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A FUNCTIONAL LIMIT THEOREM FOR LOCALLY PERTURBED RANDOM WALKS

BY

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Abstract. A particle moves randomly over the integer points of the real line. Jumps of the particle outside the membrane (a fixed "locally perturbating set") are i.i.d., have zero mean and finite variance, whereas jumps of the particle from the membrane have other distributions with finite means which may be different for different points of the membrane; furthermore, these jumps are mutually independent and independent of the jumps outside the membrane. Assuming that the particle cannot jump over the membrane, we prove that the weak scaling limit of the particle position is a skew Brownian motion with parameter $\gamma \in [-1, 1]$. The path of a skew Brownian motion is obtained by taking each excursion of a reflected Brownian motion, independently of the others, positive with probability $2^{-1}(1+\gamma)$ and negative with probability $2^{-1}(1-\gamma)$. To prove the weak convergence result, we give a new approach which is based on the martingale characterization of a skew Brownian motion. Among others, this enables us to provide the explicit formula for the parameter γ. In the previous articles, the explicit formulae for the parameter have only been obtained under the assumption that outside the membrane the particle performs unit jumps.

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1. INTRODUCTION AND MAIN RESULT

Denote by $D := D[0,\infty)$ the Skorokhod space of right-continuous real-valued functions which are defined on $[0,\infty)$ and have finite limits from the left at each positive point. We stipulate hereafter that $\Rightarrow$ denotes weak convergence of probability measures on D endowed with Skorokhod J_1-topology.

For $x \in \mathbb{R}$ and $(\xi_i)_{i\in\mathbb{N}}$ a sequence of independent identically distributed (i.i.d.) random variables which take integer values and have zero mean and finite variance $\sigma^2 > 0$, set

$$S(0) := x, \quad S(n) := x + \xi_1 + \ldots + \xi_n, \quad n \in \mathbb{N}.$$

Donsker's theorem states that

$$U_n \Rightarrow W, \quad n \to \infty, \tag{1.1}$$

where $U_n(\cdot) := \sigma^{-1} n^{-1/2} S([n\cdot])$ and $W := \big(W(t)\big)_{t\geqslant 0}$ is a Brownian motion. Like many other authors (see references below and [11]) we are interested to know how the presence of a local perturbation of $\big(S(n)\big)$ may influence (1.1).

To define a local perturbation, we need more notation. Fix any $m \in \mathbb{N}$ and set $A := \{-m, -m+1, \dots, m\}$. For $j \in A$, denote by η_j, $(\eta_{j,k})_{k\in\mathbb{N}}$ i.i.d. integer-valued random variables with distribution that may depend on j. It is assumed that the so-defined random variables are independent of (ξ_i) and that η_i and η_j are independent whenever $i \neq j$. For $x \in \mathbb{Z}$, define a random sequence $\big(X(n)\big)_{n\in\mathbb{N}_0}$ by the formulae

$$X(0) = x, \;\; X(n) = x + \sum_{k=1}^{n} \Big(\xi_k \, \mathbb{1}_{\{|X(k-1)|>m\}} + \sum_{|j|\leqslant m} \eta_{j,k} \, \mathbb{1}_{\{X(k)=j\}}\Big)$$

for $n \in \mathbb{N}$. Note that $\big(X(n)\big)_{n\in\mathbb{N}_0}$ is a homogeneous Markov chain with the transition probabilities

$$p_{ij} := \begin{cases} \mathbb{P}\{\xi = j - i\}, & |i| > m, \\ \mathbb{P}\{\eta_i = j\}, & |i| \leqslant m. \end{cases}$$

Assuming that the Markov chain $\big(X(n)\big)_{n\in\mathbb{N}_0}$ is irreducible[1], set

$$\alpha_0 := 0, \quad \alpha_k := \inf\{i > \alpha_{k-1} : X(i) \in A\}, \; k \in \mathbb{N},$$

and $Y(k) := X(\alpha_k)$, $k \in \mathbb{N}_0$. The sequence $\big(Y(k)\big)_{k\in\mathbb{N}}$ is an irreducible homogeneous Markov chain. Denote by $\pi := (\pi_i)_{i\in A}$ its unique stationary distribution. Note that $\pi_i > 0$ for all $i \in A$. In the sequel we shall use the standard notation: $\mathbb{E}_\pi(\cdot) := \sum_{i\in A} \pi_i \mathbb{E}\big(\cdot\,|Y(0) = i\big)$.

Recall that a *skew Brownian motion* $W_\beta := \big(W_\beta(t)\big)_{t\geqslant 0}$ with parameter $\beta \in [-1, 1]$ is a continuous Markov process with $W_\beta(0) = 0$ and the transition density

$$p_t(x, y) = \varphi_t(x - y) + \beta \operatorname{sign}(y) \varphi_t(|x| + |y|), \quad x, y \in \mathbb{R},$$

where $\varphi_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}$ is the density of the normal distribution with zero mean and variance t (see, e.g., [9]). The latter formula enables us to conclude that W_0, W_1 and W_{-1} have the same distributions as W, $|W|$ and $-|W|$, respectively.

[1] Here is a simple sufficient condition for irreducibility: $\mathbb{P}\{\xi_1 = 1\} > 0$, $\mathbb{P}\{\xi_1 = -1\} > 0$, $\mathbb{P}\{\eta_j = 1\} > 0$ and $\mathbb{P}\{\eta_j = -1\} > 0$ for all $j \in A$.

Our main result is the following.

THEOREM 1.1. *In addition to all the aforementioned conditions assume that* $\mathbb{E}|\eta_j| < \infty$ *for all* $j \in A$ *and that* $|\xi_1| \leqslant 2m+1$ *almost surely* (*a.s.*). *Then*

$$X_n \Rightarrow W_\gamma, \quad n \to \infty,$$

where

$$X_n(t) := \sigma^{-1} n^{-1/2} X([nt]) \quad \textit{and} \quad \gamma := \frac{\mathbb{E}_\pi\big(X(1) - X(\alpha_1)\big)}{\mathbb{E}_\pi|X(1) - X(\alpha_1)|}.$$

REMARK 1.1. *Since* x *in the definition of* $\big(X(k)\big)$ *is arbitrary*, *the theorem remains valid if we replace the set* A *with* $A - j = \{-m-j, \dots, m-j\}$ *for any* $j \in \mathbb{Z}$.

REMARK 1.2. *Since* $\mathbb{E}_\pi\big(X(1) - X(\alpha_1)\big) = \sum_{j \in A} \pi_j \mathbb{E}\eta_j$, *the condition* $\mathbb{E}\eta_j = 0$ *for all* $j \in A$ *ensures that the limit process in Theorem* 1.1 *is a Brownian motion.*

Now we review briefly some related papers. The case $A = \{0\}$, $1 - \mathbb{P}\{\eta_0 = -1\} = \mathbb{P}\{\eta_0 = 1\} = p \in [0,1]$, $\mathbb{P}\{\xi_1 = \pm 1\} = 1/2$ has received considerable attention, see [4], [7], [9], [17]. In [7] it is remarked (without proof) that if A and the distribution of ξ_1 are as above, whereas η_0 has an arbitrary distribution which is concentrated on integers and has a finite mean, then $\gamma = \mathbb{E}\eta_0/\mathbb{E}|\eta_0|$. To facilitate comparison of this equality to the formula for γ given in Theorem 1.1 we note that in the present situation the stationary distribution π is degenerate at zero. The paper [13] is concerned with the case when $A = \{0\}$, ξ_1 takes integer values (possibly more than two), has zero mean and finite variance, whereas the distribution of η_0 belongs to the domain of attraction of an α-stable distribution, $\alpha \in (0,1)$. The case when $m \in \mathbb{N}$ is arbitrary, $\mathbb{P}\{\xi_1 = \pm 1\} = 1/2$, and the variables η_j are a.s. bounded is investigated in [10], [12]. In [19] the author assumes that ξ_1 is a.s. bounded rather than having the two-point distribution. The articles [10] and [15] remove the assumption of a.s. boundedness of η_j, still assuming that the distribution of ξ_1 is a two-point one.

The rest of the paper is organized as follows. In Section 2.1 we discuss our approach (which seems to be new in the present context) which is based on the martingale characterization of a skew Brownian motion. With this being an essential ingredient, the proof of Theorem 1.1 is completed in Section 2.2. Some auxiliary results are proved in Section 3.

2. PROOF OF THEOREM 1.1

2.1. Decomposition of perturbed random walk. We shall use the following martingale characterization of a skew Brownian motion. Its proof can be found in [8], see also [18].

PROPOSITION 2.1. *Suppose that a couple* $(X, V) := (X_t, V_t)_{t \geqslant 0}$ *of continuous processes adapted to the filtration* $(\mathcal{F}_t)_{t \geqslant 0}$ *satisfies the following conditions*:

(1) $V(0) = 0$, V *is nondecreasing a.s.*;

(2) *processes* $\big(M^{\pm}(t)\big)_{t \geqslant 0}$ *defined by*

$$M^{\pm}(t) := X^{\pm}(t) - \frac{1 \pm \beta}{2} V_t, \quad t \geqslant 0,$$

are continuous martingales (*with respect to* $(\mathcal{F}_t)$) *with the predictable quadratic variations*

$$\langle M^{+} \rangle_t = \int_0^t \mathbb{1}_{\{X_s \geqslant 0\}} \, \mathrm{d}s, \quad \langle M^{-} \rangle_t = \int_0^t \mathbb{1}_{\{X_s \leqslant 0\}} \, \mathrm{d}s,$$

where $\beta \in [-1, 1]$, $X_t^{+} = X_t \vee 0$ *and* $X_t^{-} = X_t^{+} - X_t$.

Then X *is a skew Brownian motion with parameter* β.

To prove Theorem 1.1 we decompose the perturbed random walk $\big(X(n)\big)$ into the sum of three summands. Roughly speaking, these are given by the sums of jumps which are accumulated while $\big(X(n)\big)$ is staying in the sets (m, ∞), $(-\infty, -m)$ and $[-m, m]$, respectively. It turns out that the first two summands are martingales. Furthermore, their scaling limits are the martingales $M^{\pm}$ appearing in Proposition 2.1 (see Lemma 2.2 below). We analyze the third summand and its scaling limit in Lemma 2.2 and in Section 2.2.

For convenience we assume that $X(0) = 0$. The general case can be treated similarly. For $n \in \mathbb{N}_0$, let $\widetilde{X}^{\pm}(n)$ be the absolute value of $X(n)$ when $X(n)$ stays outside A, and zero otherwise, i.e., $\widetilde{X}^{\pm}(n) = \pm X(n) \, \mathbb{1}_{\{\pm X(n) > m\}}$. Further, we put $\tau_0^{\pm} = 0$ and

$$\sigma_k^{\pm} = \inf\{i > \tau_k^{\pm} : \pm X(i) > m\}, \quad \tau_{k+1}^{\pm} := \inf\{i > \sigma_k^{\pm} : \pm X(i) \leqslant m\}$$

for $k \in \mathbb{N}_0$. Note that $[\sigma_0^{+}, \tau_1^{+} - 1]$, $[\sigma_1^{+}, \tau_2^{+} - 1], \ldots$ are successive intervals of time in which $\big(X(n)\big)$ stays in (m, ∞) (or, in other words, $X(n) = X^{+}(n)$ for all n belonging to these intervals), and $[\tau_0^{+}, \sigma_0^{+} - 1]$, $[\tau_1^{+}, \sigma_1^{+} - 1], \ldots$ are successive intervals of time in which $\big(X(n)\big)$ stays in $(-\infty, m]$. The meaning of the quantities with minus superscript is similar.

The subsequent presentation is essentially based on the following equality:

$$\begin{aligned} \widetilde{X}^{\pm}(n) &= \pm \sum_{k=1}^{n} \mathbb{1}_{\{\pm X(k-1) > m\}} \, \xi_k \pm \sum_{i \geqslant 0} \big(X(\sigma_i^{\pm}) - X(\tau_i^{\pm})\big) \, \mathbb{1}_{\{\sigma_i^{\pm} \leqslant n\}} \\ &\quad \mp \sum_{i \geqslant 0} X(\tau_i^{\pm}) \, \mathbb{1}_{\{\tau_i^{\pm} \leqslant n < \sigma_i^{\pm}\}} \\ &=: M^{\pm}(n) + L^{\pm}(n) \mp \sum_{i \geqslant 0} X(\tau_i^{\pm}) \, \mathbb{1}_{\{\tau_i^{\pm} \leqslant n < \sigma_i^{\pm}\}}. \end{aligned} \tag{2.1}$$

The term $M^+(n)$ takes into account the excursions of $\big(X(n)\big)$ to the right of A. Namely, if $\sigma_j^+ \leqslant n < \sigma_{j+1}^+$, then

$$M^+(n) = (\xi_{\sigma_j^+ +1} + \ldots + \xi_{n\wedge\tau_{j+1}^+}) + \sum_{k=0}^{j-1} (\xi_{\sigma_k^+ +1} + \xi_{\sigma_k^+ +2} + \ldots + \xi_{\tau_{k+1}^+}).$$

The term $L^+(n)$ represents the cumulative effect of the excursions of $\big(X(n)\big)$ into $(-\infty, m]$. The last term in (2.1) equals $X(\tau_j^+)$ for $n = \tau_j^+, \ldots, \sigma_j^+ - 1$ and zero for $n = \sigma_j^+, \ldots, \tau_{j+1}^+ - 1$. Consequently, it is only needed to take care of the event $\{X(n) \in A\}$.

It is clear that $\widetilde{X}^+(\sigma_j^+) = X(\sigma_j^+)$ and $\widetilde{X}(\tau_j^+) = 0$ because $X(\sigma_j^+) > m$ and $X(\tau_j^+) \leqslant m$. To simplify understanding of formula (2.1), we shall now check that it gives the same answer. Indeed,

$$\begin{aligned}\widetilde{X}^+(\sigma_j^+) &= \sum_{k=1}^{\sigma_j^+} \mathbb{1}_{\{X(k-1)>m\}}\, \xi_k + \sum_{i\geqslant 0} \big(X(\sigma_i^+) - X(\tau_i^+)\big)\, \mathbb{1}_{\{\sigma_i^+ \leqslant \sigma_j^+\}} \\ &= \sum_{k=0}^{j-1} (\xi_{\sigma_k^+ +1} + \xi_{\sigma_k^+ +2} + \ldots + \xi_{\tau_{k+1}^+}) \\ &\quad + \sum_{i=0}^{j} \big(X(\sigma_i^+) - X(\tau_i^+)\big) = X(\sigma_j^+)\end{aligned}$$

because $X(\sigma_i^+) + \xi_{\sigma_j^+ +1} + \ldots + \xi_{\tau_j^+} = X(\tau_{i+1}^+)$ for $i = 0, \ldots, j-1$. Similarly,

$$\begin{aligned}\widetilde{X}^+(\tau_j^+) &= \sum_{k=1}^{\tau_j^+} \mathbb{1}_{\{X(k-1)>m\}}\, \xi_k + \sum_{i\geqslant 0} \big(X(\sigma_i^+) - X(\tau_i^+)\big)\, \mathbb{1}_{\{\sigma_i^+ \leqslant \tau_j^+\}} - X(\tau_j^+) \\ &= \sum_{k=0}^{j-1} (\xi_{\sigma_k^+ +1} + \xi_{\sigma_k^+ +2} + \ldots + \xi_{\tau_{k+1}^+}) \\ &\quad + \sum_{i=0}^{j-1} \big(X(\sigma_i^+) - X(\tau_i^+)\big) - X(\tau_j^+) = 0.\end{aligned}$$

For $n \in \mathbb{N}_0$, put

$$M_n^{\pm}(t) := \sigma^{-1} n^{-1/2} M^{\pm}([nt]), \quad L_n^{\pm}(t) := \sigma^{-1} n^{-1/2} L^{\pm}([nt]), \quad t \geqslant 0.$$

The proofs of Lemmas 2.1 and 2.2 given below are postponed to Section 3.

LEMMA 2.1. *The sequence* $(X_n^{\pm}, M_n^{\pm}, L_n^{\pm})_{n\in\mathbb{N}}$ *is weakly relatively compact on* $D([0,T], \mathbb{R}^6)$ *for each* $T > 0$. *Furthermore, each limit point* $(X_\infty^{\pm}, M_\infty^{\pm}, L_\infty^{\pm})$ *of the sequence is a continuous process satisfying*

$$\int_0^T \mathbb{1}_{\{X_\infty^{\pm}(t)=0\}}\, \mathrm{d}t = 0 \ a.s. \tag{2.2}$$

LEMMA 2.2. *Let* (n_k) *be a sequence such that*

$$(X^{\pm}_{n_k}, M^{\pm}_{n_k}, L^{\pm}_{n_k}) \Rightarrow (X^{\pm}_{\infty}, M^{\pm}_{\infty}, L^{\pm}_{\infty}), \quad k \to \infty,$$

on $D([0,T],\mathbb{R}^6)$ *for some* $T > 0$. *Then*:

(1) *the processes* $L^{\pm}_{\infty}$ *are nondecreasing a.s. and satisfy*

$$\int_0^T \mathbb{1}_{\{X^{\pm}_{\infty}(t)>0\}}\,\mathrm{d}L^{\pm}_{\infty}(t) = 0 \text{ a.s.}; \tag{2.3}$$

(2) *the processes* $\big(M^{\pm}_{\infty}(t)\big)_{t\in[0,T]}$ *are continuous martingales with respect to the filtration* $(\mathcal{F}_t)_{t\in[0,T]}$, *where* $\mathcal{F}_t := \sigma\big(X^{\pm}_{\infty}(s), M^{\pm}_{\infty}(s), L^{\pm}_{\infty}(s),\ s\in[0,t]\big)$, *with the predictable quadratic variations*

$$\langle M^{\pm}_{\infty}\rangle_t = \int_0^t \mathbb{1}_{\{X^{\pm}_{\infty}(s)>0\}}\,\mathrm{d}s. \tag{2.4}$$

2.2. Analysis of the processes $L^{\pm}_{\infty}$. If we can prove that

$$L^{+}_{\infty}(t) = \frac{1+\gamma}{1-\gamma} L^{-}_{\infty}(t) \text{ a.s.}, \tag{2.5}$$

then using (2.1), Lemma 2.1 and the fact that the absolute value of the last summand in (2.1) does not exceed m, we conclude that

$$X^{\pm}_{\infty}(t) = M^{\pm}_{\infty}(t) + \frac{1\pm\gamma}{2} V_{\infty}(t) \text{ a.s.}, \quad t \geqslant 0,$$

where

$$V_{\infty}(t) := \frac{2}{1+\gamma} L^{+}_{\infty}(t) = \frac{2}{1-\gamma} L^{-}_{\infty}(t).$$

By Lemma 2.2 and Proposition 2.1, X_{∞} is then a skew Brownian motion with parameter γ.

Recalling the notation

$$\alpha_0 := 0, \quad \alpha_k := \inf\{i > \alpha_{k-1} :\ X(i) \in A\}, \quad k\in\mathbb{N},$$

and $Y(n) = X(\alpha_n)$, $n\in\mathbb{N}$, set

$$\rho^{\pm}_k := \pm\big(Y(k+1) - Y(k)\big)\,\mathbb{1}_{\{\pm X(\alpha_k+1)\leqslant m\}} \pm \big(X(\alpha_k+1) - Y(k)\big)\,\mathbb{1}_{\{\pm X(\alpha_k+1)>m\}}, \quad k\in\mathbb{N}.$$

Thus, if the (α_k+1)st jump of $\big(X(n)\big)$ brings $\big(X(n)\big)$ to the right of m (to the set $(-\infty, m]$), then ρ^{+}_k is the magnitude of this jump, i.e., the magnitude of the $(k+1)$st jump of $\big(Y(n)\big)$.

LEMMA 2.3. *The limit relation*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \rho_k^{\pm}}{n} = \mathbb{E}_\pi\big(X(1) - X(\alpha_1)\big)^{\pm}$$

holds a.s.

The proof of the lemma is postponed until Section 3.

In view of the relations

$$\begin{aligned}\Big|L^{\pm}(n) - \sum_{k:\,\alpha_k\leqslant n} \rho_k^{\pm}\Big| = \Big| &\pm \sum_{i\geqslant 0} \big(X(\sigma_i^{\pm}) - X(\tau_i^{\pm})\big)\, \mathbb{1}_{\{\sigma_i^{\pm}\leqslant n\}} \\ &\mp \sum_{k:\,\alpha_k\leqslant n} \Big(\big(Y(k+1) - Y(k)\big)\, \mathbb{1}_{\{\pm X(\alpha_k+1)\leqslant m\}} \\ &+ \big(X(\alpha_k+1) - Y(k)\big)\, \mathbb{1}_{\{\pm X(\alpha_k+1)> m\}}\Big)\Big| \\ \leqslant{}& 2m\end{aligned}$$

and Lemma 3.1 (a) below, we can invoke Lemma 2.3 to infer that

$$\lim_{n\to\infty} \frac{L^+(n)}{L^-(n)} = \frac{\mathbb{E}_\pi\big(X(1) - X(\alpha_1)\big)^+}{\mathbb{E}_\pi\big(X(1) - X(\alpha_1)\big)^-} = \frac{1+\gamma}{1-\gamma} \text{ a.s.,}$$

thereby proving (2.5). The proof of Theorem 1.1 is complete.

3. PROOFS OF AUXILIARY RESULTS

For $n \in \mathbb{N}_0$, denote by $\nu(n)$ the sojourn time in A of $\big(X(k)\big)_{0\leqslant k\leqslant n}$, i.e.,

$$\nu(n) := \sum_{k=0}^{n} \mathbb{1}_{\{|X(k)|\leqslant m\}}.$$

LEMMA 3.1. *We have*:

(a) $\lim_{n\to\infty} \nu(n) = \infty$ *a.s.*;

(b) $\mathbb{E}\nu(n) = O(\sqrt{n})$ *as* $n \to \infty$.

P r o o f. Part (a) is obvious. Passing to the proof of part (b), for each $j \in A = \{-m, \ldots, m\}$, we set

$$\zeta_0^{(j)} := \inf\{i \in \mathbb{N} : X(i) = j\}$$

and

$$\widetilde{\zeta}_k^{(j)} = \inf\{i > \zeta_k^{(j)} : |X(i)| > m\}, \quad \zeta_{k+1}^{(j)} = \inf\{i > \widetilde{\zeta}_k^{(j)} : X(i) = j\}, \quad k \in \mathbb{N},$$

with the standard convention that the infimum of the empty set equals $+\infty$. Plainly, the so-defined random variables are stopping times with respect to filtration generated by $\big(X(k)\big)_{k\in\mathbb{N}_0}$. Moreover, the random vectors $\{(\widetilde{\zeta}_k^{(j)}-\zeta_k^{(j)},\zeta_{k+1}^{(j)}-\widetilde{\zeta}_k^{(j)})\}_{k\in\mathbb{N}}$ are i.i.d.

For typographical ease, we assume that $|X(0)|=|x|>m$ hereafter. If the first entrance into A following the $(l-1)$st exit from A, $l\in\mathbb{N}$, occurs at the state j_l, then

$$\nu(n)\leqslant\sum_{l\geqslant 1}(\widetilde{\zeta}_{l-1}^{(j_l)}-\zeta_{l-1}^{(j_l)})\,\mathbb{1}_{\{\zeta_{l-1}^{(j_l)}\leqslant n\}}\quad\text{a.s.}$$

Hence

$$\begin{aligned}\nu(n)&\leqslant\sum_{|j|\leqslant m}\sum_{k\geqslant 0}(\widetilde{\zeta}_k^{(j)}-\zeta_k^{(j)})\,\mathbb{1}_{\{\zeta_k^{(j)}\leqslant n\}}\\&\leqslant\sum_{|j|\leqslant m}\sum_{k\geqslant 0}(\widetilde{\zeta}_k^{(j)}-\zeta_k^{(j)})\,\mathbb{1}_{\{(\zeta_1^{(j)}-\widetilde{\zeta}_0^{(j)})+\ldots+(\zeta_k^{(j)}-\widetilde{\zeta}_{k-1}^{(j)})\leqslant n\}},\end{aligned}$$

and consequently

$$\mathbb{E}\nu(n)\leqslant\sum_{|j|\leqslant m}\mathbb{E}(\widetilde{\zeta}_0^{(j)}-\zeta_0^{(j)})\sum_{k\geqslant 0}\mathbb{P}\{(\zeta_1^{(j)}-\widetilde{\zeta}_0^{(j)})+\ldots+(\zeta_k^{(j)}-\widetilde{\zeta}_{k-1}^{(j)})\leqslant n\}$$

because, for $k\in\mathbb{N}$, $\widetilde{\zeta}_k^{(j)}-\zeta_k^{(j)}$ is independent of $\mathbb{1}_{\{(\zeta_1^{(j)}-\widetilde{\zeta}_0^{(j)})+\ldots+(\zeta_k^{(j)}-\widetilde{\zeta}_{k-1}^{(j)})\leqslant n\}}$ and has the same distribution as $\widetilde{\zeta}_0^{(j)}-\zeta_0^{(j)}$. Thus, to complete the proof, it suffices to check that, for fixed $j\in A$,

$$\mathbb{E}(\widetilde{\zeta}_0^{(j)}-\zeta_0^{(j)})<\infty \tag{3.1}$$

and

$$\limsup_{n\to\infty}n^{-1/2}\sum_{k\geqslant 0}\mathbb{P}\{(\zeta_1^{(j)}-\widetilde{\zeta}_0^{(j)})+\ldots+(\zeta_k^{(j)}-\widetilde{\zeta}_{k-1}^{(j)})\leqslant n\}<\infty. \tag{3.2}$$

Proof of (3.1). By using the mathematical induction we can check that

$$\mathbb{P}\{\widetilde{\zeta}_0^{(j)}-\zeta_0^{(j)}>s\}\leqslant\mathbb{P}\{|\eta_j+j|\leqslant m\}(\mathbb{P}\{\min_*|\eta_k+k|\leqslant m\})^{s-1},\quad s\in\mathbb{N},$$

where we write $\min_*$ to mean that the minimum is taken over all integers $k\in[-m,m]$ for which $\mathbb{P}\{|\eta_k+k|\leqslant m\}<1$. Such indices k do exist in view of the irreducibility. Thus, not only does (3.1) hold, but also some exponential moments of $\widetilde{\zeta}_0^{(j)}-\zeta_0^{(j)}$ are finite.

Proof of (3.2). Noting that

$$\begin{aligned}\{\pm X(\widetilde{\zeta}_0^{(j)})>m,\ \pm\xi_{\widetilde{\zeta}_0^{(j)}+1}\geqslant 0,\ldots,\pm\xi_{\widetilde{\zeta}_0^{(j)}+1}\pm\ldots\pm\xi_{\widetilde{\zeta}_0^{(j)}+n}\geqslant 0\}&\\ \subseteq\{\pm X(\widetilde{\zeta}_0^{(j)})>m,\ \zeta_1^{(j)}-\widetilde{\zeta}_0^{(j)}>n\}&\end{aligned}$$

for $n \in \mathbb{N}$ and setting $p_j := \mathbb{P}\{X(\widetilde{\zeta}_0^{(j)}) > m\}$, we arrive at

$$\begin{aligned}(3.3)\quad \mathbb{P}\{\zeta_1^{(j)} - \widetilde{\zeta}_0^{(j)} > n\} &\\ \geqslant p_j \mathbb{P}\{\xi_{\widetilde{\zeta}_0^{(j)}+1} \geqslant 0, \ldots, \xi_{\widetilde{\zeta}_0^{(j)}+1} + \ldots + \xi_{\widetilde{\zeta}_0^{(j)}+n} \geqslant 0\} &\\ + (1-p_j)\mathbb{P}\{\xi_{\widetilde{\zeta}_0^{(j)}+1} \leqslant 0, \ldots, \xi_{\widetilde{\zeta}_0^{(j)}+1} + \ldots + \xi_{\widetilde{\zeta}_0^{(j)}+n} \leqslant 0\}.\end{aligned}$$

Observe that $(\xi_{\widetilde{\zeta}_0^{(j)}+1} + \ldots + \xi_{\widetilde{\zeta}_0^{(j)}+k})_{k\in\mathbb{N}}$ is a standard random walk. Its jumps have zero mean and finite variance because these have the same distribution as ξ_1. Hence

$$(3.4)\quad \lim_{n\to\infty} n^{1/2}\mathbb{P}\{\xi_{\widetilde{\zeta}_0^{(j)}+1} \geqslant 0, \ldots, \xi_{\widetilde{\zeta}_0^{(j)}+1} + \ldots + \xi_{\widetilde{\zeta}_0^{(j)}+n} \geqslant 0\} = c_+ \in (0,\infty),$$

$$(3.5)\quad \lim_{n\to\infty} n^{1/2}\mathbb{P}\{\xi_{\widetilde{\zeta}_0^{(j)}+1} \leqslant 0, \ldots, \xi_{\widetilde{\zeta}_0^{(j)}+1} + \ldots + \xi_{\widetilde{\zeta}_0^{(j)}+n} \leqslant 0\} = c_- \in (0,\infty)$$

(see, for instance, pp. 381–382 in [2]). Using Erickson's inequality (Lemma 1 in [6]) we infer that

$$\begin{aligned}\sum_{k\geqslant 0} \mathbb{P}\{(\zeta_1^{(j)} - \widetilde{\zeta}_0^{(j)}) + \ldots + (\zeta_k^{(j)} - \widetilde{\zeta}_{k-1}^{(j)}) \leqslant n\} &\leqslant \frac{2n}{\mathbb{E}\big((\zeta_1^{(j)} - \widetilde{\zeta}_0^{(j)}) \wedge n\big)}\\ &\leqslant \frac{2}{\mathbb{P}\{\zeta_1^{(j)} - \widetilde{\zeta}_0^{(j)} > n\}},\end{aligned}$$

which in combination with (3.3)–(3.5) gives

$$\begin{aligned}\limsup_{n\to\infty} n^{-1/2} \sum_{k\geqslant 0} \mathbb{P}\{(\zeta_1^{(j)} - \widetilde{\zeta}_0^{(j)}) + \ldots + (\zeta_k^{(j)} - \widetilde{\zeta}_{k-1}^{(j)}) \leqslant n\}&\\ \leqslant \frac{2}{p_j c_+ + (1-p_j)c_-} < \infty.&\end{aligned}$$

The proof of Lemma 3.1 is complete. ■

P r o o f o f L e m m a 2.1. Weak relative compactness and continuity of the limit follow if we can check that either of the sequences $(X_n^\pm)$, $(M_n^\pm)$ and $(L_n^\pm)$ is weakly relatively compact, and that their weak limit points are continuous processes. Actually, verification for $(L_n^\pm)$ is not needed, for (a) the absolute value of the last summand in (2.1) does not exceed m; (b) $\sup_{t\geqslant 0} |X_n^\pm(t) - \widetilde{X}_n^\pm(t)| \leqslant \sigma^{-1} m n^{-1/2}$, where

$$\widetilde{X}_n^\pm(t) := \sigma^{-1} n^{-1/2} \widetilde{X}^\pm([nt]), \quad t \geqslant 0.$$

Further, it is clear that instead of $(X_n^\pm)$ and $(M_n^\pm)$ we can work with (X_n) and (M_n), where, as usual, $M_n := M_n^+ - M_n^-$.

According to Theorem 15.5 in [1] it suffices to prove that

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}\{\sup_{|t-s|\leqslant\delta,\, t,s\in[0,T]}|X_n(t)-X_n(s)|>\varepsilon\}=0$$

and that

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}\{\sup_{|t-s|\leqslant\delta,\, t,s\in[0,T]}|M_n(t)-M_n(s)|>\varepsilon\}=0$$

for any $\varepsilon>0$ or, which is equivalent,

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}\Big\{\sup_{|i-j|\leqslant[\delta n],\, i,j\in[0,[nT]]}|X(i)-X(j)|>\varepsilon\sigma\sqrt{n}\Big\}=0, \tag{3.6}$$

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}\Big\{\sup_{|i-j|\leqslant[\delta n],\, i,j\in[0,[nT]]}|M(i)-M(j)|>\varepsilon\sigma\sqrt{n}\Big\}=0. \tag{3.7}$$

Furthermore, if (X_n) and/or (M_n) converge along a subsequence, the corresponding limits have continuous versions.

Define a random sequence $\big(X^*(k)\big)_{k\in\mathbb{N}_0}$ by

$$X^*(k):=S\big(r(k)\big)+\sum_{|j|\leqslant m}\sum_{i=1}^{r_j(k)}\eta_{j,i},\quad k\in\mathbb{N}_0, \tag{3.8}$$

where

$$r(0):=0,\quad r(n):=\sum_{i=0}^{n-1}\mathbb{1}_{\{|X^*(i)|>m\}}$$

and, for each $j\in A=\{-m,\dots,m\}$,

$$r_j(0):=0,\quad r_j(n)=\sum_{i=0}^{n-1}\mathbb{1}_{\{X^*(i)=j\}}.$$

Then $\big(X^*(k)\big)_{k\in\mathbb{N}_0}$ is a Markov chain with $X^*(0)=x$ and the same transition probabilities as the Markov chain $\big(X(k)\big)_{k\in\mathbb{N}_0}$. Hence the distributions of the two Markov chains are the same. This implies in particular that

$$\sum_{|j|\leqslant m}r_j(n)\overset{d}{=}\nu(n-1)=\sum_{k=0}^{n-1}\mathbb{1}_{\{|X(k)|\leqslant m\}} \tag{3.9}$$

for each $n\in\mathbb{N}$, where $\overset{d}{=}$ denotes equality of distributions. Further, observe that

$$M(n)=\sum_{k=1}^{n}\xi_k\,\mathbb{1}_{\{|X(k-1)|>m\}}=\sum_{i=1}^{n}\big(X(i)-X(i-1)\big)\,\mathbb{1}_{\{|X(i-1)|>m\}}$$

and

$$M^*(n) := S\big(r(n)\big) - x = \sum_{i=1}^{n} \big(X^*(i) - X^*(i-1)\big)\, \mathbb{1}_{\{|X^*(i-1)|>m\}}$$

for $n \in \mathbb{N}_0$. Since the sequences $\big(X(n)\big)_{n\in\mathbb{N}_0}$ and $\big(X^*(n)\big)_{n\in\mathbb{N}_0}$ have the same distribution, so have $\big(M(n)\big)_{n\in\mathbb{N}_0}$ and $\big(M^*(n)\big)_{n\in\mathbb{N}_0}$.

Relation (3.7) is a consequence of the following reasoning:

$$\begin{aligned}
&\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}\Big\{\sup_{|i-j|\leqslant[\delta n],\, i,j\in[0,[nT]]}|M(i)-M(j)| > \varepsilon\sigma\sqrt{n}\Big\}\\
&\qquad = \lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}\Big\{\sup_{|i-j|\leqslant[\delta n],\, i,j\in[0,[nT]]}|M^*(i)-M^*(j)| > \varepsilon\sigma\sqrt{n}\Big\}\\
&\qquad \leqslant \lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}\Big\{\sup_{|i-j|\leqslant[\delta n],\, i,j\in[0,[nT]]}|S(j)-S(i)| > \varepsilon\sigma\sqrt{n}\Big\} = 0,
\end{aligned}$$

where the last equality is implied by (1.1).

Turning to the proof of (3.6), we first show that, for any $0 \leqslant i, j \leqslant [nT]$,

$$\sup_{|i-j|\leqslant[\delta n]}|X^*(i)-X^*(j)| \leqslant 2m + 2\sup_{|i-j|\leqslant[\delta n]}|S(i)-S(j)| + \max_{|l|\leqslant m}\max_{1\leqslant k\leqslant r_l([nT])}|\eta_{l,k}| \text{ a.s.} \tag{3.10}$$

By symmetry it is sufficient to investigate the case $0 \leqslant i < j \leqslant [nT]$.

If $|X^*(i)| \leqslant m$ and $|X^*(j)| \leqslant m$, then $|X^*(i) - X^*(j)| \leqslant 2m$ a.s.

If $j - i \leqslant [\delta n]$ and $|X^*(k)| > m$ for all $k \in \{i, \dots, j\}$, then $|X^*(i) - X^*(j)| \leqslant \sup_{|i'-j'|\leqslant[\delta n]} |S(i') - S(j')|$.

Finally, assume that $j - i \leqslant [\delta n]$, $X^*(i) > m$ and $X^*(j) < -m$ (the case $X^*(i) < -m$ and $X^*(j) > m$ can be treated analogously). Set

$$\alpha := \inf\{k>i : X^*(k)\in[-m,m]\} \quad \text{and} \quad \beta := \sup\{k<j : X^*(k)\in[-m,m]\}.$$

Then

$$\begin{aligned}
|X^*(i)-X^*(j)| &\leqslant |X^*(i)-X^*(\alpha)| + |X^*(\alpha)-X^*(\beta)|\\
&\qquad + |X^*(\beta)-X^*(\beta+1)| + |X^*(\beta+1)-X^*(j)|\\
&\leqslant 2\sup_{|i'-j'|\leqslant[\delta n]}|S(i')-S(j')| + 2m\\
&\qquad + \max_{|l|\leqslant m}\max_{1\leqslant k\leqslant r_l([nT])}|\eta_{l,k}|.
\end{aligned}$$

Thus, (3.10) holds, which entails

$$\begin{aligned}\mathbb{P}\Big\{&\sup_{|i-j|\leqslant[\delta n],\,i,j\in[0,[nT]]}|X(i)-X(j)|>\varepsilon\sigma\sqrt{n}\Big\}\\&=\mathbb{P}\Big\{\sup_{|i-j|\leqslant[\delta n],\,i,j\in[0,[nT]]}|X^*(i)-X^*(j)|>\varepsilon\sigma\sqrt{n}\Big\}\\&\leqslant\mathbb{P}\Big\{2m+2\sup_{|i-j|\leqslant[\delta n],\,i,j\in[0,[nT]]}|S(i)-S(j)|\\&\quad+\max_{|l|\leqslant m}\max_{1\leqslant k\leqslant r_l([nT])}|\eta_{l,k}|>\varepsilon\sigma\sqrt{n}\Big\}.\end{aligned}$$

In view of (1.1), to complete the proof of (3.6), it remains to check that

$$n^{-1/2}\max_{|l|\leqslant m}\max_{1\leqslant k\leqslant r_l([nT])}|\eta_{l,\,k}|\xrightarrow{P}0,\quad n\to\infty. \tag{3.11}$$

Using Boole's inequality (twice) and Markov's inequality yields

$$\begin{aligned}\mathbb{P}\Big\{&n^{-1/2}\max_{|l|\leqslant m}\max_{1\leqslant k\leqslant r_l([nT])}|\eta_{l,\,k}|>\varepsilon\sqrt{n}\Big\}\\&\leqslant\mathbb{P}\Big\{\sum_{|j|\leqslant m}r_j([nT])>x\sqrt{n}\Big\}+\sum_{|l|\leqslant m}\mathbb{P}\Big\{\max_{1\leqslant k\leqslant[x\sqrt{n}]+1}|\eta_{l,\,k}|>\varepsilon\sqrt{n}\Big\}\\&\leqslant x^{-1}n^{-1/2}\mathbb{E}\sum_{|j|\leqslant m}r_j([nT])+([x\sqrt{n}]+1)\sum_{|l|\leqslant m}\mathbb{P}\{|\eta_{l,\,1}|>\varepsilon\sqrt{n}\}.\end{aligned}$$

Sending first $n\to\infty$ (taking into account (3.9) together with Lemma 3.1 and the assumption $\lim_{n\to\infty}n\mathbb{P}\{|\eta_{l,1}|>n\}=0$) and then $x\to\infty$, we arrive at (3.11).

It remains to prove (2.2). To this end, note that any limit point $(X_\infty^\pm,M_\infty^\pm,L_\infty^\pm)$ satisfies

$$\begin{aligned}X_\infty(t):=X_\infty^+(t)-X_\infty^-(t)&=M_\infty^+(t)-M_\infty^-(t)+L_\infty^+(t)-L_\infty^-(t)\\&=:M_\infty(t)+L_\infty(t).\end{aligned}$$

Representation (3.8) together with Lemma 3.1 implies that M_∞ is a Brownian motion. Another appeal to (3.8) allows us to conclude that L_∞ is a continuous process of locally bounded variation. Hence (2.2) follows from the occupation time formula (Corollary 1.6 of Chapter 6 in [16]) because $\langle X_\infty\rangle(t)=\langle M_\infty\rangle(t)=t$ (see Proposition 1.18 of Chapter 4 in [16]). The proof of Lemma 2.1 is complete. ■

Proof of Lemma 2.2. (1) Since the converging processes $L_n^\pm$ are a.s. nondecreasing, so are $L_\infty^\pm$.

For each $\varepsilon>0$, denote by $f_\varepsilon(x)$ a continuous nonnegative function such that $f_\varepsilon(x)=1$ for $x\geqslant\varepsilon$ and $f_\varepsilon(x)=0$ for $x\leqslant\varepsilon/2$. To prove (2.3), it is sufficient to check that

$$\int_0^T f_\varepsilon\big(X_\infty^\pm(s)\big)\mathrm{d}L_\infty^\pm(s)=0\text{ a.s.}$$

for each $\varepsilon > 0$ and then use $\lim_{\varepsilon\to 0} f_\varepsilon(x) = \mathbb{1}_{(0,\infty)}(x)$ together with Lebesgue's dominated convergence theorem.

By Skorokhod's representation theorem there exist versions of the original processes which converge a.s. Furthermore, the convergence is locally uniform, for the limit processes are a.s. continuous. Hence we have (for versions)

$$\int_0^T f_\varepsilon\big(X_\infty^\pm(s)\big)\mathrm{d}L_\infty^\pm(s) = \lim_{k\to\infty}\int_0^T f_\varepsilon\big(X_{n_k}^\pm(s)\big)\mathrm{d}L_{n_k}^\pm(s) = 0 \text{ a.s.},$$

as desired.

(2) We give only the proof for M_∞^+. We have to check that

(I) $\big(M_\infty^+(t)\big)_{t\in[0,\,T]}$ is a martingale;

(II) $\big(\big(M_\infty^+(t)\big)^2 - A(t)\big)_{t\in[0,\,T]}$ is a martingale where

$$A(t) := \int_0^t \mathbb{1}_{\{X_\infty^+(s)>0\}}\,\mathrm{d}s, \qquad t \geqslant 0.$$

We concentrate on the proof of (II), for the proof of (I) is similar but simpler. Set $X_\infty := X_\infty^+ - X_\infty^-$. Observe that the σ-algebra $\sigma\big(X_\infty(s), s \leqslant t\big)$ is generated by a family of random variables

$$\big\{f\big(X_\infty(t_1),\ldots,X_\infty(t_j)\big) \mid j\in\mathbb{N},\ 0\leqslant t_1 < t_2 < \ldots < t_j \leqslant t,\ f\in C_b(\mathbb{R}^j)\big\},$$

where $C_b(\mathbb{R}^j)$ is the set of bounded continuous real-valued functions defined on $\mathbb{R}^j$. It thus suffices to verify

$$\text{(3.12)}\quad \mathbb{E}f\big(X_\infty(t_1),\ldots,X_\infty(t_j)\big) \times \Big(\big(M_\infty^+(t)\big)^2 - A(t) - \big(M_\infty^+(t_j)\big)^2 + A(t_j)\Big) = 0$$

for any $t\in[0,T]$, and $j\in\mathbb{N}$, any $0\leqslant t_1 < t_2 < \ldots < t_j \leqslant t$ and any function $f\in C_b(\mathbb{R}^j)$.

Put $\mathcal{F}_0 := \{\emptyset,\Omega\}$, $\mathcal{F}_k := \sigma\big(X(i),\xi_i\big)_{1\leqslant i\leqslant k}$, $k\in\mathbb{N}$, and

$$\mathcal{E}_k(n) := (\sigma^2 n)^{-1}\big(\sum_{i=1}^k \mathbb{1}_{\{X(i-1)>m\}}\,\xi_i\big)^2 - n^{-1}\sum_{i=1}^k \mathbb{1}_{\{X(i-1)>m\}}.$$

Since $\big(\mathcal{E}_k(n)\big)_{k\in\mathbb{N}_0}$ is a martingale with respect to $(\mathcal{F}_i)_{i\in\mathbb{N}_0}$, we infer that

$$\begin{aligned}\mathbb{E}\Big(\big(M_n^+(t^{(n)})\big)^2 - \int_0^{t^{(n)}} \mathbb{1}_{\{X_n^+(s)>0\}}\,\mathrm{d}s\Big|\mathcal{F}_{[nt_j]}\Big) &= \mathbb{E}(\mathcal{E}_{[nt]}|\mathcal{F}_{[nt_j]}) = \mathcal{E}_{[nt_j]}\\ &= \big(M_n^+(t_j^{(n)})\big)^2 - \int_0^{t_j^{(n)}} \mathbb{1}_{\{X_n^+(s)>0\}}\,\mathrm{d}s,\end{aligned}$$

where $t_k^{(n)} := [nt_k]/n$, $k = 1, \dots, j$, $t^{(n)} := [nt]/n$. Hence

$$\mathbb{E} f\big(X_n(t_1^{(n)}), \dots, X_n(t_j^{(n)})\big) \times \Big(\big(M_n^+(t^{(n)})\big)^2 - \big(M_n^+(t_j^{(n)})\big)^2 - \int\limits_{t_j^{(n)}}^{t^{(n)}} \mathbb{1}_{\{X_n^+(s)>0\}}\, \mathrm{d}s\Big) = 0. \tag{3.13}$$

The sequence $\big(\big(M_n^+(t^{(n)})\big)^2 - \big(M_n^+(t_j^{(n)})\big)^2 - \int_{t_j^{(n)}}^{t^{(n)}} \mathbb{1}_{\{X_n^+(s)>0\}}\, \mathrm{d}s\big)_{n\in\mathbb{N}}$, with $s \in [0, T]$ fixed, is uniformly integrable if we can show that

$$\sup_{k\in\mathbb{N}} \mathbb{E}\mathcal{E}^2_{[ns]}(n) < \infty. \tag{3.14}$$

The expression under the expectation sign in (3.13), with n replaced by n_k, converges weakly, as $k \to \infty$, to the expression under the expectation sign in (3.12), whence equality (3.12) follows by the aforementioned uniform integrability.

While proving (3.14), we assume, for simplicity, that $s = 1$. By the Marcinkiewicz–Zygmund inequality for martingales (Theorem 9 in [3]),

$$\mathbb{E}\mathcal{E}_n^2(n) \leqslant C\mathbb{E} \sum_{k=1}^{n} Z_k(n)^2 \tag{3.15}$$

for some constant $C > 0$ which does not depend on n, where $\big(Z_k(n)\big)_{k\in\mathbb{N}}$ are martingale differences defined by

$$\begin{aligned} Z_k(n) := (\sigma^2 n)^{-1}\big((\xi_k^2 - \sigma^2)\, \mathbb{1}_{\{X(k-1)>m\}} \\ + 2\xi_k\, \mathbb{1}_{\{X(k-1)>m\}} \sum_{i=1}^{k-1} \xi_i\, \mathbb{1}_{\{X(i-1)>m\}}\big) \end{aligned}$$

for $k \in \mathbb{N}$ (with the convention that $\sum_{i=1}^{0} \ldots = 0$). Setting

$$r := \mathbb{E}(\xi_1^2 - \sigma^2)^2 < \infty,$$

we have

$$\begin{aligned} \sigma^4 n^2 \mathbb{E} Z_k(n)^2 &\leqslant 2\Big(\mathbb{E}(\xi_k^2 - \sigma^2)\, \mathbb{1}_{\{X(k-1)>m\}} \\ &\quad + 4\mathbb{E}\xi_k^2\, \mathbb{1}_{\{X(k-1)>m\}} \big(\sum_{i=1}^{k-1} \xi_i\, \mathbb{1}_{\{X(i-1)>m\}}\big)^2\Big) \\ &\leqslant 2\Big(r + 4\sigma^2 \mathbb{E}\big(\sum_{i=1}^{k-1} \xi_i\, \mathbb{1}_{\{X(i-1)>m\}}\big)^2\Big) \leqslant 2\big(r + 4\sigma^4(k-1)\big). \end{aligned}$$

Using the last inequality and (3.15), we get (3.14). Thus the proof of Lemma 2.2 is complete. ■

P r o o f o f L e m m a 2.3. Fix $x \in A$. It suffices to prove that the convergence holds $\mathbb{P}_x$-a.s. rather than a.s. The subsequent proof is similar to the proof of the strong law of large numbers for Markov chains (see, for instance, p. 87 in [5]). We only treat $\rho_k := \rho_k^+$.

Put $T_x^{(0)} := 0$ and, for $k \in \mathbb{N}$, denote by $T_x^{(k)}$ the time of the kth return of (Y_j) to x. Also, for $k \in \mathbb{N}$, we set $\theta_k(x) := \sum_{j=T_x^{(k-1)}}^{T_x^{(k)}-1} \rho_j$ and observe that the random variables $\theta_1(x), \theta_2(x), \ldots$ are independent and $\mathbb{P}_x$-identically distributed. We have

$$\begin{aligned}
&\mathbb{E}_x\theta_1(x) = \\
&= \sum_{y\in A}\sum_{j\geqslant 0} \mathbb{E}_x\Big(\big(Y(j+1)-Y(j)\big)\,\mathbb{1}_{\{X(\alpha_j+1)\leqslant m\}} \\
&\quad + \big(X(\alpha_j+1)-Y(j)\big)\,\mathbb{1}_{\{X(\alpha_j+1)>m\}}\,|Y(j)=y\Big)\mathbb{P}\{Y(j)=y, T_x^{(1)}>j\} \\
&= \sum_{y\in A}\mathbb{E}\Big(\big(Y(1)-Y(0)\big)\,\mathbb{1}_{\{X(1)\leqslant m\}} \\
&\quad + \big(X(1)-Y(0)\big)\,\mathbb{1}_{\{X(1)>m\}}\,|Y(0)=y\Big)\sum_{j\geqslant 0}\mathbb{P}\{Y(j)=y, T_x^{(1)}>j\} \\
&= \mathbb{E}_xT_x^{(1)}\sum_{y\in A}\pi_y\mathbb{E}\Big(\big(Y(1)-Y(0)\big)\,\mathbb{1}_{\{X(1)\leqslant m\}} \\
&\quad + \big(X(1)-Y(0)\big)\,\mathbb{1}_{\{X(1)>m\}}\,|Y(0)=y\Big) \\
&= \mathbb{E}_xT_x^{(1)}\mathbb{E}_\pi\Big(\big(X(\alpha_1)-Y(0)\big)\,\mathbb{1}_{\{X(1)\leqslant m\}} + \big(X(1)-Y(0)\big)\,\mathbb{1}_{\{X(1)>m\}}\Big) \\
&= \mathbb{E}_xT_x^{(1)}\mathbb{E}_\pi\big(X(1)-X(\alpha_1)\big)\,\mathbb{1}_{\{X(1)>m\}} \\
&= \mathbb{E}_xT_x^{(1)}\mathbb{E}_\pi\big(X(1)-X(\alpha_1)\big)^+
\end{aligned}$$

having utilized Theorem 8.2 on p. 84 in [5] for the third equality, the last equality being a consequence of the fact that on the event $\{X(1) < -m\}$ one has $X(1) < X(\alpha_1)$, while on $\{X(1) \in [-m, m]\}$ one has $X(1) = X(\alpha_1)$. Using the strong law of large numbers for random walks and renewal processes, we obtain

$$\frac{\sum_{k=1}^{N_n(x)}\theta_k(x)}{n} = \frac{N_n(x)}{n}\,\frac{\sum_{k=1}^{N_n(x)}\theta_k(x)}{N_n(x)} \to \mathbb{E}_\pi\big(X(1)-X(\alpha_1)\big)^+, \quad n\to\infty,$$

$\mathbb{P}_x$-a.s., where $N_n(x) := \#\{k \in \mathbb{N} : T_x^{(k)} \leqslant n\}$. It remains to note that

$$\Big|\sum_{k=1}^{n}\rho_k - \sum_{k=1}^{N_n(x)}\theta_k(x)\Big| \leqslant |\theta_{N_n(x)+1}(x)| \leqslant \max_{1\leqslant j\leqslant n+1}|\theta_j(x)|,$$

and that, as $n \to \infty$, the right-hand side divided by n converges to zero $\mathbb{P}_x$-a.s. in view of $\mathbb{E}|\theta_1(x)| < \infty$ and the Borel–Cantelli lemma. ∎

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REFERENCES

[1] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York 1968.

[2] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*, Cambridge University Press, Cambridge 1989.

[3] D. L. Burkholder, *Martingale transforms*, Ann. Math. Statist. 37 (1966), pp. 1494–1504.

[4] A. S. Cherny, A. N. Shiryaev, and M. Yor, *Limit behavior of the 'horizontal-vertical' random walk and some extensions of the Donsker–Prokhorov invariance principle*, Theory Probab. Appl. 47 (2002), pp. 377–394.

[5] R. Durrett, *Essentials of Stochastic Processes*, Springer, New York 1999.

[6] K. B. Erickson, *The strong law of large numbers when the mean is undefined*, Trans. Amer. Math. Soc. 185 (1973), pp. 371–381.

[7] J. M. Harrison and L. A. Shepp, *On skew Brownian motion*, Ann. Probab. 9 (1981), pp. 309–313.

[8] A. M. Kulik, *A limit theorem for diffusions on graphs with variable configuration*, preprint available at arXiv: math/0701632 (2007).

[9] A. Lejay, *On the constructions of the skew Brownian motion*, Probab. Surv. 3 (2006), pp. 413–466.

[10] R. A. Minlos and E. A. Zhizhina, *Limit diffusion process for a non-homogeneous random walk on a one-dimensional lattice*, Russian Math. Surveys 52 (1997), pp. 327–340.

[11] D. Paulin and D. Szász, *Locally perturbed random walks with unbounded jumps*, J. Stat. Phys. 141 (2010), pp. 1116–1130.

[12] A. Yu. Pilipenko and Yu. E. Pryhod'ko, *Limit behavior of symmetric random walks with a membrane*, Theory Probab. Math. Statist. 85 (2012), pp. 93–105.

[13] A. Yu. Pilipenko and Yu. E. Prykhodko, *Limit behavior of a simple random walk with non-integrable jump from a barrier*, Theory Stoch. Process. 19 (35) (2014), pp. 52–61.

[14] A. Pilipenko and Yu. Prykhodko, *On the limit behavior of a sequence of Markov processes perturbed in a neighborhood of the singular point*, Ukrainian Math. J. 67 (2015), pp. 564–583.

[15] A. Pilipenko and L. Sakhanenko, *On a limit behavior of a one-dimensional random walk with non-integrable impurity*, Theory Stoch. Process. 20 (36) (2015), pp. 97–104.

[16] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, third edition, Springer, Berlin 1999.

[17] D. Szász and A. Telcs, *Random walk in an inhomogeneous medium with local impurities*, J. Stat. Phys. 26 (1981), pp. 527–537.

[18] B. S. Tsirelson, *Triple points: from non-Brownian filtrations to harmonic measures*, Geom. Funct. Anal. 7 (1997), pp. 1096–1142.

[19] D. A. Yarotskii, *Invariance principle for nonhomogeneous random walks on the grid* $\mathbb{Z}^1$, Math. Notes 66 (1999), pp. 372–383.

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