STRONG LAW OF LARGE NUMBERS FOR RANDOM VARIABLES WITH MULTIDIMENSIONAL INDICES

BY

AGNIESZKA M. GDULA (LUBLIN) AND ANDRZEJ KRAJKA (LUBLIN)

Abstract. Let \( \{X_n; n \in V \subset \mathbb{N}^2\} \) be a two-dimensional random field of independent identically distributed random variables indexed by some subset \( V \) of lattice \( \mathbb{N}^2 \). For some sets \( V \) the strong law of large numbers

\[
\lim_{n \to \infty; n \in V} \frac{\sum_{k \in V; \|k\|_\infty \leq n} X_k}{\|n\|} = \mu \text{ a.s.}
\]

is equivalent to

\[
EX_1 = \mu \quad \text{and} \quad \sum_{n \in V} P[|X_1| > |n|] < \infty.
\]

In this paper we characterize such sets \( V \).

2010 AMS Mathematics Subject Classification: Primary: 60F15; Secondary: 60G50, 60G60.

Key words and phrases: Strong law of large numbers, sums of random fields, multidimensional index.

1. INTRODUCTION

Let \( \{X_n; n = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d\} \) be a family of independent identically distributed random variables indexed by \( \mathbb{N}^d \)-vectors, and let us put

\[
S_{\underline{m}} = \sum_{\underline{k} \leq \underline{m}} X_{\underline{k}}, \quad \underline{n} \in \mathbb{N}^d,
\]

where \( \underline{k} \leq \underline{m} \) iff \( k_j \leq m_j, j = 1, 2, \ldots, d \). In this paper we investigate the almost sure behavior of the sums \( S_{\underline{n}} \) when \( |\underline{n}| = \prod_{j=1}^d n_j \to \infty \), i.e., the strong law of large numbers (SLLN).
In the case of \( d = 1 \) the classical Kolmogorov’s SLLN result asserts that

\[
\frac{S_n}{n} \rightarrow \mu \text{ a.s.}
\]

is equivalent to

\[
EX = \mu, \quad E|X| < \infty,
\]

where here and in what follows \( X = X_1 \). The proof of Kolmogorov’s SLLN is based on the fact that for \( d = 1 \) the relation (1.1) is equivalent to

\[
\forall \epsilon > 0 \quad P \left( \left| \frac{S_n}{n} - \mu \right| \geq \epsilon, \text{ infinitely often} \right) = 0.
\]

This is not the case if \( d > 1 \); since (1.1) is weaker than (1.3) even for i.i.d. random fields. Fortunately, Smythe [8] (Proposition 3.1, p. 913) observed that for i.i.d. random fields satisfying \( E|X| < \infty \) (this is obviously necessary for (1.1) to hold) relations (1.1) and (1.3) are equivalent. Moreover, Smythe [2] proved that (1.3) is equivalent to

\[
EX = \mu, \quad E|X|(\log_+ |X|)^{d-1} < \infty.
\]

Let us notice that the sufficiency of (1.4) was obtained in a more general setting of non-commutative ergodic transformations much earlier by Dunford [1] (see also Zygmund [10]).

It was Gabriel [2] who first observed that if we replace the whole lattice \( \mathbb{N}^d \) with a sector \( \mathbb{V}_d = \{ n : \theta n_i \leq n_j < \theta^{-1} n_i, i \neq j, i, j = 1, 2, \ldots, d \} \), then the situation is completely analogous to the one-dimensional case, namely \( E|X| < +\infty \) if and only if

\[
\lim_{V} \frac{S_n}{n} \text{ exists a.s.,}
\]

and then the limit is, of course, equal to \( EX \). Here \( \lim_{V} c_n = c_0 \) means that for every \( \epsilon > 0 \) we have \( |c_n - c_0| < \epsilon \) for all but a finite number of \( n \in V \). (We refer also to [3] for the sectorial Marcinkiewicz–Zygmund laws of large numbers.)

Later, Klesov and Rychlik [6] and Indlekofer and Klesov [4] proved that for a large class of subsets \( V \subset \mathbb{N}^d \) the SLLN along \( V \), i.e.

\[
\lim_{V} \frac{S_n}{n} = EX \text{ a.s.,}
\]

is equivalent to

\[
\sum_{n \in V} P[|X| \geq \lfloor n \rfloor] < +\infty.
\]
The relation (1.6) can be written in terms of the Dirichlet divisors. For $V \subset \mathbb{N}^d$ let us define

$$\tau_V(n) = \text{card}\{k \in V : |k| = n\}, \quad T_V(x) = \sum_{k \leq x} \tau_V(k).$$

By the very definition we have

$$\sum_{n \in V} P[|X| \geq |n|] = ET_V(|X|),$$

hence (1.6) can be verified if we are able to determine the asymptotics of $T_V$. For example, using methods of number theory, one can show that

$$T_{N^d}(x) \sim nw^d_{d-1}(\log x),$$

where $w_{k-1}$ is a polynomial of degree $k - 1$. This in turn leads to (1.4) as a necessary and sufficient condition for (1.1) and rediscovers a result of Smythe [8].

In fact, the results of [4] and [6] were proved for the case $d = 2$ only. We shall describe them briefly. Let us introduce the following classes of nonnegative functions on $\mathbb{N}$:

$$F_1 \overset{\text{def}}{=} \{f : f \nearrow, x \leq f(x), f(x)/x \nearrow\},$$

$$G_1 \overset{\text{def}}{=} \{g : g \nearrow, g(x) \leq x, g(x)/x \searrow\},$$

$$F_2 \overset{\text{def}}{=} \{f : f \text{ is nondecreasing}, x \leq f(x)\},$$

$$G_2 \overset{\text{def}}{=} \{g : g \text{ is nondecreasing}, g(x) \leq x\}.$$

By $C(F_i, G_i), i = 1, 2$, we will denote the class of subsets $V \subset \mathbb{N}^2$ of the form

$$V = V(f, g) = \{(n_1, n_2) : g(n_1) \leq n_2 \leq f(n_1)\},$$

where $f \in F_i, g \in G_i$. Then the main result of [3] states that the class $C(F_1, G_1)$ consists of good sets, i.e. such that (1.5) is equivalent to (1.6), while the paper [5] proves that a larger class $C(F_2, G_2)$ has this property as well.

The purpose of the present paper is to indicate some other classes of subsets of $\mathbb{N}^2$, which are determined by classes of functions $F_j$ and $G_j$, exhibiting less regularity in comparison with $C(F_2, G_2)$, but still containing $C(F_2, G_2)$. In the next section we provide three theorems, each exploiting a different direction, as Example 2.1 shows:

(i) We smooth out the boundaries from up and down and evaluate the difference of series (1.6) for these boundaries.

(ii) We introduce the usual order for the boundaries with a finite number of oscillations.
(iii) We smooth on the boundaries from the bottom and evaluate the measure of area between the smoothed and original boundaries.

Throughout the paper, $c$ denotes the generic constants different in different places, perhaps. All functions in the families $F$ and $G$ considered in this paper always satisfy additionally $f(x) \geq x, x \in \mathbb{R}_+$, and $0 < g(x) \leq x, x \in \mathbb{R}_+$, respectively. We will use the inverse function for not necessarily strictly monotone and continuous functions putting $f^{-1}(y) = \inf \{x \in \mathbb{R}_+ : f(x - 0) \leq y \leq f(x + 0)\}$ and $f^{-1}(y) = \sup \{x \in \mathbb{R}_+ : f(x - 0) \leq y \leq f(x + 0)\}$. Furthermore, for an arbitrary graph $\Gamma = \{(x, f(x)) : x \in X\}$, where $X \subset \mathbb{R}$, we define the $\mathbb{N}^2$ boundary of $\Gamma$ by

$$\partial \Delta_f = \{(i, j) \in \mathbb{N}^2 : \exists (i_1, j_1), (i_2, j_2) \in \{(i, j), (i+1, j), (i, j+1), (i+1, j+1)\} f(i_1) < j_1, f(i_2) > j_2\}$$

(obviously, this definition obeys the case when $f$ is a function). In the whole paper we note $x \vee y = \max\{x, y\}, x \wedge y = \min\{x, y\}, \log_+ x = \max\{\log x, 0\}$, and $\log x$ denotes the natural logarithm.

2. MAIN RESULTS

For an arbitrary function $f \in \mathbb{R}^\mathbb{R}_+$, we put

$$\underline{f}(x) = \inf_{u \geq x} f(u), \quad \overline{f}(x) = \sup_{0 \leq u \leq x} f(u).$$

It is easy to check that

(i) $\underline{f}(x)$ is nondecreasing, $\overline{f}(x)$ is nondecreasing,

(ii) $\underline{f}(x) \leq f(x) \leq \overline{f}(x), x \in \mathbb{R}_+$,

(iii) for $f(x)$ nondecreasing or $f(x)$ nonincreasing, $\underline{f}(x) = f(x) = \overline{f}(x)$.

Furthermore, for two functions $f, g$ we put

$$\nabla = \nabla(f, g) = V(\overline{f}, \overline{g}),$$

$$\overline{\nabla} = \nabla(f, g) = V(\underline{f}, \underline{g})$$

(for fixed $f, g$ we will often omit arguments), and for arbitrary families of the functions $F$ and $G$ let us define

$$C(F, G) = \{V(f, g) : f \in F, g \in G\},$$

$$C(F, G) = \{\nabla(f, g) : f \in F, g \in G\}.$$

Moreover, let us define the families of the functions $\{F_3, G_3\}$ as follows:

$$F_3 = \left\{ f : \int_0^\infty \frac{\overline{f}(x) \vee e}{x \vee 1} dx < \infty \right\}, \quad G_3 = \left\{ g : \int_0^\infty \frac{\underline{g}(x) \vee e}{x \vee 1} dx < \infty \right\}.$$
**Theorem 2.1.** The class $C(F_3, G_3)$ consists of good sets.

Let $B_f(y)$ denote the minimal family of connected subsets of the set $\{(x, y) : f(x) < y\}$ (minimal means that for every $B_1 \in B_f(y), B_2 \in B_f(y), B_1 \neq B_2, B_1 \cup B_2$ is disconnected). Let us note that all sets of the family $B_f(y)$ are subsets $[0, y] \times \{y\}$. Furthermore, let $K_f(y) := \text{card}\{B_f(y)\}$. Let us define

$$F_4 = \{f : \sup_{n \in \mathbb{N}} K_f(n) < \infty\}, \quad G_4 = \{g : \sup_{n \in \mathbb{N}} K_g(n) < \infty\}.$$

**Theorem 2.2.** The class $C(F_4, G_4)$ consists of good sets.

Now we consider the families:

$$F_6 = \left\{ f : \forall x, y \in (f(x), f(x)) \cap \mathbb{N} \left\{ [y - f(x)] \log_+ (x[y - f(x)]) \leq cy \right. \right. \left. \left. \text{or } [f^{-1}(y) - x] \log_+ (y[f^{-1}(y) - x]) \leq cx \right\} \right\},$$

$$G_6 = \left\{ g : \forall x, y \in (g(x), g(x)) \cap \mathbb{N} \left\{ [g(x) - y] \log_+ (x[g(x) - y]) \leq cy \right. \right. \left. \left. \text{or } [g^{-1}(y) - x] \log_+ (y[g^{-1}(y) - x]) \leq cx \right\} \right\},$$

$$F_7 = \left\{ f : \forall x, y \in (f(x), f(x)) \cap \mathbb{N} \left\{ [f^{-1}(y) - f^{-1}(y)] \log_+ (y[f^{-1}(y) - f^{-1}(y)]) \leq cf^{-1}(y) \right\} \right\},$$

$$G_7 = \left\{ g : \forall x, y \in (g(x), g(x)) \cap \mathbb{N} \left\{ [g^{-1}(y) - g^{-1}(y)] \log_+ (y[g^{-1}(y) - g^{-1}(y)]) \leq cg^{-1}(y) \right\} \right\}.$$

**Theorem 2.3.** The class $C(F_5, G_5)$ consists of good sets.

It is obvious that if $F \subset F', G \subset G'$, and the class $C(F', G')$ consists of good sets, then the class $C(F, G)$ consists also of good sets.

**Remark 2.1.** The following inclusions are true:

$$F_6 \cup F_7 \subset F_5, \quad G_6 \cup G_7 \subset G_5.$$

Because for $f$ nondecreasing and $g$ nondecreasing we have $f = \overline{f}, g = g = \overline{g}$ and $K_f(y) = 1, K_g(y) = 1$, we get

**Corollary 2.1.** The following inclusions are true:

$$F_1 \subset F_2 \subset F_i \quad \text{and} \quad G_1 \subset G_2 \subset G_i \quad \text{for } i = 3, 4, 5, 6, 7.$$

Therefore, all our Theorems 2.1, 2.2 generalize the main results of [3] and [6].
EXAMPLE 2.1. We will consider the class of functions

\[ f(x) = u(x) + g(x) \cos(h(x)\pi) \]

for nondecreasing positive functions \( g \) and \( u \), with \( u(x) \geq x \), and an arbitrary function \( h \). Notice that we always have \( \bar{f}(x) = u(x) + g(x) \) and \( f(x) = u(x) \).

(i) If \( u(x) = 2^x(\log_2 x)^2 \), \( g(x) = 2^x \), \( h(x) = 2^x(\log_2 x)^2 \), \( x \in \mathbb{R} \), then the assumptions of Theorem 2.1 are satisfied, but those of Theorems 2.2 and 2.3 fail.

(ii) If \( u(x) = x \), \( g(x) = x \), \( h(x) = (x - 2^k)/2^{k-1} \), \( x \in \mathbb{R} \), \( k = \lceil \log_2 x \rceil \), then the assumptions of Theorem 2.2 hold, but those of Theorems 2.1 and 2.3 fail.

(iii) If \( u(x) = x \), \( g(x) = x/\log x \), \( h(x) = 2^x \), \( x \in \mathbb{R} \), then the assumptions of Theorem 2.3 are satisfied, but those of Theorems 2.1 and 2.2 fail.

3. PROOFS

Proof of Theorem 2.1. From Theorem 1 in [4] we infer that for arbitrary families of the functions \( F, G \) the conditions for both the classes \( C(F, G) \) and \( \overline{C}(F, G) \) to consist of good sets are satisfied, i.e.

\[ (\sum_{n \in V} P|X| \geq |n|) < \infty \text{ and } EX = \mu \Leftrightarrow \lim_{V} \frac{S_n}{|n|} = \mu, \]

and

\[ (\sum_{n \in V} P|X| \geq |n|) < \infty \text{ and } EX = \mu \Leftrightarrow \lim_{V} \frac{S_n}{|n|} = \mu. \]

If additionally we show that, for every fixed \( f \in F_3, g \in G_3 \),

\[ \sum_{n \in V \setminus V} P|X| \geq |n| < \infty, \]

then the assertion follows from the chain of implications

\[ (\sum_{n \in V} P|X| \geq |n|) < \infty \text{ and } EX = \mu \Leftrightarrow (\sum_{n \in V} P|X| \geq |n|) < \infty \text{ and } EX = \mu \]

\[ \implies \left( \lim_{V} \frac{S_n}{|n|} = \mu \right) \implies \left( \lim_{V} \frac{S_n}{|n|} = \mu \right) \implies \left( \lim_{V} \frac{S_n}{|n|} = \mu \right) \]

\[ (\sum_{n \in V} P|X| \geq |n|) < \infty \text{ and } EX = \mu \Leftrightarrow (\sum_{n \in V} P|X| \geq |n|) < \infty \text{ and } EX = \mu, \]

so that it is enough to prove (3.1). From the above considerations we may and do assume that \( EX = \mu \), i.e. \( E|X| < \infty \).
Because for each nonincreasing function \( h \) and nondecreasing \( t \) we have

\[
\sum_{n=1}^{\infty} h(n) \leq \int h(x) \land h(1) \, dx, \quad \sum_{n \in \partial \Delta_1} P(|X| \geq |n|) \leq E \sqrt{|X|}
\]

(for the last inequality see the proof of Lemma 2 in [3]), and

\[
\sum_{n \in V \setminus V} P(|X| \geq |n|) \leq \sum_{n \in V \setminus V} E|X|/|n|,
\]

we obtain

\[
\sum_{n \in V \setminus V} P(|X| \geq |n|) \leq E|X| \
\int \int_{\{x \in \mathbb{R}^2 : f(x_1) \leq x_2 \leq f(x_1)\}} \frac{1}{(x_1 \lor 1)(x_2 \lor 1)} \, dx_1 \, dx_2 
\]

\[
+ E|X| \int \int_{\{x \in \mathbb{R}^2 : g(x_1) \leq x_2 \leq g(x_1)\}} \frac{1}{(x_1 \lor 1)(x_2 \lor 1)} \, dx_1 \, dx_2 
\]

\[
+ \sum_{n \in \partial \Delta_f} P(|X| \geq |n|) + \sum_{n \in \partial \Delta_g} P(|X| \geq |n|) 
\]

\[
+ \sum_{n \in \partial \Delta_f} P(|X| \geq |n|) + \sum_{n \in \partial \Delta_g} P(|X| \geq |n|) 
\]

\[
\leq E|X|I_1 + E|X|I_2 + 4E \sqrt{|X|}, \text{ say.}
\]

Now we show how to evaluate \( I_1 \).

First we remark that because for \( 0 \leq a \leq b < \infty \) we have

\[
\int_a^b \frac{1}{x \lor 1} \, dx = \begin{cases} 
\log(b/a) & \text{if } 1 \leq a \leq b, \\
\log(b) + (1 - a) & \text{if } a < 1 \leq b, \\
b - a & \text{if } a \leq b \leq 1,
\end{cases}
\]

and for \( a < 1 \) we get \( \log \frac{b \lor e}{a \lor 1} \geq 1 \), the following inequality holds true:

\[
\int_a^b \frac{1}{x \lor 1} \, dx \leq 2 \log \frac{b \lor e}{a \lor 1}.
\]

Therefore,

\[
I_1 \leq \int_0^{\infty} f(x_1) \frac{1}{x_2 \lor 1} \, dx_2 \int_0^{\infty} f(x_1) \frac{1}{x_1 \lor 1} \, dx_1 \leq 2 \int_0^{\infty} \frac{\log \frac{f(x) \lor e}{f(x) \lor 1}}{x \lor 1} \, dx < \infty,
\]

and similarly for \( I_2 < \infty \). \( \blacksquare \)
2.2 Let us notice that the functions $f$ and $g$ from the families $F_4$ and $G_4$, respectively, can be discontinuous. If, e.g., $f(x_0 - 0) = y_0 < y_1 = f(x_0 + 0)$, then we “complete” the definition putting $f(x_0) = [y_0, y_1]$ (the whole interval $[y_0, y_1]$). Obviously, at this moment $\Gamma = \{(x, f(x)), x \in \mathbb{R}\}$ is not a function, but a continuous graph, and $f$ is a relation. However, we will write later “function $f$”, so that it does not cause misunderstanding. We say that the piecewise continuous graph $\{(x, f(x)), x \in X\}$ for $X \subset \mathbb{R}$ satisfies the condition $G$ if

**CONDITION G.** If $\{(x, f(x)), x \in (x_0, x_1)\}$ and $\{(x, f(x)), x \in (x_2, x_3)\}$ are two pieces where the graph is continuous and $x_1 \leq x_2$, then $f(x_0) \leq f(x_3)$.

For such graphs we have

**PROPOSITION 3.1.** Let $\{(x, f(x)), x \in X\}$, where $X \subset \mathbb{R}$, be a piecewise nonincreasing graph satisfying the condition $G$. Then

\[
\sum_{(i,j) \in \partial \Delta_f} P[|X| > ij] \leq 4E|X|.
\]

**Proof of Proposition 3.1.** By $Q(i, j)$ we denote the square $\{(x, y) \in \mathbb{R}^2 : i < x \leq i + 1, j \leq y < j + 1\}$.

Let us consider one piece of the graph $\Gamma = \{(x, f(x)), x \in (x_0, x_1)\}$ on which the graph is continuous (and it is not continuous or even does not exist at $x_1$).

The boundary of this piece of the graph can be expressed as a subset $P_1$ (may be empty) of the path $P = [(i, j), \ldots, (i + k, j - l)]$ for some positive integers $i, j, k, l$, where if $(i_1, j_1)$ and $(i_2, j_2)$ are subsequent points, then $(i_2, j_2)$ is equal to $(i_1 + 1, j_1)$ or $(i_1, j_1 - 1)$, or $(i_1 + 1, j_1 - 1)$ according to the way the graph $\Gamma$ “goes out” from $Q(i_1, j_1)$ and “enters” $Q(i_2, j_2)$. If the graph $\Gamma$ does not “enter” the interior $Q(i_2, j_2)$, then $(i_2, j_2) \notin P_1$, but obviously $(i_2, j_2) \in P$.

For such paths $P$ and $P_1$ we construct a function $H$ defined on $\Delta_f$ and taking values in $\{(x, 1) : x \in \mathbb{N}\} \cup \{(1, y) : y \in \mathbb{N}\}$ as follows:

\[
H((i_1, j_1)) = (i_1, 1),
\]

\[
H((i_k, j_k)) = \begin{cases} (i_k, 1) & \text{if } i_k > i_{k-1}, \\ (1, j_k) & \text{if } i_k = i_{k-1}. \end{cases}
\]

On the piece $(x_0, x_1)$ we have

\[
H(\Delta_{f_{|x \in (x_0, x_1)}}) \subset \{(i, 1), (i+1, 1), \ldots, (i+k, 1), (1, j), (1, j-1), \ldots, (1, j-l)\},
\]

and $H$ is the injective function (in this area), where $f_{|x \in (x_0, x_1)}$ denotes the restriction of the function $f$ to the interval $(x_0, x_1)$. Obviously, because for every point...
Without loss of generality we assume \((i, j) \in (\mathbb{N}\backslash\{0\})^2\) we have \(ij > \max\{i, j\}\), it follows that

\[
(3.3) \quad \sum_{(i,j) \in \Delta f, x \in (x_0, x_1)} P[|X| > ij] \leq \sum_{(i,j) \in H(\Delta f, x \in (x_0, x_1))} P[|X| > ij].
\]

It may happen then that one continuous piece of the graph \(\Gamma\) has a path of boundaries \([(i, j), \ldots, (i + k, j - l)]\), whereas the next continuous piece of the graph contains a point \((i + k, j)\), and in this case the projection \(H\) may transform \((i + k, j)\) into the existing point \((i + k, 1)\) or \((1, j)\); consequently,

\[
(3.4) \quad \sum_{(i,j) \in \partial f} P[|X| > ij] \leq 2 \sum_{(i,j) \in H(\partial f)} P[|X| > ij] \leq 4 \sum_{i=1}^{\infty} P[|X| > i] = 4E|X|,
\]

which completes the proof. ■

**Proof of Theorem** 2.2. Without loss of generality we assume \(EX = 0\).

We consider only the sector \(\{ (m, n) \in \mathbb{R}^2 : m \leq n \}\) and the family of functions \(F_4\) since in the case \(G_4\) the proof runs similarly. For the function \(f : \mathbb{R} \to \mathbb{R}\), such that \(f(x) > x\) and every \(y \in \mathbb{R}\), we define the partition of the interval \([0, y]\) by \(B_f(y)\) and \(A_f(y)\) as follows:

\[
B_f(y) = ([0, x_1] \times \{y\}) \cup ([x_2, x_3] \times \{y\}) \cup \ldots \cup ([x_{K_f(y)-1}, x_{K_f(y)}] \times \{y\})
\]

\[
= \bigcup_{k=1}^{K_f(y)} B_k(f, n),
\]

\[
A_f(y) = ([x_1, x_2] \times \{y\}) \cup ([x_3, x_4] \times \{y\}) \cup \ldots \cup ([x_{K_f(y)}, y] \times \{y\})
\]

\[
= \bigcup_{k=1}^{K_f(y)} A_k(f, y), \quad 0 < x_1 < x_2 < x_3 < \ldots < x_{K_f(y)} < y,
\]

for some finite (the definition of the family \(F_4\) integers \(K_f(y)\) \(\in \mathbb{N}\). We put \(K = \sup\{K_f(y) : y \in \mathbb{R}\}\). For each \(y\) we complete the families \(B(f, y) = \{B_k(f, y), 1 \leq k \leq K_f(y)\}\) putting \(B_k(f, y) = \emptyset\) for \(k = K_f(y) + 1, K_f(y) + 2, \ldots, K\).

Immediately, from the definition of this family we have the property

\[
\forall y_1 < y_2 \forall 1 \leq i \leq K \exists 1 \leq j \leq KB_i(f, y_1) \subset B_j(f, y_2).
\]

Thus, on the base of the family \(B(f, y)\) we define the family

\[
\Gamma_k(y) = \bigcup_{i=1}^{K} \bigcup_{1 \leq t \leq y : j : B_i(f, t) \subset B_i(f, y), 1 \leq j \leq K} B_j(f, t), \quad 1 \leq k \leq K.
\]
Furthermore, for every \(1 \leq k \leq K\) we put

\[
A(k) = \bigcup_{y \in \mathbb{R}} A_k(f, y), \quad k = 1, 2, 3, \ldots, K.
\]

We explain the introduced families in Figure 1.

**Figure 1.** The partition of the graph on the areas \(A(i), 1 \leq i \leq K\)

It is easy to check that Lemma 1 and the proof of Theorem 1 in [4] hold for the sequences \(\{n_k, k \in \mathbb{N}\} \subset A(k)\) and the increasing sequences of sums of random variables

\[
Y_n(k) = \sum_{m \in \Gamma_k(n_2) \cap \mathbb{N}^2} X_m = \sum_{m \in [1,n_1] \times [1,n_2] \cap B} X_m, \quad n \in A(k),
\]

iff only \(A(k)\) is not bounded for \(k = 1, 2, 3, \ldots, K\). Some comments are required about the fulfilling of Lemma 2 in [4] for the boundaries of our sets \(A(k)\). The boundary of such sets can be divided by at most \(K\) graphs \(\Xi_i, 1 \leq i \leq K\), piecewise continuous and increasing (in Figure II we mark three such graphs: \(a, b\) and \(c\), respectively) and at most \(K\) graphs \(\Upsilon_i, 1 \leq i \leq K\), piecewise continuous and decreasing (in Figure II we mark two such graphs: \(d\) and \(e\), respectively). For each graph from the family \(\Xi_i, 1 \leq i \leq K\), we intermediately use Lemma 2 of [4], whereas for the graphs from the family \(\Upsilon_i, 1 \leq i \leq K\), we use our Proposition 3.1.

Thus, using the notation of [4],

\[
\lim_{n \in A(k)} Y_n(k)_{[1,n_1] \times [1,n_2] \cap B} = 0, \quad k = 1, 2, 3, \ldots, K,
\]
and because each subsequence $\mathcal{N} = \{n_i \in A, i \in \mathbb{N}\}$ can be divided into $K$ subsequences $\mathcal{N} \cap A(k)$, the assertion holds. ■

Note that in the above proof we use only the definitions of $\{A_i(f, y), B_i(f, y), \Gamma_i(y)\}$ for integer $y$'s. Therefore, we restrict ourselves in the definitions of $F_i$ and $G_i$, and $K_f(y)$ and $K_g(y)$ for integer $y$'s, only.

**Proof of Theorem**

We show that if

$$\lim_{V} \frac{S_{n_i}}{n} = EX,$$

then

$$\lim_{V} \frac{S_{n_i}}{n} = EX.$$  

Obviously, (3.5) follows from Theorem 1 in [4]. Then we have $E|X| < \infty$. Furthermore, we define four functions:

$$M_1 : \begin{cases} 
V \to V, \\
M_1((k_1, k_2)) = (k_1, \lfloor f(k_1) \rfloor), 
\end{cases}$$

$$M_2 : \begin{cases} 
V \to V, \\
M_2((k_1, k_2)) = (\lfloor f(k_2) \rfloor, k_2), 
\end{cases}$$

$$M_3 : \begin{cases} 
V \to V, \\
M_3((k_1, k_2)) = (k_1, \lfloor g(k_1) \rfloor), 
\end{cases}$$

$$M_4 : \begin{cases} 
V \to V, \\
M_4((k_1, k_2)) = (\lfloor g^{-1}(k_2) \rfloor, k_2). 
\end{cases}$$

Obviously, as $M_i(k_1, k_2) \in V, i = 1, 2, 3, 4$, from (3.5) we have

$$\lim_{|n| \to \infty} \frac{S_{M_i(n)}}{|M_i(n)|} = EX, \quad i = 1, 2, 3, 4.$$  

Let the sequence $\{n_k = (n_{1,k}, n_{2,k}), k \in \mathbb{N}\} \subset V \setminus \mathbb{V}$ be such that $|n_k| \to \infty$, and let

$$\{n_k, k \in \mathbb{N}\} = \bigcup_{i=1}^4 \{n_k^{(i)} = (n_{1,k}^{(i)}, n_{2,k}^{(i)}), k \in \mathbb{N}\}$$

be four subsequences such that

$$[f(n_{1,k}^{(1)}) - f(n_{1,k}^{(1)})] \log_+ (n_{1,k}^{(1)} [f(n_{1,k}^{(1)}) - f(n_{1,k}^{(1)})]) \leq cf(n_{1,k}^{(1)}),$$

$$[f^{-1}(n_{2,k}^{(2)}) - f^{-1}(n_{2,k}^{(2)})] \log_+ (n_{2,k}^{(2)} [f^{-1}(n_{2,k}^{(2)}) - f^{-1}(n_{2,k}^{(2)})]) \leq cf^{-1}(n_{2,k}^{(2)}),$$

$$[g(n_{1,k}^{(3)}) - g(n_{1,k}^{(3)})] \log_+ (n_{1,k}^{(3)} [g(n_{1,k}^{(3)}) - g(n_{1,k}^{(3)})]) \leq cg(n_{1,k}^{(3)}),$$

$$[g^{-1}(n_{2,k}^{(4)}) - g^{-1}(n_{2,k}^{(4)})] \log_+ (n_{2,k}^{(4)} [g^{-1}(n_{2,k}^{(4)}) - g^{-1}(n_{2,k}^{(4)})]) \leq cg^{-1}(n_{2,k}^{(4)}).$$
At least one of the above-defined subsequences is infinite (we denote the set of such subsequences by \(I\)).

Let us remark that for \(x > y > 0\) we have \([x] - [y] \leq [x - y]\). Indeed, if \(x - y\) is an integer, then \([x] - [y] = x - y = [x - y]\). On the other hand, since for arbitrary \(z \in (0, 2)\) we have \(\lfloor z \rfloor \leq 1\), it follows that

\[
[x] - [y] = [x - y] = [x - y + \{y\}]
\]

\[
= \lfloor [x - y] + \{x - y\} + \{y\}\rfloor = [x - y] + \lfloor \{x - y\} + \{y\}\rfloor
\]

\[
\leq [x - y] + 1 = [x - y].
\]

Therefore, the subsequences defined as above satisfy

\[
\limsup_{k \to \infty} \frac{(|\xi_k| - |M_i(\eta_k)|) \log \left( |\eta_k| - |M_i(\eta_k)| \right) \lor 1}{|\eta_k|} < c < \infty, \quad i \in I,
\]

and, in consequence, because \(\lim_{V \to \infty} \log \left( |\eta_k| - |M_i(\eta_k)| \right) = +\infty\) or \(|\xi_k| = |M_i(\eta_k)|, k \in \mathbb{N}\), we obtain

\[
\limsup_{k \to \infty} \frac{|M_i(\eta_k)|}{|\xi_k|} = 1, \quad i \in I.
\]

On the other hand, let us remark that

\[
S_{\eta} - S_{M_i(\eta)} \sim S_{\eta - M_i(\eta)},
\]

and from Theorem 1 in [5] we have

\[
\lim_{k \to \infty} \frac{S_{\eta_k} - ES_{\eta_k} - S_{\eta_k}(\xi_k) + ES_{\eta_k}(\xi_k)}{\log \left( |\eta_k| - |M_i(\eta_k)| \right) \lor 1} = 0, \quad i \in I.
\]

Because for \(i \in I\)

\[
\lim_{k \to \infty} \frac{-ES_{\eta_k} + ES_{\eta_k}(\xi_k)}{\log \left( |\eta_k| - |M_i(\eta_k)| \right) \lor 1} = \lim_{k \to \infty} \frac{-EX}{\log \left( |\eta_k| - |M_i(\eta_k)| \right) \lor 1} = 0,
\]
and
\[ \lim_{k \to \infty} \frac{S_{n_k}^{(i)}}{|n_k^{(i)}|} = \]

\[ \lim_{k \to \infty} \left\{ \frac{S_{M_i(n_k^{(i)})}}{|M_i(n_k^{(i)})|} + \frac{S_{n_k^{(i)}} - S_{M_i(n_k^{(i)})}}{|n_k^{(i)}| - |M_i(n_k^{(i)})|} \left( \log_+ \left( |n_k^{(i)}| - |M_i(n_k^{(i)})| \right) \lor 1 \right) \right\} \]

\[ = EX \cdot 1 + 0 \cdot c = EX, \quad i \in I, \]

and, in consequence,

(3.11) \[ \lim_{k \to \infty} \frac{S_{n_k}}{n_k} = EX, \]

the proof is completed.  

**Proof of Example 2.1** In all the three cases we have
\[ \int_0^\infty \frac{\log \left( \frac{f(x)^{1/e}}{f(x)^{1/(1)}} \right)}{x \lor 1} dx = \int_0^\infty \log \left( \frac{u(x) + g(x)}{u(x)^{1/e}} \right) \frac{1}{x \lor 1} dx, \]

\[ [f(x) - f(x)] \log_+ \left( x[f(x) - f(x)] \right) - [g(x) \cos (h(x) \pi)] \log_+ \left( x[g(x) \cos (h(x) \pi)] \right). \]

In the case (i), because \( \log(1 + x) \leq x \), we have
\[ \int_1^{\infty} \log \left( 1 + 1/(\log x)^2 \right) dx \leq \int_1^{\infty} \frac{1}{x(\log x)^2} dx < \infty. \]

Let us define the sequence \( \{x_n, n \geq 1\} \) divergent to infinity, so that, for \( i \geq 1, 2^{x_i}(\log x_i)^2 \in \mathbb{N} \) (it is possible as the function \( 2^{x}(\log x)^2 \) is continuously increasing to infinity for \( x > 1 \). Then for every constant \( c \) there exists \( i_0 \) such that, for every \( i > i_0 \),
\[ \left[ 2^{x_i} \cos \left( 2^{x_i}(\log x_i)^2 \pi \right) \right] \log_+ \left( x_i \left[ 2^{x_i} \cos \left( 2^{x_i}(\log x_i)^2 \pi \right) \right] \right) = 2^{x_i} \log x_i + x_i 2^{x_i} \log 2 \geq c(2^{x_i}(\log x_i)^2 + 2^{x_i}); \]
thus the assumptions of Theorem 2.1 are satisfied, whereas the assumptions of Theorem 2.3 fail. Let us remark that, for arbitrary \( x \in \mathbb{N} \) in the interval \((x, y)\); the function \( f \) has at least \( 2^y (\log y)^2 - 2^x (\log x)^2 - 2 \) oscillations, where \( 2^y (\log y)^2 = 2^x [(\log x)^2 + 1] \). Therefore, for \( y > e \),

\[
K_f(y) \geq 2^y (\log y)^2 - 2^x (\log x)^2 - 2 \geq 2^x - 2,
\]

and \( K_f(y) \to \infty \) as \( y \to \infty \), so that the assumptions of Theorem 2.2 fail.

In the case (ii) we have

\[
\int_1^\infty \frac{\log(2)}{x} \, dx = \infty.
\]

Furthermore, it is easy to check that \(|\cos (h(x)\pi)|\) is equal to one only for \( x = 2^k \) or \( x = 3 \cdot 2^{k-1} \) and it is equal to zero only for \( x = 5 \cdot 2^{k-2} \) and \( x = 7 \cdot 2^{k-2} \) for \( k \in \mathbb{N} \). Thus, in the interval \( x \in [2^k, 2^{k+1}] \) the function \( f \) has two local minima at \( x = 5 \cdot 2^{k-2} \) and \( x = 7 \cdot 2^{k-2} \) equal to \( 5 \cdot 2^{k-2} \) and \( 7 \cdot 2^{k-2} \), respectively, and two local maxima at \( x = 2^k \) and \( x = 3 \cdot 2^{k-1} \) equal to \( 2^{k+1} \) and \( 3 \cdot 2^k \), respectively, so that for every \( x \in \mathbb{R} \) we have \( K_f(x) \leq 4 \), and the assumptions of Theorem 2.3 are fulfilled. Taking \( x = k \in \mathbb{N} \), we see that for every constant \( c \) there exists a sufficiently large \( k \in \mathbb{N} \) such that

\[
[k\cos(k\pi)] \log_+ (k |\cos(k\pi)|) = 2k \log k > ck;
\]

thus the assumptions of Theorem 2.3 fail.

In the case (iii) we have

\[
\int_1^\infty \frac{\log(1 + 1/\log x)}{x} \, dx = \infty,
\]

so that the assumptions of Theorem 2.3 fail. Failure of the assumptions of Theorem 2.2 follows from analogous considerations to those for the point (i). From

\[
\frac{x}{\log x} |\cos(2^x \pi)| \log \left( \frac{x^2}{\log x} |\cos(2^x \pi)| \right) \leq \frac{x}{\log x} \log x^2 = 2x \leq 2 \left( x + \frac{x}{\log x} |\cos(2^x \pi)| \right)
\]

we see that the assumptions of Theorem 2.3 are satisfied with \( c = 2 \).

Acknowledgments. The authors gratefully acknowledge many helpful suggestions of the referee during the preparation of the paper.
REFERENCES


Agnieszka M. Gdula  
Institute of Mathematics  
Maria Curie-Skłodowska University  
pl. Marii Curie-Skłodowskiej 1  
20-031 Lublin, Poland  
E-mail: gdula.agnieszka@gmail.com

Andrzej Krajka  
Institute of Computer Sciences  
Maria Curie-Skłodowska University  
pl. Marii Curie-Sklodowskiej 1  
20-031 Lublin, Poland  
E-mail: akrajka@gmail.com

Received on 1.7.2014; revised version on 14.8.2016