SOME PROPERTIES OF THE EMPTINESS TIME OF A DAM

BY

B. KOPOCIŃSKI (WROCLAW)

Abstract. We investigate the emptiness time $T(c)$ of a dam initiated by content $c \geq 0$, assuming that the total input process is a compound Poisson one and that the release rate function is given. We prove that if the release function is increasing (decreasing), then the expected value $ET(c), c \geq 0$, is concave (convex). We also give an estimation of the expected value of the emptiness time under some deformations of inputs.

1. Introduction. Let us consider the content process of a dam generated by a compound Poisson process and a general release function. Let $\tau = (\tau_1, \tau_2, \ldots)$ denote the sequence of input instances forming the Poisson process with some parameter $\lambda$ and let $A = (A_1, A_2, \ldots)$ denote the sequence of inputs being a sequence of independent random variables with some distribution function $F$. The release rate function is denoted by $r = r(z), z \geq 0$, and it is assumed that $r(0) = 0, r(z) > 0, z > 0$.

In this note we consider the emptiness time of the dam initiated by content $c \geq 0$. The unconditional emptiness time of the dam is equal to $T(A) = T$, where $A$ is a random variable independent of the input process and distributed according to $F$.

It is obvious that for $r(z) = 1, z > 0$, the content process of the dam reduces to the virtual waiting process in the queueing system $M/G/1$ generated by $\tau$ and $A$. In that case we have

$$T(c) = c + T_1 + T_2 + \ldots + T_{N(c)}, \quad c \geq 0,$$

where $N(c), c \geq 0$, is the Poisson process with parameter $\lambda$, and $T_1, T_2, \ldots$ are independent random variables distributed as $T$.

Equation (1) implies that in the queueing process the value $ET(c) = (1 + \lambda ET)c, c \geq 0$, is a linear function in $c$. Thus $ET = (1 + \lambda ET)EA$, and $ET = EA/(1 - \lambda EA)$ depends only upon the expectations $E\tau_1 = 1/\lambda$ and $EA$. 
This note is an attempt to generalize these results. We prove that if the function \( r \) is increasing (decreasing), then the function \( ET(c) \), \( c \geq 0 \), is concave (convex). This property enables us to characterize the consequence of the deformations of the inputs on the emptiness time of the dam. This result extends the paper [3] where the consequences of deformations of the content process were analyzed.

2. The emptiness time. Let us introduce after Harrison and Resnick [2] the functions necessary to a description of the emptiness time process:

\[
R(c) = \int_0^c \frac{1}{r(u)} \, du, \quad z(c, t) = R^{-1}(R(c) - t), \quad c \geq 0, \ 0 \leq t \leq R(c),
\]

gives the content \( z(c, t) \) of the dam initiated by content \( c \) and with no inputs in the time interval \([0, R(c)]\).

The stochastic process \( T(c) \), \( c \geq 0 \), satisfies the equality

\[
T(c) = \begin{cases} 
R(c) & \text{if } \tau_1 > R(c), \\
\tau_1 + T_1(z(c, \tau_1) + A_1) & \text{if } \tau_1 \leq R(c), 
\end{cases}
\]

where \( T_1(c) \), \( c \geq 0 \), is the emptiness time process generated by the jump moments \( \tau_1 = (\tau_2, \tau_3, \ldots) \), the inputs \( A_1 = (A_2, A_3, \ldots) \), and the release function \( r \).

It is easy to prove that (1) and (2) are equivalent in the case \( r(z) = 1 \), \( z > 0 \).

In this note we deal with the expected value \( \theta(c) = ET(c) \), \( c \geq 0 \). Taking in (2) the expected value with respect to \( \tau_1 \) we get

\[
E_{\tau_1} T(c) = R(c) e^{-\lambda R(c)} + \int_0^{R(c)} \left( u + T_1(z(c, u) + A_1) \right) \lambda e^{-\lambda u} \, du
\]

\[
= \frac{1}{\lambda} \left( 1 - e^{-\lambda R(c)} \right) + \int_0^{R(c)} T_1(z(c, u) + A_1) \lambda e^{-\lambda u} \, du, \quad c \geq 0.
\]

Hence, for the function \( \theta(c) \), \( c \geq 0 \), we get the integral equation

\[
\theta(c) = \frac{1}{\lambda} \left( 1 - e^{-\lambda R(c)} \right) + \int_0^{R(c)} E_A \theta(z(c, u) + A) \lambda e^{-\lambda u} \, du, \quad c \geq 0.
\]

Equation (3) may be solved in a standard manner (see [1]). To that purpose define the nonnegative operator \( \varphi \) to operate on nondecreasing function \( f(c) \), \( c \geq 0 \), having the form

\[
\varphi(f)(c) = \int_0^{R(c)} E_A f(z(c, u) + A) \lambda e^{-\lambda u} \, du, \quad c \geq 0.
\]
Emptiness time of a dam

If we define $\phi^0(f) = f$, $\phi^n(f) = \phi(\phi^{n-1}(f))$, $n = 1, 2, \ldots$, then the solution of (3), provided it exists, takes form

$$
\theta(c) = \sum_{n=0}^{\infty} \phi^n(1 - e^{-\lambda R(c)})(c), \quad c \geq 0.
$$

The equation solution (4) is rather no suitable for subsequent considerations including the simplest case of the release function $r(z) = 1$, $z > 0$. In the sequel we reformulate equation (3) to the differential form.

**Lemma 1.** The function $\theta(c)$, $c \geq 0$, satisfies the equation

$$
\theta'(c) = \frac{1}{r(c)}(1 - \lambda \theta(c) + \lambda E_A \theta(c + A)), \quad c \geq 0.
$$

**Proof.** For the function $z = z(c, u)$, $c \geq 0$, $0 \leq u \leq R(c)$, we have

$$
\frac{dz}{dc} = \frac{r(z)}{r(c)}, \quad \frac{dz}{du} = -r(z).
$$

Differentiation of (3) gives

$$
\theta'(c) = e^{-\lambda R(c)} \frac{1}{r(c)} + E_A \theta(z(c, R(c)) + A) \lambda e^{-\lambda R(c)} \frac{1}{r(c)} +
$$

$$
+ \int_0^{R(c)} E_A \theta'(z(c, u) + A) \frac{r(z)}{r(c)} \lambda e^{-\lambda u} du
$$

$$
= \frac{1}{r(c)} \left[ e^{-\lambda R(c)} + \lambda E_A \theta(A) e^{-\lambda R(c)} -
$$

$$
- \int_0^{R(c)} \left( \frac{d}{du} E_A \theta(z(c, u) + A) \lambda e^{-\lambda u} du \right) \right].
$$

Integrating by parts we obtain

$$
\int_0^{R(c)} \left( \frac{d}{du} E_A \theta(z(c, u) + A) \right) e^{-\lambda u} du
$$

$$
= E_A \theta(A) e^{-\lambda R(c)} - E_A \theta(c + A) + \int_0^{R(c)} E_A \theta(z(c, u) + A) \lambda e^{-\lambda u}
$$

$$
= E_A \theta(A) e^{-\lambda R(c)} - E_A \theta(c + A) + \theta(c) - \frac{1}{\lambda} (1 - e^{-\lambda R(c)}).
$$

Substituting the above into (5) we get Lemma 1.

Let $T(a, b)$, $a \geq b \geq 0$, denote the first passage time in the content process of the dam from state $a$ to state $b$. But $T(a, 0) = T(a)$ and $T(c + a) = T(c + a, c) + T(c)$, $c \geq 0$, so that

$$
\theta(c + a) = \theta(c + a, c) + \theta(c) \geq \theta(c), \quad \text{where } \theta(c + a, c) = E T(c + a, c) \geq 0.
$$
THEOREM 1. If the function $r$ is increasing (decreasing), then the function $\theta(c)$, $c \geq 0$, is concave (convex).

Proof. In the proof we restrict our considerations to the case of the increasing function $r$. Let us consider the difference $D = E_A \theta'(c + A) - \theta'(c)$. From Lemma 1 we have

$$D = E_A \frac{1}{r(c+A)} (1 - \lambda \theta(c + A) + \lambda E_A \theta(c + A + A')) -$$

$$- \frac{1}{r(c)} (1 - \lambda \theta(c) + \lambda E_A \theta(c + A'))$$

$$\leq \frac{\lambda}{r(c)} E_A E_A' (\theta(c + A + A') - \theta(c + A') - \theta(c + A) + \theta(c))$$

$$= \frac{\lambda}{r(c)} E_A E_A' (\theta(c + A + A', c) - \theta(c + A, c) - \theta(c + A', c))$$

$$= \frac{\lambda}{r(c)} E_A E_A' (\theta(c + A + A', c + A) - \theta(c + A', c)).$$

For any release function $r$ and $a \geq 0$ define $r_a(0) = 0$, $r_a(z) = r(z + a)$, $z > 0$. Let us consider the process $T_a(c)$, $c \geq 0$, generated by the release function $r_a$ and the random sequences $A$ and $\tau$. The characteristics of this process will be indexed by the parameter $a$.

The equality $R_a(b) = R(a + b) - R(a)$ implies $z_a(b, t) + a = z(a + b, t)$, $t \geq 0$, and, in consequence, $T_a(c + b, c) = T(c + a + b, c + a)$, $a, b, c \geq 0$. For the increasing function $r$ and every $a \geq 0$ we have $r_a(z) \geq r(z)$, $z \geq 0$. Thus $z_a(c, t) \leq z(c, t)$, $t \geq 0$, whence $T_a(c + b, c) \leq T(c + b, c)$, $a, b, c \geq 0$. Finally,

$$\theta(c + a + b, c + a) = \theta_a(c + b, c) \leq \theta(c + b, c).$$

Substituting the above into (6) we have $D \leq 0$, which, by Lemma 1, completes the proof of Theorem 1.

3. The deformation of inputs. Let us consider the sequence of inputs $A_n^{**} = A_n + A_n$, $n = 1, 2, \ldots$, being a deformation of the sequence $A$. Assume that $(A_n, A_n)$, $n = 1, 2, \ldots$, are independent, $A_n + A_n \geq 0$, $E A_n | A_n = 0$, $n = 1, 2, \ldots$. The emptiness time process of the dam under the deformation assumption is indexed by two asterisks. Theorem 1 and Jensen's inequality applied to (3) lead to the following result:

THEOREM 2. If in the model of the dam the release function $r$ is increasing (decreasing), then the deformation of inputs decreases (increases) the emptiness time of the dam in expectation:

$$ET^{**}(c) \leq ET(c) \quad (ET^{**}(c) \geq ET(c)), \quad c \geq 0.$$
REFERENCES


Institute of Mathematics, Wrocław University
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

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