( H \in (1/2, 3/4) \). Let \( \tilde{\theta}_n \) be the estimator of \( \theta \) defined in (1.2). Then, almost surely, for all \( z \in \mathbb{R} \),

\[
\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{1}_{((\sqrt{n}/\sigma(H,\theta))(\theta - \tilde{\theta}_k) \leq z)} \xrightarrow{a.s.} \phi(z),
\]

or, equivalently, for any bounded and continuous function \( \varphi \),

\[
\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \varphi\left(\frac{\sqrt{n}}{\sigma(H,\theta)}(\theta - \tilde{\theta}_k)\right) \xrightarrow{a.s.} \mathbb{E}(\varphi(N)),
\]

where \( \sigma(H, \theta) > 0 \) is a constant depending only on \( H \) and \( \theta \).

**Proof.** Setting

\[
Q_n := \frac{1}{n} \sum_{i=1}^{n} X_{i-1}^2,
\]

we can write

\begin{align}
\tilde{\theta}_n &= \left(\frac{Q_n}{HT(2H)}\right)^{-1/(2H)}, \\
\hat{\theta}_n &\xrightarrow{a.s.} \theta
\end{align}

Let us recall that (see [7]), as \( n \to \infty \),

\begin{align}
\tilde{\theta}_n &\xrightarrow{a.s.} \theta \\
\frac{\sqrt{n}}{\sigma(H, \theta)}(\theta - \tilde{\theta}_n) &\xrightarrow{d} \mathcal{N}(0, 1).
\end{align}
We have

\[
\frac{\sqrt{n}}{\sigma(H, \theta)}(\theta - \bar{\theta}_n) = \xi_n^{-1/(2H) - 1}\frac{\sqrt{n}}{2H \sigma(H, \theta)} \left( \frac{Q_n}{\Gamma(2H)} - \theta^{-2H} \right),
\]

where \( \xi_n \) is a random variable between \( Q_n/(\Gamma(2H)) \) and \( \theta^{-2H} \). The convergence (4.20) leads to \( \theta^{-2H-1}\xi_n^{-1/(2H)} \to 1 \) almost surely as \( n \to \infty \). Then, using Theorem 3.1 it suffices to show that

\[
\left\{ \frac{\theta^{2H+1}\sqrt{n}}{2H \sigma(H, \theta)} \left( \frac{Q_n}{\Gamma(2H)} - \theta^{-2H} \right) \right\}_{n \geq 1}
\]

satisfies the ASCLT. On the other hand,

\[
\frac{\theta^{2H+1}\sqrt{n}}{2H \sigma(H, \theta)} \left( \frac{Q_n - \mathbb{E}Q_n}{\Gamma(2H)} - \theta^{-2H} \right) = G_n + R_n,
\]

where

\[
G_n = \frac{\theta^{2H+1}\sqrt{n}}{2H \sigma(H, \theta)} \left( \frac{Q_n - \mathbb{E}Q_n}{\Gamma(2H)} \right) \in \mathcal{H}_2,
\]

and from (7) it follows that

\[
R_n = \frac{\theta^{2H+1}\sqrt{n}}{2H \sigma(H, \theta)} \left( \frac{\mathbb{E}Q_n}{\Gamma(2H)} - \theta^{-2H} \right)
\]

converges to zero as \( n \to \infty \). Hence, using Theorem 3.2 it remains to prove that \( \{G_n\}_{n \geq 1} \) satisfies the ASCLT. The conditions (4.1) and (4.2) are satisfied by using the following estimates inspired by Hu and Song (7):

\[
\mathbb{E}[\|D\mathcal{G}_n\|_H^2] \leq \mathbb{E}[\|D\mathcal{G}_n\|_H^2] \leq C(\theta, H) \frac{1}{n^{8H-6}},
\]

and for all \( k \leq l \)

\[
|\mathbb{E}[G_k G_l]| \leq C(\theta, H) \sqrt{\frac{k}{l}}.
\]

Thus the proof of Theorem 4.3 is completed. 

REFERENCES


