Abstract. Let \( \{X_i(t), t \geq 0\} \), \( 1 \leq i \leq n \), be mutually independent and identically distributed centered stationary Gaussian processes. Under some mild assumptions on the covariance function, we derive an asymptotic expansion of

\[
P\left( \sup_{t \in [0, u \cdot m_r(u)]} X_r(t) \leq u \right) \quad \text{as} \quad u \to \infty,
\]

where

\[
m_r(u) = \left( P\left( \sup_{t \in [0,1]} X_r(t) > u \right) \right)^{-1} \left( 1 + o(1) \right),
\]

and \( \{X_r(t), t \geq 0\} \) is the \( r \)th order statistic process of \( \{X_i(t), t \geq 0\} \), \( 1 \leq i, r \leq n \). As an application of the derived result, we analyze the asymptotics of supremum of the order statistic process of stationary Gaussian processes over random intervals.

**2010 AMS Mathematics Subject Classification:** Primary: 60G15; Secondary: 60G70.

**Key words and phrases:** Asymptotic, Gaussian processes, order statistic, stationarity, supremum.

1. **INTRODUCTION**

Let \( \{X(t) : t \geq 0\} \) be a centered stationary Gaussian process with continuous sample paths. One of the classical results in extreme value theory states that, under some mild conditions on the covariance function of \( X \),

\[
\lim_{u \to \infty} P\left( \sup_{t \in [0, u \cdot m(u)]} X(t) \leq u \right) = e^{-x}
\]

for \( x > 0 \) and \( m(u) = P\left( \sup_{t \in [0,1]} X(t) > u \right)^{-1} \); see, e.g., Leadbetter et al. [11], Theorem 12.3.4; Arendarczyk and Dębiecki [34], Lemma 4.3; Tan and Hashorva [13], Lemma 3.3.

---

* This work was supported by the FP7 project RARE-318984.
Consider a vector-valued Gaussian stochastic process \( \{X(t) : t \geq 0\} \), where \( X(t) = (X_1(t), \ldots, X_n(t)) \) with \( \{X_i(t) : t \geq 0\} \), \( i = 1, \ldots, n \), being mutually independent copies of \( \{X(t) : t \geq 0\} \). Denote by \( \{X_r(t) : t \geq 0\} \), \( r = 1, 2, \ldots, n \), the \( r \)th smallest order statistic process, i.e., for each \( t \geq 0 \),

\[
(1.2) \quad X_{(1)}(t) = \min_{1 \leq i \leq n} X_i(t) \leq X_{(2)}(t) \leq \ldots \leq \max_{1 \leq i \leq n} X_i(t) = X_{(n)}(t).
\]

In this contribution we derive a counterpart of (1.1) for \( \{X_{(r)}(t) : t \geq 0\} \).

One of important motivations to analyze asymptotic properties of extremes of order statistic processes is their relation with the conjunction problem. Following [14], the set of conjunctions \( C_{T,u} \) is defined as

\[
C_{T,u} := \{t \in [0, T]: \min_{1 \leq i \leq n} X_i(t) > u\},
\]

so

\[
\mathbb{P}(C_{T,u} = \emptyset) = \mathbb{P}\left( \sup_{t \in [0, T]} \min_{1 \leq i \leq n} X_i(t) \leq u \right).
\]

We refer to [2], [3], [6], [9], [14] for recent results on asymptotic properties of \( \mathbb{P}(C_{T,u} \neq \emptyset) \).

As an application of the obtained result we provide the exact asymptotics of

\[
\mathbb{P}\left( \sup_{t \in [0, T]} X_{(r)}(t) > u \right) \quad \text{as} \quad u \to \infty
\]

for \( T \) being a nonnegative random variable independent of \( X(t) \). The obtained asymptotics extends the recent results of Arendarczyk and Dębicki [4].

2. PRELIMINARIES

Suppose that \( X(t) = (X_1(t), \ldots, X_n(t)) \) and \( \{X_i(t) : t \geq 0\} \), \( i = 1, \ldots, n \), are mutually independent centered stationary Gaussian processes with covariance function \( r(t) \) satisfying the following conditions:

(A1) \( r(t) = 1 - t^\alpha + o(t^\alpha) \) as \( t \to 0 \);

(A2) \( r(t) < 1 \) if \( t > 0 \);

(A3) \( r(t) \log t \to 0 \) as \( t \to \infty \).

Following Dębicki et al. [9], let us introduce the generalized Pickands constant as

\[
\mathcal{H}_{\alpha, k} = \lim_{S \to \infty} S^{-1} \mathcal{H}_{\alpha,k}(S) \in (0, \infty),
\]

where

\[
\mathcal{H}_{\alpha,k}(S) = \int_{\mathbb{R}^n} \exp \left( \sum_{i=1}^{k} w_j \right) \mathbb{P}\left( \sup_{t \in [0,S]} \min_{1 \leq i \leq k} \left( \sqrt{2} B^{(i)}_\alpha(t) - t^\alpha - w_i \right) > 0 \right) dw \in (0, \infty),
\]
Extremes of order statistics

and $B_\alpha^{(i)}$, $i = 1, \ldots, n$, are mutually independent standard fractional Brownian motions with Hurst index $\alpha/2 \in (0, 1]$, i.e., centered Gaussian processes with stationary increments and variance function $t^\alpha$.

Let

$$
m_r(u) := \frac{(2\pi)^{(n+1-r)/2}}{c_{n,r-1} H_{\alpha,n+1-r}} u^{n+1-r-2/\alpha} \exp\left(\frac{n + 1 - r}{2} u^2\right),
$$

where

$$
c_{n,r-1} = \frac{n!}{(r-1)!(n+1-r)!}.
$$

It follows from Theorem 2.2 in [8] that, for each $T > 0$ and $1 \leq r \leq n$,

$$
P\left(\sup_{t \in [0, T]} X_r(t) > u\right) = \frac{T}{m_r(u)} \left(1 + o(1)\right) \quad \text{as } u \to \infty,
$$

where $\Psi(u) = \frac{1}{\sqrt{2\pi}} \int_u^{\infty} \exp(-x^2/2)dx$.

3. MAIN RESULTS

The following theorem constitutes the main result of this contribution.

**Theorem 3.1.** Let $\{X_j(t), t \geq 0\}$ be independent and identically distributed centered stationary Gaussian processes with covariance function $r(t)$ satisfying the conditions (A1)–(A3) and assume that $0 < A < B < \infty$ and $x > 0$. Then

$$
P\left(\sup_{t \in [0, x m_r(u)]} X_r(t) \leq u\right) \to e^{-x} \quad \text{as } u \to \infty,
$$

uniformly for $x \in [A, B]$.

Let $T$ be a nonnegative random variable which is independent of $X$. In the following theorem we discuss the asymptotic behavior of $P\left(\sup_{t \in [0, T]} X_r(t) > u\right)$ as $u \to \infty$. It appears that the qualitative form of the asymptotics strongly depends on heaviness of the tail of $T$.

**Theorem 3.2.** Let $\{X_j(t), t \geq 0\}$ be independent and identically distributed centered stationary Gaussian processes with covariance function $r(t)$ satisfying the conditions (A1)–(A3), and let $T$ be a nonnegative random variable independent of $X$.

(i) If $ET < \infty$, then, as $u \to \infty$,

$$
P\left(\sup_{t \in [0, T]} X_r(t) > u\right) = ET c_{n,r-1} H_{\alpha,n+1-r} u^{2/\alpha} \left(\Psi(u)\right)^{n+1-r} \left(1 + o(1)\right).
$$
(ii) If $T$ has a regularly varying tail distribution at infinity with index $\lambda \in (0, 1)$, then, as $u \to \infty$,

\[
\mathbb{P}\left( \sup_{t \in [0, T]} X_{(r)}(t) > u \right) = \Gamma(1 - \lambda)\mathbb{P}(T > m_r(u))(1 + o(1)).
\]

(iii) If $T$ has a slowly varying tail distribution at infinity, then, as $u \to \infty$,

\[
\mathbb{P}\left( \sup_{t \in [0, T]} X_{(r)}(t) > u \right) = \mathbb{P}(T > m_r(u))(1 + o(1)).
\]

The proofs of Theorems 3.1 and 3.2 are given in Section 4.

4. PROOFS

Before proceeding to the proofs of Theorems 3.1 and 3.2, we give some preliminary lemmas. Let us put $T_r = x m_r(u)$ and $n_r = \lfloor T_r \rfloor$. For any $\varepsilon \in (0, 1)$ and $1 \leq l \leq n_r$, we write $I_l = [l - 1 + \varepsilon, l]$ and $I_l^c = [l - 1, l - 1 + \varepsilon]$.

**Lemma 4.1.** For each $B > A > 0$,

\[
\lim_{u \to \infty} \left| \mathbb{P}\left( \sup_{t \in [0, n_r]} X_{(r)}(t) \leq u \right) - \mathbb{P}\left( \sup_{t \in \bigcup_{l=1}^{n_r} I_l} X_{(r)}(t) \leq u \right) \right| \leq \rho_1(\varepsilon),
\]

uniformly for $x \in [A, B]$, where $\rho_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.

**Proof.** Suppose that $x \in [A, B]$. By stationarity, Bonferroni’s inequality (see, e.g., [14]) and (2.2), we have

\[
0 \leq \mathbb{P}\left( \sup_{t \in \bigcup_{l=1}^{n_r} I_l} X_{(r)}(t) \leq u \right) - \mathbb{P}\left( \sup_{t \in [0, n_r]} X_{(r)}(t) \leq u \right)
= \mathbb{P}\left( \sup_{t \in [0, n_r]} X_{(r)}(t) > u \right) - \mathbb{P}\left( \sup_{t \in \bigcup_{l=1}^{n_r} I_l} X_{(r)}(t) > u \right)
\leq \mathbb{P}\left( \sup_{t \in \bigcup_{l=1}^{n_r} I_l^c} X_{(r)}(t) > u \right) \leq n_r \mathbb{P}\left( \sup_{t \in [0, c]} X_{(r)}(t) > u \right)
= x m_r(u) \frac{\varepsilon}{m_r(u)} (1 + o(1)) \leq B \varepsilon =: \rho_1(\varepsilon) \quad \text{as } u \to \infty.
\]

This completes the proof. ■

**Lemma 4.2.** Let $q = q(u) = au^{-2/\alpha}$ for some $a > 0$. Then

\[
\lim_{u \to \infty} \sup_{x \in [A, B]} \left| \mathbb{P}\left( \sup_{t \in \bigcup_{l=1}^{n_r} I_l} X_{(r)}(t) \leq u \right) - \mathbb{P}\left( \max_{iq \in \bigcup_{l=1}^{n_r} I_l} X_{(r)}(i q) \leq u \right) \right| \leq \rho_2(a),
\]

uniformly for $x \in [A, B]$, where $\rho_2(a) \to 0$ as $a \to 0$. 
Proof. Since $X_i(t)$ are independent and identically distributed, we obtain
\[
P(\max_{iq \in I_1} X_i^{(r)}(iq) > u)
= \mathbb{P}\left( \bigcup_{iq \in I_1} \bigcup_{j=n-r+1}^n \{ \exists k_1, \ldots, k_j, X_{k_1}(iq) > u, \ldots, X_{k_j}(iq) > u \} \right)
\]
= \mathbb{P}\left( \bigcup_{iq \in I_1} \bigcup_{j=n-r+1}^n \{ \exists k_1, \ldots, k_j, X_{k_1}(iq) > u, \ldots, X_{k_j}(iq) > u, X_k(iq) \leq u, k \neq k_1, \ldots, k_j \} \right)
\]
= \sum_{j=n-r+1}^n \mathbb{P}(\exists iq \in I_1, X_1(iq) > u, \ldots, X_j(iq) > u, X_k(iq) \leq u, k > j)
\]
= \sum_{j=n-r+1}^n \mathbb{P}(\max_{iq \in I_1 1 \leq i \leq j} X_i(iq) > u) (1 + o(1)).

Following Dębicki et al. \cite{debicki2013extremes} we define
\[
H'_{\alpha,j}(a) = \frac{1}{a} \mathbb{P}\left( \max_{k \geq 1 1 \leq m \leq j} (\sqrt{2} B^{(m)}_\alpha(ak) - (ak)^a + \eta_m) \leq 0 \right),
\]
where $j = 1, 2, \ldots, n$, and $\{B^{(m)}_\alpha, t \geq 0\}, m \geq 1$, are independent and identically distributed standard fractional Brownian motions which are further independent of independent unit exponential random variables $\eta_m$. Using analogous arguments to those in the proof of Theorem 1.1 in Dębicki et al. \cite{debicki2013extremes} or Lemma 1 in Albin and Choi \cite{albin2006extremes}, we have
\[
\mathbb{P}(\max_{iq \in I_1} X_i^{(r)}(iq) > u) = \sum_{j=n-r+1}^n \frac{H'_{\alpha,j}(a)}{\mathcal{H}_{\alpha,j}} \frac{1 - \varepsilon}{m_{n+1-j}(u)}
= \frac{H'_{\alpha,n+1-r}(a)}{\mathcal{H}_{\alpha,n+1-r}} \frac{1 - \varepsilon}{m_r(u)} (1 + o(1)) \quad \text{as } u \to \infty,
\]
where $H'_{\alpha,k}(a) \to \mathcal{H}_{\alpha,k}$ as $a \to 0$. Therefore, by stationarity, we obtain
\[
0 \leq \mathbb{P}(\max_{iq \in I_1^{n-r} t} X_i^{(r)}(iq) \leq u) - \mathbb{P}(\sup_{t \in I_1^{n-r} t} X_i^{(r)}(t) \leq u)
\leq n_r \mathbb{P}(\max_{1 \leq i \leq n_r} \mathbb{P}(\max_{iq \in I_i} X_i(iq) \leq u) - \mathbb{P}(\sup_{t \in I_i} X_i(t) \leq u))
\leq n_r \mathbb{P}(X_i^{(r)}(0) > u) + n_r \mathbb{P}(\sup_{t \in [0,1-\varepsilon]} X_i^{(r)}(t) > u)
\leq n_r \mathbb{P}(\max_{iq \in [0,1-\varepsilon]} X_i^{(r)}(iq) > u)
= \varepsilon m_r(u) \left( \frac{1}{m_r(u)} + \frac{1 - \varepsilon}{m_r(u)} - \frac{H'_{\alpha,n+1-r}(a)}{\mathcal{H}_{\alpha,n+1-r}} \frac{1 - \varepsilon}{m_r(u)} \right) (1 + o(1))
\leq B \left( 1 - \frac{H'_{\alpha,n+1-r}(a)}{\mathcal{H}_{\alpha,n+1-r}} \right) =: \rho_2(a),
\]
where the penultimate expression is due to (2.20). Since \( \rho_2(a) \to 0 \) as \( a \to 0 \), the proof is completed. \( \blacksquare \)

For each \( 1 \leq j \leq n \), let \( \{X_j^{(k)}(t), t \geq 0\}_{k=1}^{\infty} \) be a sequence of independent and identically distributed centered stationary Gaussian processes that satisfy the conditions (A1)–(A3). Define

\[
Y_j(t) = X_j^{(k)}(t) \quad \text{if} \quad t \in [k-1, k),
\]

and, for \( t \geq 0 \),

\[
Y_{(1)}(t) = \min_{1 \leq j \leq n} Y_j(t) \leq Y_{(2)}(t) \leq \ldots \leq \max_{1 \leq j \leq n} Y_j(t) = Y_{(n)}(t).
\]

**Lemma 4.3.** We have

\[
\lim_{u \to \infty} \left| \mathbb{P}\left( \sup_{iq \in \bigcup_{t=1}^{n^r} I_t} X_{(r)}(iq) \leq u \right) - \mathbb{P}\left( \sup_{iq \in \bigcup_{t=1}^{n^r} I_t} Y_{(r)}(iq) \leq u \right) \right| = 0.
\]

**Proof.** Define \( A = \mathbb{N} \cap \bigcup_{t=1}^{n^r} I_t q^{-1} = \{i_1, i_2, \ldots, i_d\} \), where \( 1 \leq i_1 < i_2 < \ldots < i_d < \infty \), and observe that

\[
\Delta_{(r)} = \left| \mathbb{P}\left( \sup_{i \in A} X_{(r)}(iq) \leq u \right) - \mathbb{P}\left( \sup_{iq \in \bigcup_{t=1}^{n^r} I_t} Y_{(r)}(iq) \leq u \right) \right|.
\]

For \( i \in A \) and \( 1 \leq j \leq n \), we put \( X_{ij} = X_j(iq) \) and \( Y_{ij} = Y_j(iq) = X_j^{([iq]+1)}(iq) \).

Note that

\[
\sigma_{ij,ilk}^X = \mathbb{E} X_{ij} X_{ilk} = \mathbb{E} X_j(iq) X_k(lq) = r((i-l)q) \mathbb{I}\{j = k\} := \sigma_{ilk}^X \mathbb{I}\{j = k\},
\]

\[
\sigma_{ij,ilk}^Y = \mathbb{E} Y_{ij} Y_{ilk} = \mathbb{E} X_j^{([iq]+1)}(iq) X_k^{([lq]+1)}(lq) = r((i-l)q) \mathbb{I}\{iq \in [lq]\} \mathbb{I}\{j = k\} := \sigma_{ilk}^Y \mathbb{I}\{j = k\}.
\]

It follows from Theorem 2.4 in [7] that

\[
\Delta_{(r)} \leq \frac{n(c_{n-1,r-1})^2}{(2\pi)^{n+1-r}} u^{-2(n-r)} \sum_{i,j \in A, i \neq j} \left| A_{il}^{(r)} \right| \exp \left( -\frac{(n+1-r)u^2}{1 + \rho_{il}} \right),
\]

where

\[
\rho_{il} = \max(\sigma_{ilk}^X, |\sigma_{ilk}^Y|) = |r((i-l)q)|,
\]

\[
A_{il}^{(r)} = \frac{\sigma_{ilk}^X}{\sigma_{ilk}^Y} \frac{(1 + |h|)^2(n-r)}{(1 - h^2)^{(n+1-r)/2}} dh
\]

\[
= \int_0^1 \frac{r((i-l)q)}{(1 - h^2)^{(n+1-r)/2}} dh \mathbb{I}\{|iq| \neq |lq|\}.
\]
Since \( \delta := \sup \{|r(t)|, t \geq \varepsilon \} < 1 \), for \( i, l \in A \) satisfying \(|iq| \neq |lq|\), one has \(|(i-l)q| \geq \varepsilon\), and \(|r((i-l)q)| \leq \delta < 1\). Notice that the integrand in the definition of \( A_u^{(r)} \) is continuous and bounded on \([0, \delta]\), so there exists a constant \( K_1 \) such that

\[
|A_u^{(r)}| \leq K_1|r((i-l)q)|\mathbb{I}\{|iq| \neq |lq|\}.
\]

Hence,

\[
\Delta(r) \leq n(c_n^{-1}r^{-1})^2\frac{K_1}{2\pi} \frac{u^{-2(n-r)}T_r^r}{q} \sum_{q \in kq \in T_r} |r(kq)| \exp \left( -\frac{(n+1-r)u^2}{1+|r(kq)|} \right)
\]

\[
= n(c_n^{-1}r^{-1})^2\frac{K_1}{2\pi} \frac{u^{-2(n-r)}T_r^r}{q} \sum_{q \in kq \in T_r^q} |r(kq)| \exp \left( -\frac{(n+1-r)u^2}{1+|r(kq)|} \right)
\]

\[
+ n(c_n^{-1}r^{-1})^2\frac{K_1}{2\pi} \frac{u^{-2(n-r)}T_r^r}{q} \sum_{q \in kq \in T_r^\beta} |r(kq)| \exp \left( -\frac{(n+1-r)u^2}{1+|r(kq)|} \right)
\]

\[
: = \mathbb{P}_1 + \mathbb{P}_2,
\]

where \( 0 < \beta < (1-\delta)/(1+\delta) \).

First, we prove that \( \mathbb{P}_1 \to 0 \) as \( u \to \infty \). Indeed,

\[
\mathbb{P}_1 \leq n(c_n^{-1}r^{-1})^2\frac{K_1}{2\pi} \frac{u^{-2(n-r)}T_r^{\beta+1}}{q^2} \exp \left( -\frac{(n+1-r)u^2}{1+\delta} \right)
\]

\[
= n(c_n^{-1}r^{-1})^2\frac{K_1}{2\pi} \frac{u^{4(\alpha-2(n-r))}T_r^{\beta+1}}{q^2} \exp \left( -\frac{(n+1-r)u^2}{2} \right) 2/(1+\delta)
\]

\[
\leq K_2 u^{4(\alpha-2(n-r)+(\beta+1)(n+1-r-2/\alpha)} \exp \left( \frac{(n+1-r)u^2}{2} \right)^{\beta-(1-\delta)/(1+\delta)}
\]

\[
\to 0 \quad \text{as } u \to \infty.
\]

In order to show that \( \mathbb{P}_2 \to 0 \), we put \( \delta(t) = \sup \{|r(s) \log s|, s \geq t\} \). By (A3), we have \(|r(t)| \leq \delta(t)/\log t \) and \( \delta(t) \downarrow 0 \) as \( t \to \infty \). Moreover,

\[
\log T_r = \frac{n+1-r}{2} u^2 (1 + o(1)) \quad \text{for } kq > T_r^\beta.
\]

Thus,

\[
\exp \left( -\frac{(n+1-r)u^2}{1+|r(kq)|} \right) \leq \exp \left( -\frac{(n+1-r)u^2}{1+\delta(T_r^\beta/\log T_r^\beta)} \right)
\]

\[
\leq K_3 \exp \left( -\frac{(n+1-r)u^2}{2} \right).
\]
Hence,

$$
\mathbb{P}_2 \leq \left\{ K_4 u^{-2(n-r)} \frac{T_r^2}{q^2} \exp\left(- (n + 1 - r)u^2 \right) \frac{1}{\log T_r^\beta} \right\} \\
\times \frac{q}{T_r} \sum_{T_r^\beta < kq < T_r} |r(kq)| \log(kq)
$$

$$
\leq K_5 u^{-2(n-r)} \frac{u^{2(n+1-r-2/\alpha)}}{u^{4/\alpha}} \exp\left( (n + 1 - r)u^2 \right) \frac{1}{u^2} \\
\times \frac{q}{T_r} \sum_{T_r^\beta < kq < T_r} |r(kq)| \log(kq)
$$

$$
\leq K_5 \frac{q}{T_r} \sum_{T_r^\beta < kq < T_r} |r(kq)| \log(kq) \to 0 \quad \text{as} \quad u \to \infty.
$$

This completes the proof. \[ \square \]

**Lemma 4.4.** We have

$$
\limsup_{u \to \infty} \left| \mathbb{P}\left( \sup_{iq \in \bigcup_{l=1}^{n_r} I_l} Y_{(r)}(iq) \leq u \right) - \mathbb{P}\left( \sup_{t \in [0, n_r]} Y_{(r)}(t) \leq u \right) \right| \leq x \left( \rho_3(a) + \varepsilon \right),
$$

where $\rho_3(a) \to 0$ as $a \to 0$.

**Proof.** Since $I_l$, $l = 1, 2, \ldots, n_r$, are disjoint, $\{Y_{(r)}(t), t \in I_l\}$ are independent, and, by stationarity,

$$
0 \leq \mathbb{P}\left( \sup_{iq \in \bigcup_{l=1}^{n_r} I_l} Y_{(r)}(iq) \leq u \right) - \mathbb{P}\left( \sup_{t \in [0, n_r]} Y_{(r)}(t) \leq u \right)
$$

$$
= \mathbb{P}\left( \sup_{iq \in [0, 1]} Y_{(r)}(iq) \leq u \right)^{n_r} - \mathbb{P}\left( \sup_{t \in [0, 1]} Y_{(r)}(t) \leq u \right)^{n_r}
$$

$$
\leq n_r \left( \mathbb{P}\left( \sup_{iq \in I_l} Y_{(r)}(iq) \leq u \right) - \mathbb{P}\left( \sup_{t \in I_l} Y_{(r)}(t) \leq u \right) \right)
$$

$$
\leq n_r \left( \mathbb{P}(Y_{(r)}(0) > u) + \mathbb{P}\left( \sup_{iq \in [0, 1]} Y_{(r)}(iq) \leq u \right) - \mathbb{P}\left( \sup_{t \in [0, 1]} Y_{(r)}(t) \leq u \right) \right)
$$

$$
= x m_r(u) \left( o \left( \frac{1}{m_r(u)} \right) + \left( 1 - \frac{\mathcal{H}_{\alpha,n}^{\prime,1-r}(a)}{\mathcal{H}_{\alpha,n}^{\prime,1-r}} \right) \frac{1 - \varepsilon}{m_r(u)} \right) (1 + o(1))
$$

$$
\leq x \left( 1 - \frac{\mathcal{H}_{\alpha,n}^{\prime,1-r}(a)}{\mathcal{H}_{\alpha,n}^{\prime,1-r}} \right) =: x \rho_3(a),
$$
where \( \rho_3(a) \to 0 \) as \( a \to 0 \). Moreover,

\[
0 \leq P\left( \sup_{t \in \bigcup_{i=1}^{n_r} I_i} Y_{(r)}(t) \leq u \right) - P\left( \sup_{t \in [0, n_r]} Y_{(r)}(t) \leq u \right) \\
\leq P\left( \sup_{t \in [0, 1-e]} Y_{(r)}(t) \leq u \right)^{n_r} - P\left( \sup_{t \in [0, 1]} Y_{(r)}(t) \leq u \right)^{n_r} \\
\leq n_r P\left( \sup_{t \in [0, 1]} Y_{(r)}(t) > u \right) \\
= x m_r(u) \frac{\varepsilon}{m_r(u)} \left( 1 + o(1) \right) = x \varepsilon (1 + o(1)).
\]

The combination of the above displays completes the proof. ■

**Lemma 4.5.** We have

\[
\lim_{u \to \infty} P\left( \sup_{t \in [0, n_r]} Y_{(r)}(t) \leq u \right) = e^{-x}.
\]

**Proof.** Since

\[
P\left( \sup_{t \in [0, n_r]} Y_{(r)}(t) \leq u \right) = P\left( \sup_{t \in [0, 1]} X_{(r)}(t) \leq u \right)^{n_r} \\
= \left( 1 - P\left( \sup_{t \in [0, 1]} X_{(r)}(t) > u \right) \right)^{n_r} \\
= \left( 1 - m_r(u)^{-1} \right)^{x m_r(u)} \left( 1 + o(1) \right) \to e^{-x},
\]

the proof is completed. ■

**Proof of Theorem 3.1.** The proof of the theorem follows directly from Lemmas 4.1–4.5. ■

**Lemma 4.6.** For any \( S > 0 \), we have

\[
(4.3) \quad P\left( \sup_{t \in [0, S u^{-2/\alpha}]} X_{(r)}(t) > u \right) = c_{n,r-1} H_{\alpha, n+1-r}(S) (\Psi(u))^{n+1-r} (1 + o(1))
\]

as \( u \to \infty \).

The proof of Lemma 4.6 follows line-by-line the same reasoning as the proof of Theorem 2.2 in [8], and thus we omit it.

**Proof of Theorem 3.2.** (i) For any \( t, u, S > 0 \), let us put

\[
N_t = \left\lfloor \frac{t}{S u^{-2/\alpha}} \right\rfloor \quad \text{and} \quad \Delta_k = [k S u^{-2/\alpha}, (k+1) S u^{-2/\alpha}] \quad \text{with} \quad k = 0, 1, \ldots, N_t.
\]
Upper bound. By stationarity of the process \( \{X_r(t), t \geq 0\} \) and Lemma 4.6, we obtain

\[
\begin{align*}
\mathbb{P}
\left( \sup_{t \in [0,T]} X_r(t) > u \right) &= \int_0^\infty \mathbb{P}
\left( \sup_{s \in [0,t]} X_r(s) > u \right) d\mathbb{P}(T \leq t) \\
&\leq \mathbb{P}
\left( \sup_{s \in \Delta_0} X_r(s) > u \right) \left( \frac{u^{2/\alpha}}{S} \int_0^\infty t d\mathbb{P}(T \leq t) + 1 \right) \\
&= \frac{\mathcal{H}_{\alpha,n+1-r}(S)}{S} c_{n,r-1} \mathbb{E} T u^{2/\alpha} (\Psi(u))^{n+1-r} (1 + o(1))
\end{align*}
\]

as \( u \to \infty \). Thus, letting \( S \to \infty \), we get

\[
\mathbb{P}
\left( \sup_{t \in [0,T]} X_r(t) > u \right) = c_{n,r-1} \mathcal{H}_{\alpha,n+1-r} u^{2/\alpha} \mathbb{E} T (\Psi(u))^{n+1-r} (1 + o(1)).
\]

Lower bound. By Bonferroni’s inequality, we have

\[
(4.4) \quad \mathbb{P}
\left( \sup_{t \in [0,T]} X_r(t) > u \right) = \int_0^\infty \mathbb{P}
\left( \sup_{s \in [0,t]} X_r(s) > u \right) d\mathbb{P}(T \leq t) \\
\geq \int_0^u \mathbb{P}
\left( \sup_{s \in [0,t]} X_r(s) > u \right) d\mathbb{P}(T \leq t) \\
\geq \mathbb{P}
\left( \sup_{s \in \Delta_0} X_r(s) > u \right) \left( \frac{u^{2/\alpha}}{S} \int_0^u t d\mathbb{P}(T \leq t) - 1 \right) \\
- \sum_{0 \leq i < j \in N_t} \mathbb{P}
\left( \sup_{s \in \Delta_i} X_r(s) > u, \sup_{s \in \Delta_j} X_r(s) > u \right) d\mathbb{P}(T \leq t) \\
=: I_1 - I_2.
\]

Note that

\[
I_1 = \frac{\mathcal{H}_{\alpha,n+1-r}(S)}{S} c_{n,r-1} \mathbb{E} T u^{2/\alpha} (\Psi(u))^{n+1-r} (1 + o(1))
\]

as \( u \to \infty \). Thus, letting \( S \to \infty \), we obtain

\[
(4.5) \quad I_1 \geq c_{n,r-1} \mathcal{H}_{\alpha,n+1-r} u^{2/\alpha} \mathbb{E} T (\Psi(u))^{n+1-r}.
\]

Hence, in order to complete the proof it suffices to show that \( I_2 = o(I_1) \) as \( u \to \infty \).
Indeed, we have

\[ I_2 = \int_0^u \sum_{k=1}^{N_t} (N_t - k) \mathbb{P} \left( \sup_{s \in \Delta_0} X_{(r)}(s) > u, \sup_{s \in \Delta_k} X_{(r)}(s) > u \right) d\mathbb{P}(T \leq t) \]

\[ \leq \frac{u^{2/\alpha}}{S} \int_0^u t d\mathbb{P}(T \leq t) \sum_{k=1}^{N_u} \mathbb{P} \left( \sup_{s \in \Delta_0} X_{(r)}(s) > u, \sup_{s \in \Delta_k} X_{(r)}(s) > u \right) \]

\[ \leq \frac{u^{2/\alpha}}{S} ET \sum_{k=1}^{N_u} \mathbb{P} \left( \sup_{s \in \Delta_0} X_{(r)}(s) > u, \sup_{s \in \Delta_k} X_{(r)}(s) > u \right) \]

\[ \leq c_{n,r-1} \frac{u^{2/\alpha}}{S} ET \sum_{k=1}^{N_u} \mathbb{P} \left( \sup_{s \in \Delta_0} \min_{1 \leq i \leq n+1-r} X_i(s) > u, \sup_{s \in \Delta_k} \min_{1 \leq i \leq n+1-r} X_i(s) > u \right) \]

\[ + c_{n,r-1} \frac{u^{2/\alpha}}{S} ET \sum_{k=1}^{N_u} \mathbb{P} \left( \sup_{s \in \Delta_0} \min_{1 \leq i \leq n+1-r} X_i(s) > u, \sup_{s \in \Delta_k} X_{(r)}(s) > u, \sup_{s \in \Delta_k} \min_{1 \leq i \leq n+1-r} X_i(s) \leq u \right) \]

\[ =: I_{21} + I_{22}. \]

Since

\[ \sum_{k=1}^{N_u} \mathbb{P} \left( \sup_{s \in \Delta_0} \min_{1 \leq i \leq n+1-r} X_i(s) > u, \sup_{s \in \Delta_k} \min_{1 \leq i \leq n+1-r} X_i(s) \leq u, \sup_{s \in \Delta_k} X_{(r)}(s) > u \right) \]

\[ \leq N_u \mathbb{P} \left( \sup_{s \in \Delta_0} X_1(s) > u \right)^{n+2-r}, \]

we get \( I_{22} = o(I_1) \) as \( u \to \infty \). Moreover, using the relations

\[ I_{21} \leq c_{n,r-1} \frac{u^{2/\alpha}}{S} ET \sum_{k=1}^{N_u} \mathbb{P} \left( \sup_{s \in \Delta_0} X_1(s) > u, \sup_{s \in \Delta_k} X_1(s) > u \right)^{n+r-1} \]

\[ \leq c_{n,r-1} u^{2/\alpha} ET \left( \frac{1}{S^{1/(n+r-1)}} \sum_{k=1}^{N_u} \mathbb{P} \left( \sup_{s \in \Delta_0} X_1(s) > u, \sup_{s \in \Delta_k} X_1(s) > u \right) \right)^{n+r-1}, \]

we are left with finding a tight asymptotic bound for

\[ \frac{1}{S^{1/(n+r-1)}} \sum_{k=1}^{N_u} \mathbb{P} \left( \sup_{s \in \Delta_0} X_1(s) > u, \sup_{s \in \Delta_k} X_1(s) > u \right), \]

which follows by the same argument as that given in the proof of Theorem D.2 in [12] (see also the proof of Theorem 3.1 in [3]), with the minor exception that the
first term in the above summand is bounded by
\[
P\left( \sup_{s \in \Delta_0} X_1(s) > u, \sup_{s \in \Delta_1} X_1(s) > u \right) 
\leq P\left( \sup_{s \in [0,S u^{-2/\alpha}]} X_1(s) > u, \sup_{[S+S^{1/(2(n+r-1))} u^{-2/\alpha}, (2S+S^{1/(2(n+r-1))} u^{-2/\alpha}]} X_1(s) > u \right) 
+ P\left( \sup_{s \in [0,S^{1/(2(n+r-1))} u^{-2/\alpha}]} X_1(s) > u \right).
\]

This completes the proof of Theorem 3.1(i).

(ii) For any \(0 < A < B < \infty\) and sufficiently large \(u\), we make the following decomposition:
\[
P\left( \sup_{t \in [0,T]} X_{(r)}(t) > u \right) 
= (\int_0^{Am_r(u)} + \int_{Am_r(u)}^{Bm_r(u)} + \int_{Bm_r(u)}^{\infty}) P\left( \sup_{s \in [0,t]} X_{(r)}(s) > u \right) dP(T > t) 
= I_1 + I_2 + I_3.
\]

We analyze \(I_1, I_2, I_3\) separately.

Integral \(I_1\). Since the process \(\{X_{(r)}(t), t \geq 0\}\) is stationary, by Bonferroni’s inequality, we have
\[
I_1 \leq \int_0^{Am_r(u)} P(T > t) dt = 1 - Am_r(u)P(T > Am_r(u))(1 + o(1)) \quad \text{as} \quad u \to \infty,
\]
which, combined with (4.6) and Theorem 2.2 in [K], implies that
\[
I_1 \leq \frac{\lambda}{1-\lambda} A^{1-\lambda}P(T > m_r(u))(1 + o(1)) \quad \text{as} \quad u \to \infty.
\]
Integral $I_3$. It is straightforward that

$$I_3 \leq \mathbb{P}(T > Bm_r(u)) \left(1 + o(1)\right) = B^{-\lambda} \mathbb{P}(T > m_r(u)) \left(1 + o(1)\right) \text{ as } u \to \infty.$$ 

Integral $I_2$. For any $\varepsilon > 0$ and sufficiently large $u$, applying Theorem 3.1, we get the upper bound

$$I_2 = \int_A^B \mathbb{P}(\sup_{s \in [0,xm_r(u)]} X_{(r)}(s) > u) d\mathbb{P}(T \leq xm_r(u))$$

$$\leq (1 + \varepsilon) \int_A^B (1 - e^{-x}) d\mathbb{P}(T \leq xm_r(u))$$

$$= (1 + \varepsilon) \int_A^B e^{-x} \mathbb{P}(T > xm_r(u)) dx - (1 + \varepsilon)(1 - e^{-B}) \mathbb{P}(T > Bm_r(u))$$

$$+ (1 + \varepsilon)(1 - e^{-A}) \mathbb{P}(T > Am_r(u)),$$

and similarly we obtain the lower bound

$$I_2 \geq (1 - \varepsilon) \int_A^B e^{-x} \mathbb{P}(T > xm_r(u)) dx - (1 - \varepsilon)(1 - e^{-B}) \mathbb{P}(T > Bm_r(u))$$

$$+ (1 - \varepsilon)(1 - e^{-A}) \mathbb{P}(T > Am_r(u)).$$

Since $T$ has a regularly varying tail distribution at infinity, by Theorem 1.5.2 in [5], we get

$$\int_A^B e^{-x} \mathbb{P}(T > xm_r(u)) dx = \mathbb{P}(T > m_r(u)) \int_A^B e^{-x} x^{-\lambda} dx \left(1 + o(1)\right) \text{ as } u \to \infty.$$ 

Thus, for any $\varepsilon > 0$ and $0 < A < B < \infty$, we obtain

$$\limsup_{u \to \infty} \frac{I_2}{\mathbb{P}(T > m_r(u))} \leq (1 + \varepsilon) \left( \int_0^B x^{-\lambda} e^{-x} dx - (1 - e^{-B})B^{-\lambda} + (1 - e^{-A})A^{-\lambda} \right)$$

and

$$\liminf_{u \to \infty} \frac{I_2}{\mathbb{P}(T > m_r(u))} \leq (1 - \varepsilon) \left( \int_0^B x^{-\lambda} e^{-x} dx - (1 - e^{-B})B^{-\lambda} + (1 - e^{-A})A^{-\lambda} \right).$$
Therefore, letting $A \to 0$, $B \to \infty$, and $\varepsilon \to 0$, we find that $I_1$ and $I_3$ are negligible, and

$$I_2 = \Gamma(1 - \lambda)\mathbb{P}(T > m_r(u))(1 + o(1)) \quad \text{as } u \to \infty,$$

which completes the proof of Theorem 3.2(ii).

(iii) Lower bound. From Theorem 3.1, for any given $B > 0$, it follows that

$$\mathbb{P}\left( \sup_{t \in [0,T]} X(r)(t) > u \right) \geq \mathbb{P}\left( \sup_{s \in [0,Bm_r(u)]} X(r)(s) > u \right) \mathbb{P}(T > Bm_r(u))$$

$$= (1 - e^{-B})\mathbb{P}(T > m_r(u))(1 + o(1))$$

as $u \to \infty$. Thus, letting $B \to \infty$, we obtain the asymptotic lower bound

$$\mathbb{P}\left( \sup_{t \in [0,T]} X(r)(t) > u \right) \geq \mathbb{P}(T > m_r(u))(1 + o(1)) \quad \text{as } u \to \infty.$$

Upper bound. For given $A > 0$, we get

$$\mathbb{P}\left( \sup_{t \in [0,T]} X(r)(t) > u \right)$$

$$\leq \int_0^T \mathbb{P}\left( \sup_{s \in [0,t]} X(r)(s) > u \right) d\mathbb{P}(T \leq t) + \mathbb{P}(T > Am_r(u))$$

$$= \int_0^T \mathbb{P}\left( \sup_{s \in [0,t]} X(r)(s) > u \right) d\mathbb{P}(T \leq t) + \mathbb{P}(T > m_r(u))(1 + o(1))$$

as $u \to \infty$. Due to the stationarity of the process $\{X(r)(t), t \geq 0\}$ and Bonferroni’s inequality, we have

$$\int_0^T \mathbb{P}\left( \sup_{s \in [0,t]} X(r)(s) > u \right) d\mathbb{P}(T \leq t)$$

$$\leq \mathbb{P}\left( \sup_{s \in [0,1]} X(r)(s) > u \right) \left( \int_0^T t d\mathbb{P}(T \leq t) + 1 \right)$$

$$\leq \mathbb{P}\left( \sup_{s \in [0,1]} X(r)(s) > u \right) \left( \int_0^T \mathbb{P}(T > t) dt + 1 \right).$$

From Karamata’s theorem (see, e.g., Proposition 1.5.8 in [5]), we get

$$\int_0^T \mathbb{P}(T > t) dt = Am_r(u)\mathbb{P}(T > Am_r(u))(1 + o(1))$$
as \( u \to \infty \), which, combined with (4.7) and Theorem 2.2 in [8], implies that

\[
\mathbb{P} \left( \sup_{t \in [0,T]} X(r)(t) > u \right) \leq (1 + A) \mathbb{P} \left( T > m_r(u) \right) \left( 1 + o(1) \right)
\]

as \( u \to \infty \). Letting \( A \to 0 \), we obtain (3.4). This completes the proof of Theorem 3.2.

Acknowledgments. The author would like to thank Professor Krzysztof Dębicki and the referees for their valuable comments.

REFERENCES


Chunming Zhao
Department of Statistics, School of Mathematics
Southwest Jiaotong University
Xi’an Road 999, Xipu, Pixian
Chengdu, Sichuan 611756, PR of China
E-mail: cmzhao@swjtu.cn

Received on 17.3.2015;
revised version on 23.10.2016