1. Introduction. The paper deals with homogeneous random point processes in $\mathbb{R}^n$. Our aim is to obtain sufficient conditions for asymptotic normality of the number of points of a point process $P$ in a ball when the radius tends to infinity. These conditions are formulated in terms of Palm distributions of the point process $P$. The proofs of the theorems stated below are based on the relation between the distribution $P$ of a point process in $\mathbb{R}^n$ and its Palm distribution due to Ambartzumian [1]. In the one-dimensional case these relations reduce to the so-called Palm-Khinchin formulae [2].

2. Notation. Let $M$ be the class of all countable subsets of $\mathbb{R}^n$ such that any $m \in M$ has no cluster point in a bounded subset of $\mathbb{R}^n$.

Define $N(B, m)$ to be the number of points in $B \cap m$, where $B$ is a bounded Borel set in $\mathbb{R}^n$ and $m \in M$.

Denote by $C$ the minimal $\sigma$-algebra of subsets of $M$ containing all subsets of the form $\{m: N(B, m) = k\}$, $k = 0, 1, 2, \ldots$ Any probability measure $P$ on $C$ describes a random point process.

A random point process $P$ is said to be homogeneous if, for any $c \in C$, $P(tc)$ does not depend on $t \in T$, where $T$ denotes the group of all translations of $\mathbb{R}^n$ and

$$tc = \{m: t^{-1} m \in c\}.$$ 

Further, we assume that for every bounded Borel set $B$ in $\mathbb{R}^n$

$$E_P(N(B)) = \lambda |B|, \quad \lambda < \infty,$$

where $|B|$ is the volume of $B$. In other words, we consider the finite intensity case.
3. Main results. Let $S(v)$ be the sphere of volume $v$ centred at the origin in $\mathbb{R}^n$. We consider the random number $N(S(v))$ of points of the process in $S(v)$ for large values of $v$.

$N(S(v))$ is called \textit{asymptotically normal} if

$$
\sum_{k:(k - \lambda v)/\sqrt{\lambda v} < x} P(N(S(v)) = k) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp \left\{ -\frac{s^2}{2} \right\} ds \quad \text{as } v \to \infty, \quad x \in \mathbb{R}.
$$

$N(S(v))$ is called \textit{locally asymptotically normal in variation} if

$$
\sum_{k=0}^{\infty} \left| P(N(S(v)) = k) - \frac{1}{\sqrt{2\pi \lambda v}} \exp \left\{ -\frac{(k - \lambda v)^2}{2\lambda v} \right\} \right| \to 0 \quad \text{as } v \to \infty.
$$

$N(S(v))$ is called \textit{locally asymptotically normal} if

$$
\sup_k \sqrt{\lambda v} P(N(S(v)) = k) - \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{(k - \lambda v)^2}{2\lambda v} \right\} \to 0 \quad \text{as } v \to \infty.
$$

Note that each of the relations (3) and (4) implies (2). It is known that under the assumption (1) the limit

$$
\pi_k(v) = \lim_{h \to 0} \frac{P(\{N(S(v)) = k\} \cap \{N(S(v+h)-S(v)) = 1\})}{P(N(S(v+h)-S(v)) = 1)}
$$

exists [1].

We call this limit the \textit{spherical Palm distribution}. This limit can be interpreted as the conditional probability of $\{N(S(v)) = k\}$ under the condition that a point of the random process $P$ lies on the boundary of the sphere $S(v)$. By the \textit{Palm distribution} we usually mean the conditional probability of $\{N(S(v)) = k\}$ under the condition that there is a point of the process $P$ at the origin. The spherical Palm distribution can be found by integration of the usual Palm distribution over the boundary of the sphere $S(v)$ (see [1]).

The conditions for asymptotic normality will be given in terms of the variational distance

$$
\varrho(v) = \sum_{k=0}^{\infty} |P_k(v) - \pi_k(v)|, \quad P_k(v) = P(N(S(v)) = k).
$$

We show that the above-mentioned types of asymptotic normality of $N(S(v))$ are implied by various assumptions concerning the rate of convergence of $\varrho(v)$ to zero as $v$ tends to infinity. In this sense the condition on the rate of convergence of $\varrho(v)$ to zero can replace the usual mixing condition [3].
In the sequel we prove the following theorems:

**Theorem 1.** If
\[
\lim_{v \to \infty} \frac{1}{\sqrt{v}} \int_0^v \varrho(u) \, du = 0,
\]
then \(N(S(v))\) is asymptotically normal.

**Theorem 2.** If
\[
\int_0^\infty \varrho(u) \, du < \infty,
\]
then \(N(S(v))\) is locally asymptotically normal in variation.

**Theorem 3.** If
\[
\lim_{v \to \infty} \frac{1}{\sqrt{v}} \int_0^v u \varrho(u) \, du = 0,
\]
then \(N(S(v))\) is locally asymptotically normal.

The proofs of these theorems are based on the following Ambartsumian relations (see [1]):

\[
\frac{dP_0(v)}{dv} = -\lambda \pi_0(v),
\]

(7)
\[
\frac{dP_k(v)}{dv} = -\lambda (\pi_k(v) - \pi_{k-1}(v)), \quad k = 1, 2, \ldots,
\]

\[
P_k(0) = \begin{cases} 1 & \text{for } k = 0, \\ 0 & \text{for } k \neq 0. \end{cases}
\]

Further, without loss of generality we can assume that \(\lambda = 1\).

In the one-dimensional case, Theorems 1 and 2 were announced in [4].

The role of the Palm distribution in the problems related to asymptotic normality of \(N(S(v))\) was first noticed by R. V. Ambartumian to whom Theorem 1 should be attributed.

The author expresses his gratitude to Professor R. V. Ambartumian for suggesting the present topic.

4. **Proof of Theorem 1.** Rewrite (7) in the form

\[
\frac{dP_0(v)}{dv} = -P_0(v) + (P_0(v) - \pi_0(v)),
\]

(8)
\[
\frac{dP_k(v)}{dv} = -(P_k(v) - P_{k-1}(v)) + (P_k(v) - \pi_k(v)) + (\pi_{k-1}(v) - P_{k-1}(v)),
\]

\[k = 1, 2, \ldots\]
We introduce the generating functions
\[ \Pi_z(v) = \sum_{k=0}^{\infty} P_k(v)z^k, \quad A_z(v) = \sum_{k=0}^{\infty} (\pi_k(v) - P_k(v))z^k \]
for which from (8) we derive
\[ \frac{d\Pi_z(v)}{dv} = (z-1)\Pi_z(v) + (z-1)A_z(v), \quad \Pi_z(0) = 1. \]
Hence we get
\[ \Pi_z(v) = e^{(z-1)v} + (z-1)\int_0^v e^{(1-z)u}A_z(u)\,du. \]
The characteristic function of the distribution \( \{P_k(v)\} \) can be obtained by substituting \( z \) by \( e^{it} \). Hence it is enough to show that for fixed \( t \)
\[ \exp \{-it\sqrt{v}\} \Pi_{\exp(it/\sqrt{v})}(v) \to \exp \{-t^2/2\} \quad \text{as} \quad v \to \infty. \]
Since the Poisson distribution is asymptotically normal, it remains to show that the contribution of the second summand in (10) vanishes as \( v \) tends to infinity.
We have
\[ \left| \exp \{-it\sqrt{v}\} \left( \exp \left\{ \frac{it}{\sqrt{v}} \right\} - 1 \right) \times \right. \]
\[ \left. \times \int_0^v \exp \left\{ \exp \left( \frac{it}{\sqrt{v}} \right) - 1 \right\} (v-u) A_{\exp(it/\sqrt{v})}(u) \,du \right| \]
\[ \leq \left| \exp \left\{ \frac{it}{\sqrt{v}} \right\} - 1 \right| \left| \int_0^v \exp \left\{ \exp \left( \frac{it}{\sqrt{v}} \right) - 1 \right\} (v-u) A_{\exp(it/\sqrt{v})}(u) \,du \right| \]
\[ \leq \frac{t}{\sqrt{v}} \int_0^v \left| \cos \frac{t}{\sqrt{v}} - 1 \right| (v-u) A_{\exp(it/\sqrt{v})}(u) \,du. \]
Since
\[ \left| \cos \left( \frac{t}{\sqrt{v}} \right) - 1 \right| (v-u) \leq 0 \quad \text{for} \quad u \in (0, v) \quad \text{and} \quad \left| A_{\exp(it/\sqrt{v})}(u) \right| \leq q(u), \]
the last expression does not exceed
\[ \frac{t}{\sqrt{v}} \int_0^v q(u) \,du. \]
Hence Theorem 1 holds.

5. Proof of Theorem 2. We first show that under the assumptions of Theorem 2 we have
\[ \sum_{k=0}^{\infty} |P_k(v) - A_k(v)| \to 0 \quad \text{as} \quad v \to \infty, \]
where

\[ A_k(v) = e^{-v} \frac{v^k}{k!}, \quad k = 0, 1, 2, \ldots \]

Since the probabilities \( A_k(v) \) satisfy the equations

\[
\frac{dA_0(v)}{dv} = -A_0(v),
\]

(11)

\[
\frac{dA_k(v)}{dv} = -(A_k(v) - A_{k-1}(v)), \quad k = 1, 2, \ldots,
\]

\[ A_k(0) = \begin{cases} 1 & \text{for } k = 0, \\ 0 & \text{for } k \neq 0, \end{cases} \]

using (7) we can write

\[
\frac{d}{dv}(P_0(v) - A_0(v)) = -(P_0(v) - A_0(v)) - (\pi_0(v) - P_0(v)),
\]

(12)

\[
\frac{d}{dv}(P_k(v) - A_k(v)) = (P_{k-1}(v) - A_{k-1}(v)) - (P_k(v) - A_k(v)) + (\pi_{k-1}(v) - P_{k-1}(v)) - (\pi_k(v) - P_k(v)), \quad k = 1, 2, \ldots
\]

Putting

\[ \alpha_k(v) = P_k(v) - A_k(v), \quad \beta_k(v) = \pi_k(v) - P_k(v), \quad k = 0, 1, 2, \ldots \]

we introduce the generating functions

\[
A_z(v) = \sum_{k=0}^{\infty} \alpha_k(v) z^k, \quad B_z(v) = \sum_{k=0}^{\infty} \beta_k(v) z^k.
\]

Using (12) we obtain the differential equation

(13) \[ \frac{dA_z(v)}{dv} = (z - 1) A_z(v) + (z - 1) B_z(v), \quad A_z(0) = 0. \]

Resolving this equation we have

(14) \[ A_z(v) = (z - 1) e^{(z - 1)v} \int_0^v e^{-(z - 1)u} B_z(u) du. \]

Let \( D \) stand for the differentiation operation with respect to \( z \). Since

\[ \frac{1}{k!} D^{(k)} A_z(v) |_{z=0} = P_k(v) - A_k(v), \]
we obtain
\[ \sum_{k=0}^{\infty} |P_k(v) - A_k(v)| = \sum_{k=0}^{\infty} \frac{1}{k!} |D^{(k)} A_z(v)|_{z=0}. \]

Further, we get
\[ D^{(k)} A_z(v) = D^{(k)} [(z-1)e^{(z-1)v} \int_0^v e^{(1-z)u} B_z(u) \, du] \]
\[ = e^{-v} \int_0^v e^u D^{(k)} [e^{(v-u)} (z-1) B_z(u)] \, du. \]

Using the formula
\[ D^{(k)} e^{iz} u(z) = e^{iz} (D + \lambda)^{(k)} u(z), \]
we can write
\[ D^{(k)} A_z(v) = e^{-v} \int_0^v e^u e^{z(v-u)} (D + v - u)^{(k)} [(z-1) B_z(u)] \, du \]
\[ = e^{-v} \int_0^v e^u e^{z(v-u)} \sum_{j=0}^k \frac{k!}{j!(k-j)!} (v-u)^j D^{(k-j)} [(z-1) B_z(u)] \, du. \]

Hence
\[ D^{(k)} A_z(v) \bigg|_{k!} = e^{-v} \int_0^v e^u \sum_{j=0}^k \frac{(v-u)^j}{j!} D^{(k-j)} [(z-1) B_z(u)] \bigg|_{z=0} \, du. \]

Further, we obtain
\[ D^{(k-j)} [(z-1) B_z(u)]_{z=0} = D^{(k-j)} z B_z(u)|_{z=0} - D^{(k-j)} B_z(u) |_{z=0} \]
\[ = (k-j)! (\beta_{k-j-1}(u) - \beta_{k-j}(u)), \quad k = 0, 1, 2, \ldots, j = 0, 1, \ldots, k, \]
where we put \( \beta_{-1}(u) \equiv 0 \). Substituting (16) in (15) we have
\[ D^{(k)} A_z(v) \bigg|_{k!} = e^{-v} \int_0^v e^u \sum_{j=0}^k \frac{(v-u)^j}{j!} (\beta_{k-j-1}(u) - \beta_{k-j}(u)) \, du. \]

Hence
\[ \sum_{k=0}^{\infty} |P_k(v) - A_k(v)| = \sum_{k=0}^{\infty} \left| \frac{D^{(k)} A_z(v)}{k!} \right|_{z=0} \]
\[ = e^{-v} \sum_{k=0}^{\infty} \int_0^v e^u \sum_{j=0}^k \frac{(v-u)^j}{j!} (\beta_{k-j-1}(u) - \beta_{k-j}(u)) \, du. \]

We write
\[ \sum_{j=0}^{k} \frac{(v-u)^j}{j!} (\beta_{k-j}(u) - \beta_{k-j-1}(u)) = \sum_{j=0}^{k} \left( \frac{(v-u)^j}{j!} - \frac{(v-u)^{j-1}}{(j-1)!} \right) \beta_{k-j}(u), \]
where, by definition, \((v-u)^{-1}/(-1)! = 0\). Therefore, we get

\[
\sum_{k=0}^{\infty} |P_k(v) - A_k(v)| = e^{-v} \sum_{k=0}^{\infty} \left| e^u \sum_{j=0}^{k} \frac{(v-u)^j}{j!} \frac{(v-u)^j}{(j-1)!} \beta_{k-j}(u) du \right|
\]

\[
\leq e^{-v} \sum_{k=0}^{\infty} \int e^u \sum_{j=0}^{k} \left| \frac{(v-u)^j}{j!} \frac{(v-u)^j}{(j-1)!} \right| \phi_{k-j}(u) du
\]

\[
= e^{-v} \int e^u \sum_{j=0}^{\infty} \left| \frac{(v-u)^j}{j!} \frac{(v-u)^j}{(j-1)!} \right| \phi(u) du + e^{-v} \int e^u \sum_{j=0}^{\infty} \left| \frac{(v-u)^j}{j!} \frac{(v-u)^j}{(j-1)!} \right| \phi(u) du, 0 < v_0 < v.
\]

Consequently, we obtain

\[
e^{-v} \int e^u \sum_{j=0}^{\infty} \left| \frac{(v-u)^j}{j!} \frac{(v-u)^j}{(j-1)!} \right| \phi(u) du \leq 2 \int_{v_0}^{v} \phi(u) du
\]

and

\[
e^{-v} \int e^u \sum_{j=0}^{\infty} \left| \frac{(v-u)^j}{j!} \frac{(v-u)^j}{(j-1)!} \right| \phi(u) du
\]

\[
= e^{-v} \int e^u \sum_{j=0}^{\infty} \left( \frac{(v-u)^j}{j!} \frac{(v-u)^j}{(j-1)!} \right) \phi(u) du + e^{-v} \int e^u \sum_{j=0}^{\infty} \left( \frac{(v-u)^j}{j!} \frac{(v-u)^j}{(j-1)!} \right) \phi(u) du
\]

\[
\leq c \int_{v_0}^{v} \frac{(v-u)^{v-u}}{[v-u]!} \phi(u) du \leq \frac{\hat{c}}{\sqrt{v-v_0}} \int_{v_0}^{\infty} \phi(u) du, 0 < c, \hat{c} < \infty.
\]

In the last inequality we applied Stirling's formula. Finally, we get

\[
\sum_{k=0}^{\infty} |P_k(v) - A_k(v)| \leq \int_{v_0}^{\infty} \phi(u) du + \frac{\hat{c}}{\sqrt{v-v_0}} \int_{v_0}^{\infty} \phi(u) du.
\]

By choosing \(v_0\) and \(v\) sufficiently large the last expression can be made arbitrarily small. Since Theorem 2 is true for the Poisson distribution (see [5]), the proof is complete.

6. Proof of Theorem 3. It is sufficient to show that

\[
(17) \quad \sup_k |\sqrt{v} P_k(v) - \sqrt{v} A_k(v)| \to 0 \quad \text{as } v \to \infty.
\]

By (17) and the local limit theorem for the Poisson distribution, Theorem 3 holds.
By the converse formula for the Fourier transformation of the sequences \( \alpha_k(v) \) and (14) we have

\[
P_k(v) - A_k(v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} A_{\text{expit}_1}(v) e^{-itk} dt
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{it} - 1) \exp \{ \nu (e^{it} - 1) \} \left[ \int_{0}^{\nu} \exp \{ (1 - e^{it}) u \} B_{\text{expit}_1}(u) du \right] dt.
\]

Replacing \( t \) by \( t/\sqrt{v} \), we get

\[
\sqrt{v} P_k(v) - \sqrt{v} A_k(v) = \frac{1}{2\pi} \int_{-\pi\sqrt{v}}^{\pi\sqrt{v}} \left( \exp \left( \frac{it}{\sqrt{v}} \right) - 1 \right) \times
\]

\[
\times \exp \left\{ t \left[ \exp \left( \frac{it}{\sqrt{v}} \right) - 1 \right] \right\} \left[ \int_{0}^{\nu} \exp \left\{ 1 - \exp \left( \frac{it}{\sqrt{v}} \right) \right\} u \right] B_{\text{expit}_{1/\sqrt{v}}}(u) du \right\} dt.
\]

Further, we obtain

(18) \[
|\sqrt{v} P_k(v) - \sqrt{v} A_k(v)|
\]

\[
\leq \frac{1}{2\pi} \int_{-\pi\sqrt{v}}^{\pi\sqrt{v}} \left\{ \exp \left( \frac{it}{\sqrt{v}} \right) - 1 \right\} \left[ \int_{0}^{\nu} \exp \left\{ \exp \left( \frac{it}{\sqrt{v}} \right) - 1 \right\} (v-u) \right] q(u) du \right\} dt
\]

\[
\leq \frac{1}{\pi\sqrt{v}} \int_{0}^{\nu} q(u) \left[ \int_{0}^{\nu} t \exp \left\{ -2(v-u) \sin^2 \frac{t}{2\sqrt{v}} \right\} dt \right] du
\]

\[
= \frac{1}{\pi\sqrt{v}} \int_{0}^{\nu} q(u) \left[ \int_{0}^{\nu} t \exp \left\{ -2(v-u) \sin^2 \frac{t}{2\sqrt{v}} \right\} dt \right] du +
\]

\[
\quad + \frac{1}{\pi\sqrt{v}} \int_{0}^{\nu} q(u) \left[ \int_{0}^{\nu} t \exp \left\{ -2(v-u) \sin^2 \frac{t}{2\sqrt{v}} \right\} dt \right] du, \quad 0 < \varepsilon < 1.
\]

We now estimate the first integral in (18). Since \( 0 < t/2\sqrt{v} < \pi \varepsilon /2 \), choosing \( \varepsilon > 0 \) sufficiently small we can find \( \alpha > 0 \) such that \( \sin(t/2\sqrt{v}) > \alpha t/2\sqrt{v} \). Therefore, we obtain

\[
\frac{1}{\pi\sqrt{v}} \int_{0}^{\nu} q(u) \left[ \int_{0}^{\nu} t \exp \left\{ -2(v-u) \sin^2 \frac{t}{2\sqrt{v}} \right\} dt \right] du
\]

\[
\leq \frac{1}{\pi\sqrt{v}} \int_{0}^{\nu} q(u) \left[ \int_{0}^{\nu} t \exp \left\{ -\alpha t^2 \frac{(v-u)}{2} \right\} dt \right] du
\]

\[
= \frac{1}{\alpha \pi\sqrt{v}} \int_{0}^{\nu} q(u) \left( 1 - \exp \left\{ -\frac{\alpha^2 \varepsilon^2 (v-u)}{2} \right\} \right) du
\]

\[
= \frac{1}{\alpha \pi\sqrt{v}} \int_{0}^{\nu} q(u) \left( 1 - \exp \left\{ -\frac{\alpha^2 \varepsilon^2 (v-u)}{2} \right\} \right) du +
\]
Hence the first integral in (18) tends to zero as \( v \to \infty \).

We now estimate the second integral in (18). Clearly, we have

\[
\exp \left\{ -2 \sin^2 \frac{t}{2\sqrt{v}} \right\} < e^{-\varepsilon}, \quad \pi \sqrt{v} < t < \pi \sqrt{v}, \quad 0 < \varepsilon < 1, \quad 0 < c < \infty.
\]

Therefore, we obtain

\[
\frac{1}{\pi \sqrt{v}} \int_0^v q(u) \left[ \int_0^{\pi \sqrt{v}} t \exp \left\{ -2(v-u) \sin^2 \frac{t}{2\sqrt{v}} \right\} dt \right] du = \frac{c_3 v}{\pi \sqrt{v}} \int_0^v e^{-c(v-u)} q(u) du
\]

\[
= c_4 \sqrt{v} e^{-c v} \int_0^v e^{2u} q(u) du + c_4 \sqrt{v} e^{-c v} \int_0^v e^{cu} q(u) du,
\]

\( 0 < c_3, \quad c_4 < \infty, \quad 0 < c < 1 \).

We complete the proof of the theorem observing that

\[
\sqrt{v} e^{-c v} \int_0^v e^{2u} q(u) du \leq \sqrt{v} e^{-c v} e^{c v} \int_0^v q(u) du = v e^{-c(1-\gamma)} \frac{1}{\sqrt{v}} \int_0^v q(u) du \to 0 \quad \text{as} \quad v \to \infty
\]

and

\[
\sqrt{v} e^{-c v} \int_0^v e^{cu} q(u) du \leq \sqrt{v} \int_0^v q(u) du \leq \sqrt{v} \int_0^v \frac{u}{v} q(u) du \leq \frac{1}{\gamma} \int_0^v u q(u) du \to 0 \quad \text{as} \quad v \to \infty.
\]

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B. S. Nahapetian


Institute of Mathematics
Armenian Academy of Sciences
Barekamutian St. 24-B
Yerevan-19, 375019
Armenia, U.S.S.R.

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