Abstract. In this paper we consider the problem of sequential estimation for the stationary zero-mean Gaussian process whose spectral density is of the form \(2\pi(\lambda^2 + \theta^2)^{-1}\), where \(\theta > 0\) is an unknown parameter. We find the class of Markov stopping times determining optimal sequential estimation plans for a given function \(g(\theta)\). A sequential plan is optimal if the lower bound in the information inequality is attained. Moreover, the form of efficient sequential estimators is derived and the class of efficiently estimable functions is investigated.

1. Preliminaries. Let \(\xi(t) = \xi_\theta(t), \ t \in T = [0, \infty),\) be a separable stationary zero-mean Gaussian process with the spectral density

\[
\varphi(\lambda) = \frac{1}{2\pi(\lambda^2 + \theta^2)}, \quad -\infty < \lambda < \infty,
\]

where \(\theta \in D = (0, \infty)\) is an unknown parameter. Such a process is a Markov one and has continuous sample functions with probability 1. The covariance function of the process \(\xi_\theta(t), \ t \in T,\) is defined by

\[
B_\theta(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \varphi(\lambda) d\lambda = (2\pi)^{-1} \exp(-\theta |t|)
\]

and the variance of this process is equal to \(B_\theta(0) = (2\pi)^{-1}\).

The processes \(\{\xi_\theta(t), \ t \in T\}, \ \theta \in D,\) can be determined in the following way. Assume that \(W(t), \ t \in T,\) is a Wiener process on a probability space \((\Omega, \mathcal{F}, P)\) and \(X\) is a random variable on \((\Omega, \mathcal{F}, P),\) independent of \(W(t), \ t \in T,\) and standard normally distributed. Let \(\mathcal{F}_t = \sigma\{W(s), s \leq t; X\}\) be a
family of $\sigma$-algebras. Then, for every $\mathcal{G} \in \mathcal{D}$, a unique solution $\xi(t) = \xi_\mathcal{G}(t)$, $t \in T$, of the stochastic integral equation

$$\xi(t) - \xi(0) = - \mathcal{G} \int_0^t \xi(s) \, ds + W(t), \quad \xi(0) = \frac{1}{\sqrt{2\mathcal{G}}} X,$$

considered with respect to $W(t)$, $t \in T$, and $\mathcal{F}_t$, is the process with the above-mentioned properties.

In the sequel, if no ambiguity arises, we omit the index $\mathcal{G}$ and write simply $\xi(t)$, $t \in T$.

2. **Sufficient statistics.** An equivalent model of the processes $\{\xi_\mathcal{G}(t), t \in T\}$, $\mathcal{G} \in \mathcal{D}$, obtained by the canonical form will be useful in our considerations. Let $C$ be the space of all continuous real-valued functions $x = x(t)$, $t \in T$, and let $\mathcal{C} = \sigma \{x = x(t), t \in T\}$ denote the minimal $\sigma$-algebra consisting of all cylinder sets of $C$. By $\mu_\mathcal{G}$ we denote the measure on $(C, \mathcal{C})$ corresponding to the process $\xi_\mathcal{G}(t), t \in T$:

$$\mu_\mathcal{G}(B) = P(\xi_\mathcal{G}(\cdot) \in B), \quad B \in \mathcal{C}. $$

Let $\mu_{\mathcal{G},t}$ be the truncation of the measure $\mu_\mathcal{G}$ on $\mathcal{C}_t = \sigma \{x = x(s), s \leq t, s \in T\}$.

Let us consider the sequential statistical space $(C, \mathcal{C}_t, \{\mu_{\mathcal{G},t}, \mathcal{G} \in \mathcal{D}\}), t \in T$, corresponding to the family of processes $\{\xi_\mathcal{G}(t), t \in T\}, \mathcal{G} \in \mathcal{D}$. Let $R$ be the real line and let $\mathcal{B}_R$ denote the $\sigma$-algebra of Borel subsets of $R$. A function $Z(t, x): T \times C \to R^k$ such that for every $t \in T$ the mapping $Z(t, \cdot)$ is $(\mathcal{C}_t, \mathcal{B}_R)$-measurable will be called a $(k$-dimensional) statistic on the space $(C, \mathcal{C}_t, \{\mu_{\mathcal{G},t}, \mathcal{G} \in \mathcal{D}\}), t \in T$.

**Lemma 1.** (a) The statistical space $(C, \mathcal{C}_t, \{\mu_{\mathcal{G},t}, \mathcal{G} \in \mathcal{D}\}), t \in T$, is dominated by a measure $\mu_{\mathcal{G}_0,t}$ for some $\mathcal{G}_0 \in \mathcal{D}$.

(b) The densities $d\mu_{\mathcal{G},t}/d\mu_{\mathcal{G}_0,t}$ are defined by

$$\frac{d\mu_{\mathcal{G},t}}{d\mu_{\mathcal{G}_0,t}}(x) = \exp \left\{ \frac{1}{2} \left[ \log \mathcal{G}_0 - \log \mathcal{G}_0 - (\mathcal{G}_0 - \mathcal{G}_0) Z_1(t, x) - (\mathcal{G}_0^2 - \mathcal{G}_0^2) Z_2(t, x) \right] \right\},$$

where $Z(t, x) = (Z_1(t, x), Z_2(t, x))$, and

$$Z_1(t, x) = x^2(0) + x^2(t) - t,$$

$$Z_2(t, x) = \int_0^t x^2(s) \, ds.$$
(c) The statistic $Z(t, x) = (Z_1(t, x), Z_2(t, x))$ with $Z_1(t, x)$ and $Z_2(t, x)$ defined by (4) and (5), respectively, is a (two-dimensional) sufficient statistic on the space $(C, C_t, \{\mu_\theta, \theta \in D\})$, $t \in T$.

Proof. Using the results on absolutely continuous substitution of measures [4] or the Skorohod theorems [6] we get (a) and the formula

$$
\frac{d\mu_\theta, t}{d\mu_{\theta, 0}}(\xi_{\theta, 0}(\cdot))
$$

where $p_{\theta, 0}(\cdot; \theta)$ denotes the distribution density of values of the process $\xi_{\theta}(t)$, $t \in T$, at time $t = 0$ relative to this distribution for the process $\xi_{\theta, 0}(t), t \in T$. The function $p_{\theta, 0}(\cdot; \theta)$ is defined as $p_{\theta, 0}(\cdot; \theta) = p(\cdot; \theta)/p(\cdot; \theta_0)$, where $p(\cdot; \theta)$ is the distribution density of values of the process $\xi_{\theta}(t), t \in T$, at time $t = 0$. We have

$$
p(\xi_{\theta, 0}(0); \theta) = \frac{1}{\sigma_\theta \sqrt{2\pi}} \exp \left[ -\frac{\xi_{\theta, 0}^2(0)}{2\sigma_\theta^2} \right],
$$

where $\sigma_\theta^2 = (2\theta)^{-1}$ is the variance of the process $\xi_{\theta}(t), t \in T$. Thus

$$
p_{\theta, 0}(\xi_{\theta, 0}(0); \theta) = (\theta/\theta_0)^{1/2} \exp \left[ -(\theta - \theta_0)\xi_{\theta, 0}^2(0) \right].
$$

From Ito's formula for processes satisfying equation (2) we obtain

$$
\int_0^t \xi_{\theta, 0}(s) dW(s) = \frac{1}{2} [\xi_{\theta, 0}^2(t) - \xi_{\theta, 0}^2(0) - t] + \theta_0 \int_0^t \xi_{\theta, 0}^2(s) ds.
$$

Substituting (7) and (8) into (6) we get

$$
\frac{d\mu_\theta, t}{d\mu_{\theta, 0}}(\xi_{\theta, 0}(\cdot)) = \exp \left\{ \frac{1}{2} [\log \theta - \log \theta_0 - (\theta - \theta_0)(\xi_{\theta, 0}^2(0) + \xi_{\theta, 0}^2(t) - t) - (\theta^2 - \theta_0^2) \int_0^t \xi_{\theta, 0}^2(s) ds] \right\},
$$

which is equivalent to (3).

(c) follows from the Fisher-Neyman theorem on factorization (see, e.g., [2], Chap. II, § 2).

3. Absolute continuity of the measures generated by a Markov stopping time and a sufficient statistic. Let $\tau = \tau(x)$ be a finite Markov time with respect to the family $\mathcal{G}_t$, $t \in T$, i.e., $\tau: C \rightarrow [0, \infty]$ so that $\{x: \tau(x) \leq t\} \in \mathcal{G}_t$ for every $t \in T$ and $\mu_{\theta}(\{x: \tau(x) < \infty\}) = 1$ for all $\theta \in D$. Let $U = T \times \mathbb{R}^2$ and let $t = t(u)$ and $z = z(u) = (z_1(u), z_2(u))$ be the components of the point $u \in U$. The
pair $\mathcal{F}(x) = (\tau(x), Z(\tau(x), x))$ of $\mathcal{F}_t$-measurable functions generates for every $\mathcal{F} \in D$ the measure $m_\mathcal{F}$ on $(U, \mathcal{B}_U)$ in the standard way: for every $A \in \mathcal{B}_U$,

$$m_\mathcal{F}(A) = \mu_\mathcal{F}(\mathcal{F}^{-1}(A)) = \mu_\mathcal{F}(\{(\tau(x), Z(\tau(x), x)) \in A\}).$$

**Lemma 2.** For every finite Markov time $\tau$ there exists a $\sigma$-finite measure $m_\tau$ on $(U, \mathcal{B}_U)$ independent of $\mathcal{F}$ and such that for every $A \in \mathcal{B}_U$ and every $\mathcal{F} \in D$

$$m_\tau(A) = \int \exp \left\{ \frac{1}{2} [\log \mathcal{F} - \mathcal{F}_1(u) - \mathcal{F}_2^2(u)] \right\} m_\mathcal{F}(du).$$

**Proof.** From the modification of the Sudakov lemma obtained in [5] for right-continuous functionals it follows that the measures $m_\mathcal{F}$, $\mathcal{F} \in D$, are absolutely continuous with respect to $m_{\mathcal{F}_0}$ and

$$\frac{dm_\mathcal{F}}{dm_{\mathcal{F}_0}}(u) = h(z(u); \mathcal{F}, \mathcal{F}_0),$$

i.e. (see formula (3))

$$\frac{dm_\mathcal{F}}{dm_{\mathcal{F}_0}}(u) = \exp \left\{ \frac{1}{2} [\log \mathcal{F} - \log \mathcal{F}_0 - (\mathcal{F} - \mathcal{F}_0) z_1(u) - (\mathcal{F}_0^2 - \mathcal{F}_0 z_2(u))] \right\}.$$

Introducing the measure $m_\tau$ defined by

$$m_\tau(du) = \exp \left\{ \frac{1}{2} [-\log \mathcal{F}_0 + \mathcal{F}_0 z_1(u) + \mathcal{F}_0^2 z_2(u)] \right\} m_{\mathcal{F}_0}(du)$$

we complete the proof.

**4. Sequential plans.** Let $g(\mathcal{F})$ be a real-valued function of the parameter $\mathcal{F} \in D$. We observe the process $\xi(t)$, $t \in T$, up to time $\tau$ and want to estimate the function $g(\mathcal{F})$. A $(\mathcal{B}_U, \mathcal{B}_R)$-measurable function $f : U \to R$ will be called an estimator for $g(\mathcal{F})$.

**Definition.** By a sequential estimation plan for $g(\mathcal{F})$ we mean any pair $\delta = (\tau, f)$ consisting of a Markov time $\tau$ satisfying, for all $\mathcal{F} \in D$, the condition

$$P(0 < \tau(\xi_0) < \infty) = 1$$

and of an estimator $f$ such that, for every $\mathcal{F} \in D$,

$$E_\mathcal{F} f^2(\mathcal{F}(\xi_0)) = \int f^2(u) \exp \left\{ \frac{1}{2} [\log \mathcal{F} - \mathcal{F}_1(u) - \mathcal{F}_2^2(u)] \right\} m_\mathcal{F}(du) < \infty$$

and

$$E_\mathcal{F} f(\mathcal{F}(\xi_0)) = \int f(u) \exp \left\{ \frac{1}{2} [\log \mathcal{F} - \mathcal{F}_1(u) - \mathcal{F}_2^2(u)] \right\} m_\mathcal{F}(du) = g(\mathcal{F}).$$

It follows from (10) that the observation of the process $\xi(t)$, $t \in T$, terminates in a finite time. Condition (12) means that $f$ is an unbiased estimator for $g(\mathcal{F})$. 
From (10) and Lemma 2 we have

$$\int \exp \left\{ \frac{1}{2} \left[ \log \theta - \theta z_1(u) - \theta^2 z_2(u) \right] \right\} m_\tau(du) = 1$$

for every $\theta \in D$.

For simplicity, in the sequel we put $Z_1(\tau) = Z_1(\tau(\xi), \xi)$ and $Z_2(\tau) = Z_2(\tau(\xi), \xi)$.

Now, we formulate the following regularity conditions:
(i) $g(\theta)$ is differentiable and not identically constant on $D$;
(ii) $0 < E_\theta [1/9 - Z_1(\tau) - 28Z_2(\tau)]^2 < \infty$ for every $\theta \in D$;
(iii) the differentiation and repeated differentiation of the integral with respect to $\theta$ in identities (12) and (13), respectively, is allowed;
(iv) $E_\theta Z_2(\tau)$ is a differentiable function of the variable $\theta \in D$.

**LEMMA 3.** If the regularity conditions (i)-(iv) are satisfied for a sequential plan $(\tau, f)$, then the following identities hold:

(14) \[2g^2E_\theta Z_2(\tau) = 1 - 9E_\theta Z_1(\tau),\]

(15) \[9^2E_\theta [1/9 - Z_1(\tau) - 28Z_2(\tau)]^2 = 49^2E_\theta Z_2(\tau) + 2,\]

(16) \[E_\theta \{f(\tau, Z(\tau)) [1/9 - Z_1(\tau) - 28Z_2(\tau)] \} = 2g'(\theta),\]

(17) \[E_\theta \{Z_2(\tau) [1/9 - Z_1(\tau) - 28Z_2(\tau)] \} = 2E_\theta Z_2(\tau),\]

(18) \[9^2D_\theta Z_1(\tau) = 49^4D_\theta Z_2(\tau) + 49^2E_\theta Z_2(\tau) + 89^3E_\theta Z_2(\tau) + 2\]

($D_\theta(\cdot)$ denotes the variance evaluated at $\theta$).

A simple proof of Lemma 3 is omitted. Identity (18) is obtained from (14), (15), and (17).

Using (14)-(16) and the Schwarz inequality we obtain

**THEOREM 1.** For every sequential plan $(\tau, f)$ satisfying conditions (i)-(iii) the inequality

(19) \[D_\theta f(\tau, Z(\tau)) \geq \frac{2g^2 [g'(\theta)]^2}{1 + 2g^2 E_\theta Z_2(\tau)}\]

holds for all $\theta \in D$. The equality holds at a particular value of $\theta$ if and only if

(20) \[f(u) = c(\theta) [1/9 - z_1(u) - 29z_2(u)] + g(\theta) \text{ m_\tau-a.e., where } c(\theta) \neq 0.\]

Condition (i) implies that a sequential estimation plan $(\tau, f)$ for $g(\theta)$ cannot consist of the estimator $f(u) = \text{const} m_\tau$-a.e. Indeed, if $f(u) = \text{const} m_\tau$-a.e., then $E_\theta f(\tau, Z(\tau)) = \text{const} = g(\theta)$ for all $\theta \in D$, which contradicts the assumption.

A sequential estimation plan $(\tau, f)$ for $g(\theta)$ is said to be efficient at (a fixed value) $\theta$ if (19) becomes equality at $\theta$. The estimator $f$ is then called efficient at this value $\theta$ and the function $g(\theta)$ is efficiently estimable at the point $\theta$. 
A sequential estimation plan \((\tau, f)\) for \(g(\vartheta)\) is said to be efficient if it is efficient at each \(\vartheta \in D\). The estimator \(f\) is then called efficient and the function \(g(\vartheta)\) is efficiently estimable.

It follows from Theorem 1 that a sequential estimation plan \((\tau, f)\) for \(g(\vartheta)\) is efficient at a point \(\vartheta\) if and only if the estimator \(f\) is of the form (20).

**Theorem 2.** If \((\tau, f)\) is an efficient sequential estimation plan for \(g(\vartheta)\), then there exist constants \(\alpha_1, \alpha_2\) not both equal to zero and a constant \(\alpha_3\) such that

\[
\alpha_1 z_1(u) + \alpha_2 z_2(u) + \alpha_3 = 0 \text{ a.e.}
\]

**Proof.** By assumption we can choose points \(\vartheta_1\) and \(\vartheta_2\) in \(D (\vartheta_1 \neq \vartheta_2)\) and then we write equality (20) in the form

\[
f(u) = c(\vartheta_1) [1/\vartheta_1 - z_1(u) - 2\vartheta_1 z_2(u)] + g(\vartheta_1) \text{ a.e.}
\]

and

\[
f(u) = c(\vartheta_2) [1/\vartheta_2 - z_1(u) - 2\vartheta_2 z_2(u)] + g(\vartheta_2) \text{ a.e.},
\]

where \(c(\vartheta_1)\) and \(c(\vartheta_2)\) are both different from zero.

Subtracting one equality from the other we obtain

\[
[c(\vartheta_2) - c(\vartheta_1)] z_1(u) + 2[c(\vartheta_2) - c(\vartheta_1)] z_2(u) +\]

\[
+ c(\vartheta_1)/\vartheta_1 - c(\vartheta_2)/\vartheta_2 + g(\vartheta_1) - g(\vartheta_2) = 0 \text{ a.e.,}
\]

which completes the proof.

**Theorem 3.** In a given sequential plan \((\tau, f)\) the function \(g(\vartheta)\) is efficiently estimable at a point \(\vartheta = \vartheta^0\) if and only if it is of the form

\[
g(\vartheta) = c(\vartheta^0) \{1/\vartheta^0 - 1/\vartheta - 2(\vartheta^0 - \vartheta) E_\vartheta [Z_2(\tau)]\} + g(\vartheta^0).
\]

**Proof.** By Theorem 1 the only efficient estimators at a point \(\vartheta = \vartheta^0\) are those which take the form

\[
f(\vartheta) = c(\vartheta^0) [1/\vartheta^0 - Z_1(\tau) - 2\vartheta^0 Z_2(\tau)] + g(\vartheta^0)
\]

with probability 1, where \(c(\vartheta^0) \neq 0\). Thus the function \(g(\vartheta)\) is efficiently estimable at \(\vartheta = \vartheta^0\) if and only if it is equal to the expected value of the estimator defined by (23). Therefore

\[
g(\vartheta) = E_\vartheta [f(\vartheta)] = c(\vartheta^0) \{1/\vartheta^0 - E_\vartheta [Z_1(\tau)] - 2\vartheta^0 E_\vartheta [Z_2(\tau)]\} + g(\vartheta^0).
\]

Hence, making use of (14) we obtain (22), which completes the proof.

**Theorem 4.** If in a given sequential plan \((\tau, f)\) the function \(g(\vartheta)\) is efficiently estimable, then it must be of the form

\[
g(\vartheta) = \frac{k_0 + k_1 \vartheta + k_2 \vartheta^2}{l_1 \vartheta + l_2 \vartheta^2},
\]

where \(k_0 \neq 0\), and \(k_1, k_2, l_1, l_2\) are arbitrary constants.
Proof. Suppose that \( \vartheta_1 \) and \( \vartheta_2 \) belong to \( D \) and \( \vartheta_1 \neq \vartheta_2 \). Since the function \( g(\vartheta) \) is efficiently estimable at these points, it follows from Theorem 3 that the equalities
\[
g(\vartheta) = c(\vartheta_1) \{1/\vartheta_1 - 1/\vartheta - 2(\vartheta_1 - \vartheta) E_\vartheta [Z_2(\tau)]\} + g(\vartheta_1)
\]
and
\[
g(\vartheta) = c(\vartheta_2) \{1/\vartheta_2 - 1/\vartheta - 2(\vartheta_2 - \vartheta) E_\vartheta [Z_2(\tau)]\} + g(\vartheta_2)
\]
must hold. Eliminating \( E_\vartheta [Z_2(\tau)] \) from these equalities we get
\[
g(\vartheta) \{c(\vartheta_1) - c(\vartheta_2)\} = \delta^{-1} c(\vartheta_1) c(\vartheta_2)(\vartheta_2 - \vartheta_1) +
\]
\[
+ \delta [c(\vartheta_1) c(\vartheta_2) (1/\vartheta_1 - 1/\vartheta_2) + c(\vartheta_2) g(\vartheta_1) - c(\vartheta_1) g(\vartheta_2)] +
\]
\[
+ c(\vartheta_1) c(\vartheta_2) (\vartheta_1 - \vartheta_2)(\vartheta_1 - \vartheta_2) + \vartheta_1 c(\vartheta_1) g(\vartheta_2) - \vartheta_2 c(\vartheta_2) g(\vartheta_1).
\]
Since neither \( c(\vartheta_1) \) nor \( c(\vartheta_2) \) can be equal to zero, the coefficients \( l_1 = \vartheta_1 c(\vartheta_1) - \vartheta_2 c(\vartheta_2) \) and \( l_2 = c(\vartheta_2) - c(\vartheta_1) \) cannot vanish simultaneously and \( k_0 = c(\vartheta_1) c(\vartheta_2)(\vartheta_2 - \vartheta_1) \neq 0 \). Thus the function standing by \( g(\vartheta) \) in the above-given equality cannot vanish and, consequently, by a proper choice of coefficients we obtain formula (24).

It follows from Theorem 2 that one should seek the efficient sequential plans for a given function \( g(\vartheta) \) from the class described in Theorem 4 among the plans determined by Markov stopping times for which (21) holds.

Let us consider the Markov times
\[
\tau^{(1)}(x) = \inf \{t: Z_1(t, x) = a\},
\]
\[
\tau^{(2)}(x) = \inf \{t: Z_2(t, x) = b\}, \quad 0 < b < \infty,
\]
\[
\tau^{(3)}(x) = \inf \{t: Z_2(t, x) = c_1 Z_1(t, x) + c_2\},
\]
where \( a, b, c_1, c_2 \) are boundaries given in advance and \( Z_1(t, x), Z_2(t, x) \) are defined by (4), (5), respectively. A sequential plan determined by \( \tau^{(2)} \) will be called a fixed-energy plan.

Observe that from Ito's formula for processes satisfying the stochastic integral equation (2) it follows that the relation
\[
Z_1(t, \xi) = 2\xi^2(0) - 28Z_2(t, \xi) + 2I(t, \xi)
\]
holds with probability 1, where \( I(t, \xi) \) denotes the stochastic integral
\[
\int_0^t \xi(s) dW(s).
\]

**Lemma 4.** If \(-\infty < a < 0\), then
\[
E_\vartheta (\tau^{(1)})^n < \infty
\]
for every \( n = 1, 2, \ldots \) and all \( \vartheta \in D \).
Proof. Observe that (see formula (4) and Fig. 1)

$$\tau^{(1)}(\xi) = \inf \{ t : \xi^2(0) + \xi^2(t) = t + a \}$$

and

$$P(\tau^{(1)}(\xi) > t) \leq P(\xi^2(0) + \xi^2(t) > t + a)$$
$$\leq P\left( \xi^2(0) > \frac{t + a}{2} \right) + P\left( \xi^2(t) > \frac{t + a}{2} \right).$$

Since the process $\xi(t)$, $t \in T$, is stationary, the terms on the right-hand side of the above inequality are equal. Thus

$$P(\tau^{(1)}(\xi) > t) \leq 2P\left( \xi^2(0) > \frac{t + a}{2} \right).$$

Taking into account the fact that the random variable $\xi(0)$ is normally distributed with mean zero and variance $\sigma^2$, and using the inequality

$$P(|X| > \lambda) \leq 2\left(\frac{\lambda}{\sqrt{2\pi}}\right) \exp\left(-\frac{1}{2}\lambda^2\right), \quad \lambda > 0,$$

for the standard normally distributed random variable $X$, from (30) we obtain

$$P(\tau^{(1)}(\xi) > t) \leq 2P\left( \sqrt{2\sigma} |\xi(0)| > \sqrt{\sigma}(t + a) \right)$$
$$\leq 4\left[2\pi\sigma(t + a)\right]^{-1/2} \exp \left[-\frac{1}{2}\sigma(t + a)\right]$$

for $t > |a|$. Consequently, putting $k = 4(2\pi\sigma)^{-1/2} \exp \left(-\frac{1}{2}\sigma a\right)$ we have

$$P(\tau^{(1)}(\xi) > t) \leq k(t + a)^{-1/2} \exp \left(-\frac{1}{2}\sigma a\right),$$

for $t > |a|$. Consequently, putting $k = 4(2\pi\sigma)^{-1/2} \exp \left(-\frac{1}{2}\sigma a\right)$ we have

$$P(\tau^{(1)}(\xi) > t) \leq k(t + a)^{-1/2} \exp \left(-\frac{1}{2}\sigma a\right),$$

for $t > |a|$. Consequently, putting $k = 4(2\pi\sigma)^{-1/2} \exp \left(-\frac{1}{2}\sigma a\right)$ we have

$$P(\tau^{(1)}(\xi) > t) \leq k(t + a)^{-1/2} \exp \left(-\frac{1}{2}\sigma a\right),$$
which implies

$$\int |a| t^{a-1} P(\tau^{(1)}(\xi) > t) \, dt < \infty$$

for all $\theta \in D$. Thus the lemma is proved.

In particular, it follows from Lemma 4 that for $-\infty < a < 0$ condition (10) of the closedness of a sequential plan $(\tau^{(1)}, f)$ is satisfied. Henceforth, we shall suppose that $-\infty < a < 0$.

**Theorem 5.** A sequential plan $\delta^{(1)} = (\tau^{(1)}, f^{(1)})$ with

$$f^{(1)} = \lambda_1 Z_2(\tau^{(1)}) + \lambda_2$$

is efficient for

$$g(\theta) = \frac{\lambda_1 (1 - a\theta) + 2\lambda_2 \theta^2}{2\theta^2},$$

where $\lambda_1 \neq 0$ and $\lambda_2$ denote arbitrary constants. The variance of the estimator $f^{(1)}$ is equal to

$$D_\theta f^{(1)} = \frac{\lambda_1^2 (2 - a\theta)}{2\theta^4}.$$

**Proof.** First we show that $E_\theta Z_2^2(\tau^{(1)}) < \infty$ for all $\theta \in D$.

Using the Schwarz inequality we get

$$E_\theta Z_2(\tau^{(1)}) E_\theta \int_0^\infty \xi^2(s) \, ds = E_\theta \int_0^\infty \chi_{(s \in \tau^{(1)})} \xi^2(s) \, ds \leq l \int_0^\infty (E_\theta \chi_{(s \in \tau^{(1)})})^{1/2} \, ds = l \int_0^\infty [P(\tau^{(1)}(\xi) > s)]^{1/2} \, ds,$$

where $\chi_A$ denote the indicator function of the set $A$ and $l = (E_\theta \xi^4(s))^{1/2}$ is a positive and finite constant. Thus, by (31), we obtain $E_\theta Z_2(\tau^{(1)}) < \infty$ for all $\theta \in D$. Then it follows from the properties of stochastic integrals with random upper limits (see, e.g., [3], Part I, § 4) that $E_\theta I(\tau^{(1)}) = 0$ and

$$E_\theta I^2(\tau^{(1)}) = E_\theta Z_2(\tau^{(1)}).$$

For the sequential plan $\delta^{(1)}$ we have $Z_1(\tau^{(1)}) = a$ with probability 1, and formula (28) implies that for this plan the relation

$$2\theta Z_2(\tau^{(1)}) + a = 2\xi^2(0) + 2I(\tau^{(1)})$$

is valid with probability 1. Taking into account (35) we obtain

$$E_\theta [2\theta Z_2(\tau^{(1)}) + a]^2 \leq 8E_\theta \xi^4(0) + 8E_\theta I^2(\tau^{(1)})$$

$$= 8E_\theta \xi^4(0) + 8E_\theta Z_2(\tau^{(1)}).$$
Since $E_{\beta} \xi^4(0) < \infty$ and $E_{\beta} Z_2(\tau^{(1)}) < \infty$ for all $\beta \in D$, we have $E_{\beta} Z_2^2(\tau^{(1)}) < \infty$ for all $\beta \in D$.

Taking into account the finiteness of $E_{\beta} Z_2^2(\tau^{(1)})$ it is easy to verify that regularity condition (iii) is satisfied for the plan $\delta^{(1)}$.

By (14) and (18), for the plan $\delta^{(1)}$ we have

\begin{equation}
E_{\beta} Z_2(\tau^{(1)}) = \frac{1-\alpha \beta}{\beta^2}
\end{equation}

and

\begin{equation}
D_{\beta} Z_2(\tau^{(1)}) = \frac{2-\alpha \beta}{\beta^4},
\end{equation}

and formula (34) follows from (37).

Now we shall prove that an efficient sequential estimator $f(\tau^{(1)}, Z(\tau^{(1)}))$ in the plan $\delta^{(1)}$ is indeed of the form defined by (32) and the function (33) is the only efficiently estimable one in this plan.

Let $f(\tau^{(1)}, Z(\tau^{(1)}))$ be an efficient estimator in the plan $\delta^{(1)}$. Then it is efficient at a certain point $\beta_1 \in D$ and, by Theorem 1, takes the form

$$f(\tau^{(1)}, Z(\tau^{(1)})) = c(\beta_1) [1/\beta_1 - a - 2 \beta_1 Z_2(\tau^{(1)})] + g(\beta_1)$$

with probability 1, where $c(\beta_1) \neq 0$. Hence, if $f(\tau^{(1)}, Z(\tau^{(1)}))$ is an efficient estimator, then there exist constants $\lambda_1 \neq 0$ and $\lambda_2$ such that

$$f(\tau^{(1)}, Z(\tau^{(1)})) = \lambda_1 Z_2(\tau^{(1)}) + \lambda_2 = f^{(1)}.$$ 

Moreover, only the function

\begin{equation}
g(\beta) = E_{\beta} f^{(1)} = \lambda_1 E_{\beta} Z_2(\tau^{(1)}) + \lambda_2
\end{equation}

is efficiently estimable in the plan $\delta^{(1)}$. By (36), from (38) we obtain (33), which completes the proof of the theorem.

In particular, it follows from Theorem 5 that

$$Z_2(\tau^{(1)}) = \int_0^{\tau^{(1)}} \xi^2(s) ds$$

is an efficient sequential estimator for $g(\beta) = (1-\alpha \beta)/2 \beta^2$.

Let us now consider the fixed-energy plan. From the ergodic theorem we obtain the following lemma:

**Lemma 5.** We have

\begin{equation}
P(\tau^{(2)}(\xi_\beta) < \infty) = 1 \quad \text{for all } \beta \in D.
\end{equation}

**Theorem 6.** A sequential plan $\delta^{(2)} = (\tau^{(2)}, f^{(2)})$ with

\begin{equation}
f^{(2)} = \lambda_1 Z_1(\tau^{(2)}) + \lambda_2
\end{equation}
is efficient for

\[ g(\theta) = \frac{\lambda_1 (1 - 2b\theta^2) + \lambda_2 \theta}{\theta} \]

where \( \lambda_1 \neq 0 \) and \( \lambda_2 \) are arbitrary constants. The variance of the estimator \( f^{(2)}(\theta) \) is given by

\[ D_{\theta} f^{(2)} = \frac{2\lambda_2^2}{\theta^2 (1 + 2b\theta^2)}. \]

**Proof.** In the plan \( \delta^{(2)} \) we have \( Z_2(\tau^{(2)}) = b \) with probability 1, and relation (28) takes the form

\[ Z_1(\tau^{(2)}) - 2\xi^2(0) = 2I(\tau^{(2)}) - 2b\theta. \]

Since

\[ E_{\theta} Z_2(\tau^{(2)}) = E_{\theta} \int_0^{\tau^{(2)}} \xi^2(s) ds = b < \infty, \]

using the properties of stochastic integrals with random upper limits and relation (42), we infer in an analogous way as in Theorem 5 that \( E_{\theta} Z_1^2(\tau^{(2)}) < \infty \) for all \( \theta \in D \) and the regularity conditions for the sequential plan are satisfied.

By (14) and (18), for the plan \( \delta^{(2)} \) we have

\[ E_{\theta} Z_1(\tau^{(2)}) = \frac{1 - 2b\theta^2}{\theta}, \]

and

\[ D_{\theta} Z_1(\tau^{(2)}) = \frac{2}{\theta^2 (1 + 2b\theta^2)}. \]

Let \( f(\tau^{(2)}, Z(\tau^{(2)})) \) be an efficient estimator in the plan \( \delta^{(2)} \). Then, similarly as in Theorem 5 we infer from Theorem 1 that it is equal with probability 1 to the estimator defined by (40). By (43) we have

\[ E_{\theta} f^{(2)} = \lambda_1 E_{\theta} Z_1(\tau^{(2)}) + \lambda_2 = \frac{\lambda_1 (1 - 2b\theta^2) + \lambda_2 \theta}{\theta} = g(\theta). \]

Thus the fixed-energy plan \( \delta^{(2)} \) is efficient and \( g(\theta) \) defined by (41) is the only efficiently estimable function in this plan.

By Theorem 6 we conclude that, e.g., \( Z_1(\tau^{(2)}) = \xi^2(0) + \xi^2(\tau^{(2)}) - \tau^{(2)} \) is an efficient sequential estimator for \( g(\theta) = (1 - 2b\theta^2) \theta^{-1} \).

**Lemma 6.** If \( c_1 > 0 \) and \( c_2 > 0 \), then

\[ P(\tau^{(2)}(\xi_{\theta}) < \infty) = 1 \quad \text{for all } \theta \in D. \]
Proof. Let us observe that, by (28),
\[ \tau^{(3)}(\xi_\theta) = \inf \{ t : (1 + 2c_1 \theta) \int_0^t \xi^2(s) \, ds - 2c_1 \int_0^t \xi(s) \, dW(s) = 2c_1 \xi^2(0) + c_2 \} . \]
Put \( \alpha = 1 + 2c_1 \theta \) and \( \beta = 2c_1 \). It is easy to see that condition (44) is satisfied if, for all \( \theta \in D \),
\[ P \left( \lim_{t \to \infty} \int_0^t \xi^2(s) \, ds - \beta \int_0^t \xi(s) \, dW(s) = \infty \right) = 1. \]
Let
\[ \limsup_{T_0 \to 0 \atop 0 \leq t \leq T_0} \left[ \alpha \int_0^t \xi^2(s) \, ds - \beta \int_0^t \xi(s) \, dW(s) \right] = \eta. \]
Then for every \( K > 0 \) and all \( \theta \in D \) we have
\[ P(\eta \geq K) \geq P \left( \sup_{0 \leq t \leq T_0} \left[ \alpha \int_0^t \xi^2(s) \, ds - \beta \int_0^t \xi(s) \, dW(s) \right] \geq K \right) \]
\[ \geq P \left( \alpha \int_0^t \xi^2(s) \, ds - \beta \sup_{0 \leq t \leq T_0} \int_0^t \xi(s) \, dW(s) \geq K \right) \]
\[ \geq P \left( \alpha \int_0^t \xi^2(s) \, ds \geq 2K, \beta \sup_{0 \leq t \leq T_0} \int_0^t \xi(s) \, dW(s) \leq K \right) \]
\[ \geq P \left( \alpha \int_0^t \xi^2(s) \, ds \geq 2K \right) + P \left( \beta \sup_{0 \leq t \leq T_0} \int_0^t \xi(s) \, dW(s) \leq K \right) - 1. \]
Let \( T_0 = K^{3/2} \). Then
\[ P \left( \alpha \int_0^{T_0} \xi^2(s) \, ds \geq 2K \right) = P \left( \frac{\alpha}{K^{3/2}} \int_0^{K^{3/2}} \xi^2(s) \, ds \geq \frac{2}{\sqrt{K}} \right) \]
and by the ergodic theorem this probability tends to 1 as \( K \to \infty \). Moreover, from the well-known inequality for stochastic integrals (see, e.g., [1], Theorem 5.1.1, or [3], Part I, § 3) we get
\[ P \left( \sup_{0 \leq t \leq T_0} \int_0^t \xi(s) \, dW(s) \leq \frac{K}{\beta} \right) \geq 1 - \frac{\beta^2}{K^2} \int_0^{K^{3/2}} \mathbb{E}_\theta \xi^2(s) \, ds = 1 - \frac{\beta \gamma}{\sqrt{K}}, \]
where \( \gamma = \mathbb{E}_\theta \xi^2(s) = (2\theta)^{-1} \). Thus the second probability on the right-hand side of (45) tends also to 1 as \( K \to \infty \). We then have \( P(\eta = \infty) = 1 \), for all \( \theta \in D \), which proves the closedness of the plan.

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Institute of Mathematics
Technical University of Wrocław
Wybrzeże Wyspińskiego 27
50-370 Wrocław, Poland

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