Abstract. The main results deal with the $GI/GI/1$ queues with infinite means of the service times and interarrival times. Theorem 3.1 gives an asymptotic, in a heavy traffic situation, of the sequence of waiting times of the consecutive customers. Theorem 4.1 gives an asymptotic of stationary waiting times in a heavy traffic situation. In a special case, the asymptotic stationary waiting times have an exponential distribution (Corollary 4.1).

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1. INTRODUCTION

The paper deals with the $GI/GI/1$ queues with infinite means of the service times and interarrival times. An example of such queues is the situation when the distributions of the service times and of the interarrival times belong to the domain of attraction of the stable distributions with stable parameters smaller than one. For such systems we consider an asymptotic, in a heavy traffic situation, of the sequence of waiting times of the consecutive customers and an asymptotic of the stationary waiting times. The subject of an asymptotic of the process of the waiting times and an asymptotic of the stationary waiting times in the heavy traffic situation, when the means of the service times and interarrival times are finite, is well known and is nicely and completely presented in Whitt [9]. As far as I know this subject was not considered in the case when the means of the service times and interarrival times are infinite. Before giving a motivation to consider this subject in the last situation let us recall a description of the $GI/GI/1$ queue with FIFO discipline of the service. Such a system is described by a sequence $\{(v_k, u_k), k \geq 1\}$ of nonnegative random variables $v_k$ and $u_k$ such that the sequences $\{v_k\}$ and $\{u_k\}$ are mutually independent and each of them is the sequence of independent and identically distributed random variables. Then $v_k$ represents the service time of the $k$-th
unit, $k \geq 1$, and $u_k$ represents the interarrival time between the $k$-th and $(k + 1)$-st units, $k \geq 1$.

To give a real model of $GI/GI/1$ queue with infinite means of the service times and interarrival times let us consider two independent processes $W = \{W(t), t \geq 0\}$ and $X = \{X(t), t \geq 0\}$, where $W$ is a standard Wiener process and $X$ is the stable Lévy process with $EX(t) = 0$, nonpositive jumps and stable parameter $\alpha_0$, $1 < \alpha_0 \leq 2$. Let $t_k$, $k \geq 1$, be the first passage time of the level $x = k$ by the Wiener process and let $t'_k$ be the first passage time of the level $x = k$ by the process $X$. Let

$$v_k = t'_k - t'_{k-1}, \quad u_{k+1} \overset{df}{=} t_k - t_{k-1}, \quad k \geq 1, \quad t_0 = t'_0 = 0.$$ 

Then $\{v_k\}$ and $\{u_k\}$ are mutually independent sequences of nonnegative random variables and each of them is the sequence of independent and identically distributed random variables. Moreover, $u_k$ and $v_k$, $k \geq 1$, are stable distributed with stable parameters $\alpha = 1/2$ and $\alpha = 1/\alpha_0 < 1$, respectively. Of course, $Ev_k = Ev_k = \infty$.

Now, let us imagine that the arrival stream to $GI/GI/1$ queue and the service times of units in this system are managed by an operator in the following way. The operator observes two independent processes $W$ and $X$ and he sends the $k$-th unit, $k \geq 2$, to the system at the moment $t_{k-1}$ which is the first passage time of the level $x = k$, $k \geq 2$, by the Wiener process $W$. The $k$-th unit gets the service time $v_k = t'_k - t'_{k-1}$ which is the time between the first passage times of the levels $x = k - 1$ and $x = k$, respectively, by the process $X$. Of course, such a $GI/GI/1$ queue is with infinite means of the service times and interarrival times.

In the paper, we consider $GI/GI/1$ queues for which the distributions of the service times and the interarrival times belong to the domain of attraction of the stable distributions with stable parameter $\alpha < 1$. For such queues we give, in the next section, the sufficient conditions for existing the stationary waiting time (Theorem 2.1). In the third section, we give an asymptotic, in a heavy traffic situation, of the sequence of waiting times of the subsequent customers (Theorem 3.1). In the fourth section, we give an asymptotic of the stationary waiting times in the heavy traffic situation (Theorem 4.1).

In the paper we use the following notation: $(x)_+ = \max(0, x)$, $\Rightarrow$ means the weak convergence of distributions of random variables or the weak convergence of distributions of stochastic processes with sample paths in $D[0, \infty)$, while $D[0, \infty)$ denotes the space of real-valued functions on $[0, \infty)$, right continuous and with limits from the left. It is equipped with the Skorokhod $J_1$ topology. The weak convergence of distributions of stochastic processes with sample paths in $D[0, \infty)$ is equipped with the Skorokhod $J_1$ topology. Furthermore, for a nonnegative random variable $Y$ we use the notation $Y \sim S_\infty(\alpha, \gamma)$, which means that $Y$ is stable distributed with stable parameter $\alpha$, $0 < \alpha < 1$, scale parameter $\sigma$, $0 < \sigma < \infty$, and location
GI/GI/1 queues with infinite means of service time and interarrival time

parameter $\gamma$, $0 \leq \gamma < \infty$. In this case the Laplace transform of $Y$ is of the form

$$E \exp(-sY) = \exp(-as^\alpha - s\gamma), \quad s \geq 0,$$

where $a = \sigma^\alpha (\cos(\pi\alpha/2))^{-1}$.

2. STATIONARY WAITING TIME

Let $w_k$ denote the waiting time for the service of the $k$-th unit in the GI/GI/1 queue, with $w_1 = 0$. Then

$$w_{k+1} = S_k - \min_{0 \leq j \leq k} S_j, \quad k \geq 1,$$

where

$$S_0 = 0, \quad S_k = \sum_{j=1}^k v_j - \sum_{j=1}^k u_j = \sum_{j=1}^k (v_j - u_j), \quad k \geq 1.$$

Let $\omega$ be a random variable such that $w_k \xrightarrow{} \omega$ if this limit exists. It means that $\omega$ is a random variable with distribution being the limiting distribution of $w_k$ in the sense of the weak convergence. If this limit exists, then $\omega$ is called the stationary waiting time for the GI/GI/1 queue. Since

$$w_{k+1} = S_k - \min_{0 \leq j \leq k} S_j \xrightarrow{D} \max_{0 \leq j \leq k} S_j \equiv \tilde{w}_k, \quad k \geq 1,$$

and $\tilde{w}_k \leq \tilde{w}_{k+1}$, the stationary waiting time exists iff

$$(2.1) \quad S_k \xrightarrow{a.e.} -\infty.$$

We know that if $Ev_1 < Eu_1$, then (2.1) holds and

$$(2.2) \quad \omega = \sup_{0 \leq k < \infty} S_k.$$

From Bingham [3] we know that in the general case (without the assumption $Ev_1 < Eu_1$) the convergence (2.1) holds iff

$$(2.3) \quad \sum_{k=1}^{\infty} \frac{1}{k} P(S_k > 0) < \infty,$$

and then we have (2.2) and the characteristic function of the stationary waiting time $\omega$ is of the form

$$(2.4) \quad \varphi(z) \overset{df}{=} E \exp(i\omega) = \exp \left\{ \sum_{k=1}^{\infty} \frac{1}{k} \left( E \exp \left( iz(S_k \right.) \right) - 1 \right\}, \quad z \in R.$$

The following theorem gives sufficient conditions for (2.1) in the case when the distributions of the service times and interarrival times belong to the domain of attraction of the stable distributions.
THEOREM 2.1. Let the generating sequence \((v_k, u_k), k \geq 1\) for the GI/GI/1 queue be such that

\[
\frac{1}{k^{1/\alpha_1}} \sum_{j=1}^{k} v_j \Rightarrow \bar{v}, \quad \frac{1}{k^{1/\alpha_2}} \sum_{j=1}^{k} u_j \Rightarrow \bar{u},
\]

where the random variables \(\bar{v}\) and \(\bar{u}\) are stable distributed, with stable parameters \(\alpha_1\) and \(\alpha_2\), respectively, and \(0 < \alpha_2 < \alpha_1 < 1\). Then

\[
\bar{w}_k \xrightarrow{a.s.} \sup_{0 \leq k < \infty} \left( \sum_{j=1}^{k} v_j - \sum_{j=1}^{k} u_j \right) \equiv \omega.
\]

Proof. Because \(\{v_k\}\) and \(\{u_k\}\) are mutually independent and each of them is the sequence of independent and identically distributed random variables, by the convergences in (2.5) we get

\[
\left( \frac{1}{k^{1/\alpha_1}} \sum_{j=1}^{[kt]} v_j, \frac{1}{k^{1/\alpha_2}} \sum_{j=1}^{[kt]} u_j \right) \Rightarrow \left( L^v(t), L^u(t) \right),
\]

where \(L^v\) and \(L^u\) are mutually independent \(\alpha_1\)- and \(\alpha_2\)-stable Lévy processes, respectively. Hence we get the convergence

\[
\frac{1}{k^{1/\alpha_1}} \sum_{j=1}^{[kt]} v_j - \frac{1}{k^{1/\alpha_2}} \sum_{j=1}^{[kt]} u_j = \frac{k^{1/\alpha_1}}{k^{1/\alpha_2}} \frac{1}{k^{1/\alpha_1}} \sum_{j=1}^{[kt]} v_j - \frac{1}{k^{1/\alpha_2}} \sum_{j=1}^{[kt]} u_j \Rightarrow 0 \cdot L^v(t) - L^u(t) = -L^u(t),
\]

which, in turn, implies

\[
\sum_{j=1}^{k} v_j - \sum_{j=1}^{k} u_j \overset{p}{\rightarrow} -\infty.
\]

Since the convergence in probability of the series of mutually independent random variables implies almost sure convergence, we have the following almost sure convergence: \(\sum_{j=1}^{k} v_j - \sum_{j=1}^{k} u_j \rightarrow -\infty\). This, in turn, gives the assertion of the theorem. \(\blacksquare\)

From the above theorem we get immediately the following corollary:

COROLLARY 2.1. If the generating sequence \((v_k, u_k), k \geq 1\) for the GI/GI/1 queue is such that

\[ v_k \sim S_{\alpha_1}(\sigma_1, \bar{v}) \quad \text{and} \quad u_k \sim S_{\alpha_2}(\sigma_2, \bar{u}), \]

where \(0 < \alpha_2 < \alpha_1 < 1\) and \(\bar{v}, \bar{u} \geq 0\), then the convergences (2.5) and (2.6) hold.
3. AN ASYMPTOTIC OF WAITING TIMES IN HEAVY TRAFFIC

Let \( \{Q_n, n \geq 1\} \) be a sequence of GI/GI/1 queues, where \( Q_n \) is generated by \( \{(v_{n,k}, u_{n,k}) : k \geq 1\} \) and let \( w_{n,k} \) be the waiting time for the service of the \( k \)-th customer in the \( Q_n \) system. In this section we consider an asymptotic of \( w_{n,k} \), as \( k, n \to \infty \), in a heavy traffic situation. An example of the heavy traffic situation is the case when the stability parameters of the interarrival time distributions are greater than or close to the stability parameters of the service time distributions. Let

\[
S_{n,0} = 0, \quad S_{n,k} = \sum_{j=1}^{k} v_{n,j} - \sum_{j=1}^{k} u_{n,j} = \sum_{j=1}^{k} (v_{n,j} - u_{n,j}), \quad k \geq 1, n \geq 1.
\]

Then

\[
w_{n,k+1} = S_{n,k} - \min_{0 \leq j \leq k} S_{n,j}.
\]

Let

\[
W_n(t) = k_n^{-1/\alpha_{n,2}} w_n[k_n t] + 1, \quad t \geq 0,
\]

(3.1)

\[
V_n(t) = \frac{1}{k_n^{1/\alpha_{n,1}}} \sum_{j=1}^{[k_n t]} (v_{n,j} - \bar{v}_n), \quad U_n(t) = \frac{1}{k_n^{1/\alpha_{n,2}}} \sum_{j=1}^{[k_n t]} (u_{n,j} - \bar{u}_n), \quad t \geq 0,
\]

where \( \bar{v}_n \) and \( \bar{u}_n \), \( n \geq 1 \), are nonnegative numbers, \( \{k_n\} \) is a sequence of positive integers tending to infinity and \( 0 < \alpha_{n,1}, \alpha_{n,2} < 1 \).

**Theorem 3.1.** Let the sequences \( \{\alpha_{n,1}\}, \{\alpha_{n,2}\}, \{\bar{v}_n\}, \{\bar{u}_n\} \) and \( \{k_n\} \) be such that

\[
\alpha_{n,1} \to \alpha_1, \quad \alpha_{n,2} \to \alpha_2, \quad 0 < \alpha_1, \alpha_2 < 1,
\]

(3.2)

\[
k_n^{1/\alpha_{n,1} - 1/\alpha_{n,2}} \to a, \quad 0 \leq a < \infty,
\]

(3.3)

\[
k_n^{(\alpha_{n,2} - 1)/\alpha_{n,2}} (\bar{v}_n - \bar{u}_n) \equiv \mu_n \to \mu, \quad |\mu| < \infty.
\]

(3.4)

Furthermore, let

\[
V_n \Rightarrow V, \quad U_n \Rightarrow U
\]

in \( D[0, \infty) \) with Skorokhod \( J_1 \) topology, where \( V \) and \( U \) are \( \alpha_1 \)- and \( \alpha_2 \)-stable Lévy processes, respectively. Then

\[
W_n \Rightarrow W
\]

in \( D[0, \infty) \) with Skorokhod \( J_1 \) topology, where for each \( t \geq 0 \)

\[
W(t) = aV(t) - U(t) + \mu t - \inf_{0 \leq s \leq t} (aV(s) - U(s) + \mu s)
\]

(3.7)

\[
\overset{D}{=} \sup_{0 \leq s \leq t} (aV(s) - U(s) + \mu s).
\]
Proof. Let $f$ be the mapping of $D[0, \infty)$ into $D[0, \infty)$ defined as $f(x)(t) = x(t) - \inf_{0 \leq s \leq t} x(s)$. The mapping $f$ is continuous in $D[0, \infty)$ with Skorokhod $J_1$ topology. Notice that

$$W_n(t) = \frac{1}{k_n^{1/\alpha_n,2}} w_{[k_n t] + 1} = f(X_n)(t),$$

where

$$X_n(t) \overset{df}{=} \frac{1}{k_n^{1/\alpha_n,2}} \sum_{j=1}^{[k_n t]} v_{n,j} - \frac{1}{k_n^{1/\alpha_n,2}} \sum_{j=1}^{[k_n t]} u_{n,j}, \quad t \geq 0, \ n \geq 1.$$ 

But

$$X_n = k_n^{1/\alpha_n,1-1/\alpha_n,2} V_n - U_n + \eta_n,$$

where

$$\eta_n(t) = (\bar{v}_n - \bar{u}_n) \frac{[k_n t]}{k_n} k_n^{1-1/\alpha_n,2}, \quad t \geq 0, \ n \geq 1.$$ 

Using the mutual independence of processes $V_n, U_n$ and $\eta_n$ and the convergences in (5.5) and (5.4), we get

$$(V_n, U_n, \eta_n) \Rightarrow (V, U, \mu e),$$

in $D[0, \infty) \times D[0, \infty) \times D[0, \infty)$ with product Skorokhod $J_1$ topology, where $e(t) = t, t \geq 0$. This and the mutual independence of processes $V, U$ and $e$ give the convergence

$$k_n^{1/\alpha_n,1-1/\alpha_n,2} V_n - U_n + \eta_n \Rightarrow aV - U + \mu e$$

in $D[0, \infty)$ with Skorokhod $J_1$ topology. Now, using the continuity of the mapping $f$ in $D[0, \infty)$ with Skorokhod $J_1$ topology and Theorem 5.1 from Billingsley [2], we get the assertion of the theorem. $lacksquare$

The following remark describes the situations in which $a = 1$ or $0 < a < \infty$ or $a = 0$ in condition (3.3).

Remark 3.1. The following implications hold:

(3.8) If $\alpha_{n,2} - \alpha_{n,1} \to 0$ and $k_n \frac{\alpha_{n,2} - \alpha_{n,1}}{\alpha_{n,1} \alpha_{n,2}} \to c$, $c \in R$, then $a = 1$.

(3.9) If $\ln(k_n) \frac{\alpha_{n,1} - \alpha_{n,2}}{\alpha_{n,1} \alpha_{n,2}} \to c$ and $c = \ln a$, then $0 < a < \infty$.

(3.10) If $\alpha_{n,2} < \alpha_{n,1}$, then $a = 0$. 

Proof. Let
\[ \beta_n = \frac{1}{\alpha_{n,2}} - \frac{1}{\alpha_{n,1}} = -\frac{\alpha_{n,1} - \alpha_{n,2}}{\alpha_{n,1} \alpha_{n,2}}. \]
The implication in (3.8) follows from the equality
\[ k_n^{1/\alpha_{n,1}} / k_n^{1/\alpha_{n,2}} = k_n^{(1/k_n)k_n(-\beta_n)} \]
and from the convergences \( k_n^{1/k_n} \to 1 \) and \( k_n \beta_n \to -c, \ c \in R \). The implication in (3.9) follows from the representation
\[ k_n^{1/\alpha_{n,1}} / k_n^{1/\alpha_{n,2}} = k_n^{-\beta_n} = e^{-\ln(k_n)(-\beta_n)} \]
and the assumed convergence. The implication in (3.10) follows from the representation
\[ k_n^{1/\alpha_{n,1}} / k_n^{1/\alpha_{n,2}} = k_n^{-\beta_n}, \]
where \( \beta_n > 0 \). This completes the proof of the remark.

Corollary 3.1. If \( \{ (v_{n,k}, u_{n,k}) : k \geq 1 \} \), \( n \geq 1 \), are such that
\[ v_{n,k} \sim S_{\alpha_{n,1}}(\sigma_{n,1}^2, \bar{v}_n), \quad u_{n,k} \sim S_{\alpha_{n,2}}(\sigma_{n,2}^2, \bar{u}_n), \]
where \( \alpha_{n,1} \to \alpha, \alpha_{n,2} \to \alpha, \ 0 < \alpha < 1 \), and the convergences (3.3)–(3.5) hold, then the convergence (3.6) holds, where \( V \) and \( U \) are \( \alpha \)-stable Lévy processes.

Corollary 3.2. If \( \{ (v_{n,k}, u_{n,k}) : k \geq 1 \} \), \( n \geq 1 \), are such that
\[ v_{n,k} \sim S_{\alpha}(\sigma_1^2, \bar{v}_n), \quad u_{n,k} \sim S_{\alpha}(\sigma_2^2, \bar{u}_n), \]
then condition (3.5) holds. Furthermore, if condition (3.4) holds with \( \alpha > 0 \), then (3.6) holds with \( a = 1 \), and \( V \) and \( U \) are \( \alpha \)-stable Lévy processes.

Corollary 3.3. If \( \{ (v_{n,k}, u_{n,k}) : k \geq 1 \} \), \( n \geq 1 \), are such that
\[ v_{n,k} \sim S_{\alpha_1}(\sigma_1^2, \bar{v}_n), \quad u_{n,k} \sim S_{\alpha_2}(\sigma_2^2, \bar{u}_n) \]
for each \( t \geq 0 \), where \( 0 < \alpha_2 < \alpha_1 < 1 \), then condition (3.5) holds. Furthermore, if condition (3.4) holds with \( \mu > 0 \), then (3.6) holds with
\[ (3.11) \quad W(t) = \mu t - U(t) - \inf_{0 \leq s \leq t} (\mu s - U(s)) \overset{D}{=} \sup_{0 \leq s \leq t} (\mu s - U(s)), \]
where \( U \) is an \( \alpha \)-stable Lévy process.

Proof. The corollary follows from Theorem 5.1 with \( a = 0 \).
Remark 3.2. If $U$ is an $\alpha$-stable Lévy process with $\alpha = 1/2$ and such that $U(1) \sim S_{1/2}(\sigma, 1)$, then

$$
\sup_{0 \leq s \leq t} \left( \mu s - U(s) \right) \overset{D}{=} \frac{\mu}{\sqrt{\sigma}} \sup_{0 \leq s \leq U(t)} \left( W(s) - \frac{\sqrt{\sigma}}{\mu} s \right),
$$

where $W$ is a Wiener process. Moreover,

$$
\sup_{0 \leq s < \infty} \left( \mu s - U(s) \right) \overset{D}{=} \frac{\mu}{\sqrt{\sigma}} \sup_{0 \leq s < \infty} \left( W(s) - \frac{\sqrt{\sigma}}{\mu} s \right) \equiv M,
$$

and

$$
P(M > x) = \exp \left( -2(\sigma/\mu^2)x \right), \quad x > 0.
$$

Proof. Using Proposition 1.2.11 from [1], p. 14, we get

(3.12) \hspace{1cm} E \exp \left( -sU(1) \right) = \exp \left( -\sqrt{as} \right), \quad s > 0, \quad \text{with} \quad a = 2\sigma.

Now, by Example 1.3.21 in [II], p. 51, we get

(3.13) \hspace{1cm} E \exp \left( -sT(t) \right) = \exp \left( -t\delta \sqrt{2s} \right), \quad s > 0,

where

$$
T(t) = \inf\{ u : W = \delta t \}, \quad 0 < \delta < \infty.
$$

Consequently, from (3.12) and (3.13) we get $\delta \sqrt{2} = \sqrt{a}$, which implies $\delta = \sqrt{\sigma}$. But

$$
\sup_{0 \leq t < \infty} \left( \mu t - T(t) \right) = \sup_{0 \leq t < \infty} \left( \mu t - U(t) \right).
$$

Putting $U(t) = s$ if $W(s) = \delta t$, we get

$$
\sup_{0 \leq t < \infty} \left( \mu t - U(t) \right) = \sup_{0 \leq t < \infty} \left( \frac{\mu}{\delta} W(s) - s \right) = \frac{\mu}{\delta} \sup_{0 \leq t < \infty} \left( W(s) - \frac{\delta}{\mu} s \right) \equiv M.
$$

Using Corollary 5.1 from [3], p. 361, we get

$$
P(M > x) = \exp \left( -2(\sigma/\mu^2)x \right), \quad x > 0.
$$

This completes the proof of the remark.
4. AN ASYMPTOTIC OF STATIONARY WAITING TIME IN HEAVY TRAFFIC

Let $\omega_n$ be the stationary waiting time for the $GI/GI/1$ queue $Q_n$ generated by $\{(v_{n,k}, u_{n,k}), k \geq 1\}$, $n \geq 1$, and let

$$v_{n,k} \sim S_{\alpha_{n,1}}(\sigma_{n,1}, \bar{v}_n), \quad u_{n,k} \sim S_{\alpha_{n,2}}(\sigma_{n,2}, \bar{u}_n),$$

(4.1)

$$0 < \alpha_{n,2} < \alpha_{n,1} < 1.$$  

(4.2)

Let $F_n$ and $F$ be the distribution functions of $u_{n,k} - \bar{u}_n$ and of $u \sim S_{\alpha_2}(\sigma, 0)$, respectively. Below, $\{k_n\}$ represents a sequence of positive integers tending to infinity.

The following theorem gives the asymptotic of the stationary waiting time in a heavy traffic situation defined by conditions (4.3)–(4.5).

**Theorem 4.1.** Let $\{\alpha_{n,1}\}, \{\alpha_{n,2}\}, \{\bar{v}_n\}, \{\bar{u}_n\}$ and $\{k_n\}$ be the sequences such that

1. $0 < \alpha_{n,2} \leq \alpha_2 < \alpha_1 \leq \alpha_{n,1} < 1$,  
2. $\alpha_{n,2} \uparrow \alpha_2$,  
3. $\alpha_{n,1} \downarrow \alpha_1$,

(4.3)

$$k_n^{1/\alpha_{n,1} - 1/\alpha_{n,2}} \to 0,$$

(4.4)

$$\bar{v}_n - \bar{u}_n \to \infty,$$

(4.5)

$$k_n^{(\alpha_{n,2} - 1)/\alpha_{n,2}}(\bar{v}_n - \bar{u}_n) \equiv \mu_n \to \mu, \quad \mu > 0,$$

(4.6)

$$k_n^{(\alpha_{n,2} - 1)/\alpha_{n,2}}(2\bar{v}_n - \bar{u}_n) = a_n < a < \infty,$$

(4.7)

$$F_n(a2^{((1-1)/\alpha_{n,2})}) \leq \mathcal{A}F(a2^{((1-1)/\alpha_2)}), \quad \mathcal{A} < \infty.$$

Furthermore, let

$$V_n \Rightarrow V, \quad U_n \Rightarrow U$$

in $D[0, \infty)$ with Skorokhod $J_1$ topology, where $V$ and $U$ are $\alpha_1$- and $\alpha_2$-stable Lévy processes. Then

$$\frac{1}{k^{1/\alpha_{n,2}}\omega_n} \Rightarrow \sup_{0 \leq t < \infty} \left(\mu t - U(t)\right).$$

(4.9)

**Proof.** By Theorem 2.1 and the assumptions (4.1) and (4.2) we get

$$\omega_n = \sup_{0 \leq k < \infty} S_{n,k} = \sup_{0 \leq t < \infty} S_{n,\lfloor k_n t \rfloor}, \quad n \geq 1.$$
But
\[
\frac{1}{k_n^{1/\alpha_n}} S_n[k_n t] = \frac{1}{k_n^{1/\alpha_n}} \sum_{j=1}^{[k_n t]} v_{n,j} - \frac{1}{k_n^{1/\alpha_n}} \sum_{j=1}^{[k_n t]} u_{n,j}
\]
\[
= \frac{1}{k_n^{1/\alpha_n}} \sum_{j=1}^{[k_n t]} (v_{n,j} - \bar{v}_n) - \frac{1}{k_n^{1/\alpha_n}} \sum_{j=1}^{[k_n t]} (u_{n,j} - \bar{u}_n) \frac{[k_n t]}{k_n^{1/\alpha_n}}
\]
\[
= k_n^{(\alpha_n,2-\alpha_n)/\alpha_n,1/\alpha_n,2} V_n(t) - U_n(t) + (\bar{v}_n - \bar{u}_n) \frac{[k_n t]}{k_n^{1-1/\alpha_n,2}}.
\]

Hence and by the mutual independence of processes $V_n$ and $U_n$, together with the convergence \eqref{eq:convergence} and assumption \eqref{assumption}, we get the convergence
\[
\frac{1}{k_n^{1/\alpha_n,2}} S_n[k_n t] \Rightarrow \mu e - U
\]
in $D[0, \infty)$ with Skorokhod $J_1$ topology.

To complete the proof of the theorem, i.e. to show the convergence
\[
\frac{1}{k_n^{1/\alpha_n,2}} \omega_n \equiv \sup_{0 \leq t < \infty} \frac{1}{k_n^{1/\alpha_n,2}} S_n[k_n t] \Rightarrow \sup_{0 \leq t < \infty} \left( \mu t - U(t) \right),
\]
we need to prove the tightness of the sequence $\{\omega_n/c_n\}$, where $c_n = k_n^{1/\alpha_n,2}$ (see the Heavy Traffic Invariance Principle in \cite{8}).

Let $t_i = 2^i$, $i = 0, 1, 2, \ldots$ Then for any $c > 0$ we have
\[
P\left( \frac{1}{c_n} \omega_n > c \right) = P\left( \frac{1}{c_n} \sup_{0 \leq t < \infty} S_n[k_n t] > c \right)
\]
\[
= P\left( \bigcup_{i=0}^{\infty} \left\{ \frac{1}{c_n} \sup_{t_i \leq t < t_{i+1}} S_n[k_n t] > c \right\} \right) \leq \sum_{i=0}^{\infty} P\left( \frac{1}{c_n} \sup_{t_i \leq t < t_{i+1}} S_n[k_n t] > c \right).
\]

But
\[
\sup_{t_i \leq t < t_{i+1}} S_n[k_n t] = \sup_{t_i \leq t < t_{i+1}} (S_n[k_n t] - S_n[k_n t_i] + S_n[k_n t_i])
\]
\[
= S_n[k_n t_i] + \sup_{t_i \leq t < t_{i+1}} (S_n[k_n t] - S_n[k_n t_i]) = S_n[k_n t_i] + \sup_{0 \leq t < t_{i+1} - t_i} \tilde{S}_n[k_n t]
\]
\[
= S_n[k_n,2^i] + \sup_{0 \leq t < 2^i} \tilde{S}_n[k_n t],
\]
where
\[
\tilde{S}_n[k_n t] = \sum_{j=1}^{[k_n t]} (\bar{v}_{n,j} - \bar{u}_{n,j}).
\]
and the sequences \( \{(\tilde{v}_{n,j}, \tilde{u}_{n,j}), j \geq 1\} \) and \( \{(v_{n,j}, u_{n,j}), j \geq 1\} \) are mutually independent, \( \{(v_{n,j}, u_{n,j}), j \geq 1\} \) being a probabilistic copy of \( \{(v'_{n,j}, u_{n,j}), j \geq 1\} \).

Hence

\[
(4.10) \quad P\left(\frac{1}{c_n} \omega_n > c\right) \leq \sum_{i=0}^{\infty} P\left(\frac{1}{c_n} S_{n,2^i k_n} + \frac{1}{c_n} \sup_{0 \leq t \leq 2^i} \tilde{S}_{n,[k_n t]} > c\right).
\]

But

\[
(4.11) \quad \sup_{0 \leq t \leq 2^i} \tilde{S}_{n,[k_n t]} \leq \sum_{j=1}^{2^i k_n} (\tilde{v}_{n,j} - \bar{v}_n) + 2^i k_n \bar{v}_n = (2^i k_n)^{1/\alpha_n,1} (2^i k_n)^{-1/\alpha_n,1} \sum_{j=1}^{k_n} (\tilde{v}_{n,j} - \bar{v}_n) + 2^i k_n \bar{v}_n \]

\[
\leq \sum_{j=1}^{2^i k_n} (\tilde{v}_{n,j} - \bar{v}_n) + 2^i k_n \bar{v}_n \]

\[
\leq (2^i k_n)^{1/\alpha_n,1} \bar{v}_n + 2^i k_n \bar{v}_n,
\]

where \( \tilde{v}_n \equiv (\tilde{v}_{n,j} - \bar{v}_n) \) and \( \bar{v}_n \) is independent of \( \{(v_{n,j}, u_{n,j}), j \geq 1\} \).

Notice that

\[
(4.12) \quad S_{n,2^i k_n} = \sum_{j=1}^{2^i k_n} (v_{n,j} - \bar{v}_n) - \sum_{j=1}^{2^i k_n} (u_{n,j} - \bar{u}_n) + 2^i k_n (\bar{v}_n - \bar{u}_n)
\]

\[
= (2^i k_n)^{1/\alpha_n,1} (2^i k_n)^{-1/\alpha_n,1} \sum_{j=1}^{k_n} (v_{n,j} - \bar{v}_n)
\]

\[
- (2^i k_n)^{1/\alpha_n,2} (2^i k_n)^{-1/\alpha_n,2} \sum_{j=1}^{k_n} (u_{n,j} - \bar{u}_n) + 2^i k_n (\bar{v}_n - \bar{u}_n)
\]

\[
\equiv (2^i k_n)^{1/\alpha_n,1} v_n - (2^i k_n)^{1/\alpha_n,2} u_n + 2^i k_n (\bar{v}_n - \bar{u}_n),
\]

where \( v_n \equiv (v_{n,j} - \bar{v}_n) \equiv \tilde{v}_n, \quad u_n \equiv (u_{n,j} - \bar{u}_n) \) and \( v_n, \tilde{v}_n, u_n, \bar{u}_n \) are mutually independent. Hence, by \((4.10)-(4.11)\), we get

\[
(4.13) \quad P\left(\frac{1}{c_n} \omega_n > c\right) \leq \sum_{i=0}^{\infty} P\left(k_n^{1/\alpha_n,1-1/\alpha_n,2} 2^{i/\alpha_n,1} v_n - 2^{i/\alpha_n,2} u_n + k_n^{1/\alpha_n,1-1/\alpha_n,2} 2^{i/\alpha_n,1} \tilde{v}_n + 2^{i/\alpha_n,2} \tilde{v}_n > c\right).
\]

But

\[ v_n + \tilde{v}_n = 2^{1/\alpha_n,1} 2^{1-1/\alpha_n,1} (v_n + \tilde{v}_n) = 2^{1/\alpha_n,1} v_n. \]
Hence, since $\beta_n = 1/\alpha_{n,2} - 1/\alpha_{n,1}$, we get

\[(4.14) \quad P\left(\frac{1}{\ell_n} \omega_n > c \right) \leq \sum_{i=0}^{\infty} P\left(k_n^{-\beta_n}(2^{1/\alpha_{n,1}})^{i+1} u_n - 2^{i/\alpha_{n,2}} u_n + 2^i k_n^{1-1/\alpha_{n,2}} (2\nu_n - \overline{u}_n) > c \right)
\]

\[
\leq \sum_{i=0}^{\infty} P\left(k_n^{-\beta_n}(2^{1/\alpha_{n,1}})^{i+1} u_n - 2^{i/\alpha_{n,2}} u_n + 2^i a > c \right)
\]

\[
= \sum_{i=0}^{\infty} P\left(v_n - k_n^{-\beta_n} 2^{-1/\alpha_{n,1}} u_n - 2^{i(1-1/\alpha_{n,1})} 2^{-1/\alpha_{n,1}} a > k_n^{-\beta_n} 2^{-(i+1)/\alpha_{n,1}} c \right)
\]

\[
\leq \sum_{i=0}^{\infty} P\left(v_n - k_n^{-\beta_n} 2^{-1/\alpha_{n,1}} (2^{i} x - 2^{i(1-1/\alpha_{n,1})} a) > 0 \right).
\]

Let $f_n$ be the density of the distribution function of $u_n$, and assume that $A_{n,i}$ and $B_{n,i}$ are the following sets in $R$:

\[A_{n,i} = \{ x : 2^{i} x - 2^{i(1-1/\alpha_{n,1})} a < 0 \} = \{ x : x < 2^{i(1-1/\alpha_{n,2})} a \},\]

\[B_{n,i} = R - A_{n,i}.
\]

Furthermore, let

\[C_{n,i} = \int_{A_{n,i}} P\left(v_n - k_n^{-\beta_n} 2^{-1/\alpha_{n,1}} (2^{i} x - 2^{i(1-1/\alpha_{n,1})} a) > 0 \right) f_n(x) dx,
\]

\[D_{n,i} = \int_{B_{n,i}} P\left(v_n - k_n^{-\beta_n} 2^{-1/\alpha_{n,1}} (2^{i} x - 2^{i(1-1/\alpha_{n,1})} a) > 0 \right) f_n(x) dx.
\]

Then, using (4.14), we get

\[(4.15) \quad P\left(\frac{1}{\ell_n} \omega_n > c \right) \leq \sum_{i=0}^{\infty} C_{n,i} + \sum_{i=0}^{\infty} D_{n,i}.
\]

Since $v_n \geq 0$, we have

\[
\sum_{i=0}^{\infty} C_{n,i} = \sum_{i=0}^{\infty} \int_{A_{n,i}} f_n(x) dx = \sum_{i=0}^{\infty} P(u_n \leq a 2^{i(1-1/\alpha_{n,2})})
\]

\[
= \sum_{i=0}^{\infty} F_n(a 2^{i(1-1/\alpha_{n,2})}) \leq \sum_{i=0}^{\infty} F_n(z_{n,i}).
\]

Now, by condition (4.17) and the inequality $z_{n,i} = a 2^{i(1-1/\alpha_{n,2})} \leq a 2^{i(1-1/\alpha_2)} \equiv z_i$, we get

\[(4.16) \quad \sum_{i=0}^{\infty} C_{n,i} \leq \bar{a} \sum_{i=0}^{\infty} F(z_i) = \bar{a} \sum_{i=0}^{\infty} F(z_i) \exp(z_i^{-\alpha_2}) \exp(-z_i^{-\alpha_2}).
\]
Theorem 1 in Feller [3], p. 448, says that if $G_\alpha$ is the distribution function of the strictly stable distribution with $0 < \alpha < 1$, then the following convergence holds:

$$G_\alpha(x) \exp(x^{-\alpha}) \to 0 \quad \text{as} \quad x \to 0. \tag{4.17}$$

Hence and by the convergence, $z_i \to 0$, the sequence $F(z_i) \exp(z_i^{-\alpha_2})$ is bounded, say by $b < 0$. Therefore,

$$\sum_{i=0}^{\infty} C_{n,i} \leq b \sum_{i=0}^{\infty} \exp(-z_i^{-\alpha_2}). \tag{4.18}$$

This and the relations

$$z_i^{-\alpha_2} = a^{-\alpha_2} (2^{i(1-1/\alpha_2)})^{-\alpha_2} = a^{-\alpha_2} 2^{i(1-\alpha_2)} \geq a^{-\alpha_2} 2^{1-\alpha_2}$$

give

$$\sum_{i=0}^{\infty} \exp(-z_i^{-\alpha_2}) \leq \sum_{i=0}^{\infty} \left( \exp(-a^{-\alpha_2} 2^{1-\alpha_2}) \right)^i = (1 - \exp(-a^{-\alpha_2} 2^{1-\alpha_2}))^{-1} < \infty,$$

which implies that the series $\sum_{i=0}^{\infty} C_{n,i}$ is uniformly bounded with respect to $n$ by the finite series $b \sum_{i=0}^{\infty} \exp(-z_i^{-\alpha_2})$.

Now, using the relation $P(v_n > x) \approx x^{-\alpha_n,1}$ for large $x$ to the second component of (4.14), we get

$$\sum_{i=0}^{\infty} D_{n,i} \approx \sum_{i=0}^{\infty} \int_{B_{i,n}} P(v_n > k_n^{\beta_n} 2^{-1/\alpha_n,1} (2^{i \beta_n x} - 2^{i(1-1/\alpha_n,1)} a)) f_n(x) \, dx$$

$$\approx \sum_{i=0}^{\infty} \int_{B_{i,n}} (k_n^{\beta_n} 2^{-1/\alpha_n,1} (2^{i \beta_n x} - 2^{i(1-1/\alpha_n,1)} a))^{-\alpha_n,1} f_n(x) \, dx$$

$$= 2 k_n^{-\beta_n \alpha_{n,1}} \sum_{i=0}^{\infty} \int_{B_{i,n}} (2^{i \beta_n x} - 2^{i(1-1/\alpha_n,1)} a)^{-\alpha_n,1} f_n(x) \, dx$$

$$\leq 2 k_n^{-\beta_n \alpha_{n,1}} \sum_{i=0}^{\infty} \int_{B_{i,n}} (2^{i \beta_n x} - a)^{-\alpha_n,1} f_n(x) \, dx$$

$$\leq 2 k_n^{-\beta_n \alpha_{n,1}} \sum_{i=0}^{\infty} \int_{B_{i,n}} (2^{-\beta_n \alpha_{n,1}})^i |x - 2^{-i \beta_n a}|^{-\alpha_n,1} f_n(x) \, dx$$

$$\leq 2 k_n^{-\beta_n \alpha_{n,1}} \sum_{i=0}^{\infty} (2^{-\beta_n \alpha_{n,1}})^i E|u_n - a|^{-\alpha_n,1}$$

$$\leq 2 k_n^{-\beta_n \alpha_{n,1}} E|u_n - a|^{-\alpha_n,1} (1 - 2^{-\beta_n \alpha_{n,1}})^{-1}.$$
The above is finite because of the equation (25.5) in Sato [2], p. 162, which gives
\[ E u^{-\alpha_n,1} = \gamma_n^{-\alpha_n,1/\alpha_n,2} \frac{\Gamma(1 + \alpha_n,1/\alpha_n,2)}{\Gamma(1 + \alpha_n,1)} = \tilde{c} < \infty, \quad \text{where} \quad \gamma_n = O(1). \]

Since \( \alpha_n,2 \leq \alpha_2 \leq \alpha_1 \leq \alpha_n,2 \), we have
\[ -\beta_n \alpha_n,1 = 1 - \alpha_n,1/\alpha_n,2 \leq 1 - \alpha_1/\alpha_2, \]
which gives \( 2^{\alpha_n,1/\alpha_n,2} \geq 2^{\alpha_1/\alpha_2} \). Hence for some finite \( D < \infty \) we have
\[ (4.19) \quad \sum_{i=0}^{\infty} D_{n,i} \leq D \sum_{i=0}^{\infty} (2^{-\beta_n \alpha_n,1})^i \leq D \sum_{i=0}^{\infty} 2^{1-\alpha_1/\alpha_2} < \infty, \]
which implies that the series \( \sum_{i=0}^{\infty} D_{n,i} \) is uniformly bounded with respect to \( n \) by the convergent series. This and (4.18) imply that we can pass with \( n \) under the sum in (4.19), so we get
\[ (4.20) \quad \lim_{n} \frac{1}{c_n} \omega_n > c \leq \sum_{i=0}^{\infty} P(\kappa_{\alpha_n} (2^{1/\alpha_n,1})^{i+1} v_n - 2^{1/\alpha_n,2} u_n + 2^i a_n > c) \]
\[ = \sum_{i=0}^{\infty} P(-2^{i/\alpha_2} u + 2^i a > c), \]
where the random variable \( u \) is strictly \( \alpha_2 \)-stable. But
\[ \sum_{i=0}^{\infty} P(-2^{i/\alpha_2} u + 2^i a > c) \leq \sum_{i=0}^{\infty} P(u < 2^{i(1-1/\alpha_2)}) \]
\[ = \sum_{i=0}^{\infty} F(x_i) \exp(x_i^{-\alpha_2}) \exp(-x_i^{-\alpha_2}) \leq \tilde{d} \sum_{i=0}^{\infty} \exp(-2^{i(1-\alpha_2)}) < \infty, \]
where in view of (4.19) we have
\[ \sup_{i} F(x_i) \exp(x_i^{-\alpha_2}) < \tilde{d} < \infty. \]
This implies that the series \( \sum_{i=0}^{\infty} P(-2^{i/\alpha_2} u + 2^i a > c) \) is uniformly bounded with respect to \( c \) by the convergent series \( \tilde{d} \sum_{i=0}^{\infty} \exp(-2^{i(1-\alpha_2)}) \). Therefore, passing with \( c \to \infty \) under the sum on the right-hand side of (4.20), we get
\[ \lim_{c \to \infty} \lim_{n} P\left(\frac{1}{c_n} \omega_n > c\right) = 0, \]
which, in turn, implies that the sequence \( \left\{ \frac{1}{c_n} \omega_n \right\} \) is tight. This completes the proof of the theorem. ■
Remark 4.1. Condition (4.7) holds true when the distribution function of $(u_n - \bar{u}_n)\frac{2}{\sigma_n}$ is $F$.

By Remark 4.2 we get the following corollary.

Corollary 4.1. If $U$ is an $\alpha$-stable Lévy process with $\alpha = 1/2$ and such that $U(1) \sim S_{1/2}(\sigma, 1)$, then

$$
\sup_{0 \leq s < \infty} \left( \mu s - U(s) \right) \overset{D}{=} \frac{\mu}{\sigma} \sup_{0 \leq s < \infty} \left( W(s) - \sqrt{\frac{\sigma}{\mu}} s \right) \equiv M,
$$

where $W$ is a Wiener process and

$$
P(M > x) = \exp \left( -2(\sigma/\mu^2)x \right), \quad x \geq 0.
$$

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