CONVERGENCE OF RANDOM MEASURES AND POINT PROCESSES
ON THE PLANE

BY

RIMAS BANYŠ (VILNIUS)

Abstract. In the paper two topologies in the space of realizations of random measures on the plane are discussed. The first one is usual vague topology while another is relativized Skorohod's topology. The relation between convergence of random measure in these topologies and conditions for convergence in relativized Skorohod's topology are established.

1. Introduction. The space \( M \) of realizations of random measures on the plane may be considered as the subset of the space \( D[0, \infty)^2 \) of "continuous from above, with limits from below" functions. Thus the relativized Skorohod's topology is another candidate for topology on \( M \) instead of the usual vague topology.

The relation between convergence in distribution of random measures in vague topology and relativized Skorohod's topology (s-topology) is investigated. The conditions for relative compactness of a sequence of random measures w.r.t. convergence in the s-topology are given. They are simpler than the conditions of relative compactness of a sequence of general random fields in the space \( D[0, \infty)^2 \).

From the results obtained the conditions for the convergence of superpositions of independent point processes to the Poisson processes are derived.

2. The space \( D = D[0, \infty)^2 \). Let \( T = [0, \infty)^2 = \{t = (t_1, t_2): 0 \leq t_i < \infty, i = 1, 2\} \). For \( t = (t_1, t_2) \in T \) we define

\[
E_i^{11} = \{s = (s_1, s_2) \in T: s_1 \geq t_1, s_2 \geq t_2\}, \quad E_i^{00} = \{s \in T: s_1 < t_1, s_2 < t_2\},
\]

\[
E_i^{10} = \{s \in T: s_1 \geq t_1, s_2 < t_2\}, \quad E_i^{01} = \{s \in T: s_1 < t_1, s_2 \geq t_2\}.
\]

We equip \( T \) with the usual topology, generated by the norm

\[ ||t|| = \max(t_1, t_2). \]
Let $D = D[0, \infty)^2$ denote the space of real valued functions on $T$ which are “continuous from above, with limits from below” in the following sense: for each $t \in T$

(a) $\lim x(s)(s \to t, s \in E^j_i)$ exists for all $(i, j), i, j = 0, 1,$ and

(b) $x(t) = \lim x(s)(s \to t, s \in E^j_i)$.

Let $A$ be the class of all homeomorphisms of $T$ onto itself of the form $\lambda = (\lambda_1, \lambda_2)$ where $\lambda_i$ is a strictly increasing continuous function on $[0, \infty)$ with $\lambda_i(0) = 0, \lambda_i(\infty) = \infty$, $i = 1, 2$, and the image of a point $t = (t_1, t_2)$ under a homeomorphism $\lambda = (\lambda_1, \lambda_2)$ is $\lambda t = (\lambda_1 t_1, \lambda_2 t_2)$.

Let $x, x_1, x_2, \ldots \in D$ be given. We say that $x_n$ S- converges to $x$ if there exist $\lambda_1, \lambda_2, \ldots \in A$ such that for each real $a > 0$

$$\lim_{n \to \infty} \sup_{\|t\| \leq a} |x_n(\lambda_n t) - x(t)| = 0$$

and

$$\lim_{n \to \infty} \sup_{\|t\| \leq a} \|\lambda_n t - t\| = 0.$$ 

The topology which corresponds to S-convergence we call S-topology. It coincides with the topology discussed in [4], and is analogous to the well-known Skorohod’s topology (cf. [1], [3], [9]). With the respect to S-topology, $D$ is separable and topologically complete, and the Borel $\sigma$-algebra $\mathcal{D}$ coincides with the $\sigma$-algebra generated by the coordinate mappings (cf. [4]).

Given a probability measure $P$ on $(D, \mathcal{D})$, let $T_P$ consist of those $t \in T$ for which

$$P\{x \in D: x \text{ is continuous at } t\} = 1.$$ 

For $a = (a_1, a_2)$ let $D([0, a_1] \times [0, a_2])$ denote the space of real valued functions on

$$[0, a_1] \times [0, a_2] = \{t = (t_1, t_2) \in T: t_i \leq a_i, i = 1, 2\},$$

which are “continuous from above, with limits from below”. The space $D([0, a_1] \times [0, a_2])$ is analogous to $D[0, 1]^2$ (for $D[0, 1]^2$ cf. [2], [8], [9]).

Let $\tau_a: D \to D([0, a_1] \times [0, a_2])$ be defined by

$$\tau_a x(t) = x(t), \quad t_1 \leq a_1, t_2 \leq a_2.$$ 

From [4] we have

**Theorem 1.** Let $P, P_1, P_2, \ldots$ be probability measures on $(D, \mathcal{D})$. Then $P_n \Rightarrow P$ if and only if $P_n \tau_a^{-1} \Rightarrow P \tau_a^{-1}$ for all $a \in T_P$. (Here $\Rightarrow$ means weak convergence).

By this theorem one can restrict oneself to the space $D[0, 1]^2$ when considering the convergence in $D[0, \infty)^2$. 

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3. The space of realizations of random measures. Let $M = M(T)$ denote the class of all measures on $(T, \mathcal{B}(T))$ such that $\mu(\{0\} \times B) = \mu(B \times \{0\}) = 0$, $B \in \mathcal{B}(\mathbb{R}_+)$ and $\mu A < \infty$ for all bounded $A \in \mathcal{B}(T)$. (Here $\mathcal{B}(T)$ and $\mathcal{B}(\mathbb{R}_+)$ mean Borel $\sigma$-algebras of subsets of $T$ and $\mathbb{R}_+ = [0, \infty)$, respectively). Every $\mu \in M$ can be regarded as a function in $D = D(T)$ which we call the distribution function of $\mu$ and denote also by $\mu$. Thus, we will not distinguish between a measure and its distribution function.

Let $s$ denote the relative topology on $M$. $M$ is a closed subset of $\mathcal{B}(\mathbb{R}_+)$. Thus, topological space $(M, s)$ is separable and topologically complete. Let $\mathcal{M}_s$ denote the Borel $\sigma$-algebra of subsets of $(M, s)$. We have $\mathcal{M}_s \subset \mathcal{D}$.

Let $\mu, \mu_1, \mu_2, \ldots \in M$ be given. We say that $\mu_n \nu$-converges to $\mu$ if $\mu_n(t) \to \mu(t)$ for all continuity points $t$ of $\mu$.

One can introduce a metric corresponding to $\nu$-convergence which makes $M$ separable and complete (cf. [3], [5]). We denote by $\nu$ the corresponding topology and by $\mathcal{M}_\nu$ the Borel $\sigma$-algebra of subsets of $(M, \nu)$. $\nu$-convergence implies $\nu$-convergence but not vice versa. Thus $\mathcal{M}_\nu \subset \mathcal{M}_s$. Actually, $\mathcal{M}_\nu$ and $\mathcal{M}_s$ coincide with the $\sigma$-algebra generated by coordinate mappings (cf. [4], [5], [9]). Let us put $\mathcal{M} = \mathcal{M}_\nu = \mathcal{M}_s$.

Write

$$N = \{ \mu \in M : \mu B \in \mathbb{Z}_+ = \{0, 1, \ldots\}, B \in \mathcal{B}(T) \}.$$  

$N$ is $\nu$-closed and $\nu$-closed subset of $M$ and $N \in \mathcal{M}$.

We turn now to the problem of characterizing the compact subsets of the topological space $(M, s)$. By Theorem 1 we may restrict ourselves to the space of functions defined on $[0, 1]^2$.

Let $D_0 = D[0, 1]^2(M_0 = M[0, 1]^2)$ be the space of functions in $D$ (in $M$, respectively) restricted to $[0, 1]^2 = \{(s, t) : 0 \leq s, t \leq 1\}$ with Skorohod (relative) topology (cf. [2], [8], [9] for the space $D_0$). We may assume that all functions in $M_0$ are continuous at $(1, 1)$.

For $x \in M_0$, $t \in (0, 1)$ and $\delta > 0$ set

$$w^{(1)}(t, x) = \int_{(t-\delta,t+\delta)} [x(t+\delta, 1) - x(s, 1)] dx(s, 1)$$

and

$$w^{(2)}(t, x) = \int_{(t-\delta,t+\delta)} [x(1, t+\delta) - x(1, s)] dx(1, s)$$

**Theorem 2.** A set $A \subset M_0$ has compact closure in the Skorohod (relative) topology if and only if $\sup_{x \in A} x(1, 1) < \infty$ and

$$\limsup_{\delta \to 0} x(\delta, 1) = \limsup_{\delta \to 0} x(1, \delta) = 0,$$

(1)
(2) \[ \limsup_{\delta \to 0} \sup_{x \in A} [x(1, 1) - x(1 - \delta, 1)] = \limsup_{\delta \to 0} \sup_{x \in A} [x(1, 1) - x(1, 1 - \delta)] = 0 \]
and, for all \( t \in (0, 1) \),
\[ \limsup_{\delta \to 0} \sup_{x \in A} w_\delta^{(t)}(t, x) = 0, \quad i = 1, 2. \]

Proof. For \( x \in M_0 \) and \( 0 < \delta < 1 \), put
\[ \omega_\delta(x) = \inf \max_{i,j} \left\{ x(s^{(0)} - 0, t^{(0)} - 0) - x(s^{(i-1)}, t^{(i-1)}) \right\}, \]
where the infimum extends over the finite collections \( \{(s^{(i)}, t^{(j)})\} \) of points \( (s^{(i)}, t^{(j)}) \) satisfying the following conditions:
\[ \begin{align*}
0 &= s^{(0)} < s^{(1)} < \ldots < s^{(p)} = 1, \\
0 &= t^{(0)} < t^{(1)} < \ldots < t^{(q)} = 1, \\
(s^{(i)} - s^{(i-1)}) &> \delta, \quad (t^{(j)} - t^{(j-1)}) > \delta, \quad i = 1, \ldots, p; \quad j = 1, \ldots, q.
\end{align*} \]

If \( A \) has a compact closure, then \( \sup_{x \in A} x(1, 1) < \infty \) and (cf. [8])
\[ \limsup_{\delta \to 0} \sup_{x \in A} \omega_\delta(x) = 0. \]

Obviously, conditions (1) and (2) also hold.

Given \( \delta \) and \( \varepsilon \), decompose \([0, 1)^2 = \{(s, t): 0 \leq s, t < 1\}\) into rectangles \([s^{(i-1)}, s^{(i)}) \times [t^{(j-1)}, t^{(j)}]\), such that \( s^{(i)} - s^{(i-1)} > 2\delta \), \( t^{(j)} - t^{(j-1)} > 2\delta \) and
\[ \max_{i,j} \left\{ x(s^{(0)} - 0, t^{(0)} - 0) - x(s^{(i-1)}, t^{(i-1)}) \right\} < \omega_\delta(2\delta) + \varepsilon. \]

Then for any \( t \in (0, 1) \) either \( (t - \delta, t + \delta) \subset [s^{(i-1)}, s^{(i)}) \) for some \( i, 1 \leq i \leq p \), in which case
\[ x(t + \delta, 1) - x(t - \delta, 1) < x(s^{(i-1)}, t^{(i-1)}) - x(s^{(i-1)}, t^{(i-1)}) < \omega_\delta(2\delta) + \varepsilon, \]
or some \( s^{(i)} \) lies in the interval \( (t - \delta, t + \delta) \), in which case
\[ \begin{align*}
   &|x(s^{(i)} - 0, 1) - x(t - \delta, 1)| < \omega_\delta(2\delta) + \varepsilon, \\
   &|x(t + \delta, 1) - x(s^{(i)}, 1)| < \omega_\delta(2\delta) + \varepsilon.
\end{align*} \]

If (6) holds, we have
\[ w_\delta^{(1)}(t, x) \leq x(1, 1)[x(t + \delta, 1) - x(t - \delta, 1)] < x(1, 1)[\omega_\delta(2\delta) + \varepsilon]. \]

If (7) holds, we can divide the interval \( (t - \delta, t + \delta) \) into subintervals \( (t - \delta, s^{(i)}) \) and \([s^{(i)}, t + \delta]). \) Then we have
\[ w_\delta^{(1)}(t, x) < 2x(1, 1)[\omega_\delta(2\delta) + \varepsilon]. \]

By (8) and (9) we have \( w_\delta^{(1)}(t, x) \leq 2K\omega_\delta(2\delta), \) where
\[ K = \sup_{x \in A} x(1, 1). \]
Analogously, \( w^{(2)}_\delta(t, x) \leq 2K\omega'_\delta(2\delta) \). Thus, (5) implies (4) which proves the necessity.

To prove the sufficiency, define

\[
A_\epsilon(\delta) = \max_{i=1,2} A^{(i)}_\epsilon(\delta),
\]

where

\[
A^{(1)}_\epsilon(\delta) = \sup \{ \min \| x(t, \cdot) - x(s, \cdot) \|, \| x(u, \cdot) - x(t, \cdot) \|, s \leq t \leq u, u - s \leq \delta \},
\]

\[
\| x(t, \cdot) - x(s, \cdot) \| = \sup_u \| x(t, u) - x(s, u) \|;
\]

the modulus \( A^{(2)}_\epsilon(\delta) \) being defined analogously.

It suffices to show that (cf. [2], [7])

\[
\lim_{\delta \to 0} \sup_{x \in A} A^{(i)}_\epsilon(\delta) = 0, \quad i = 1, 2.
\]

For arbitrary \( \delta > 0 \), \( t \in (0, 1) \) and \( t^* \in (t - \delta, t + \delta) \), we have

\[
w^{(1)}_\delta(t, x) = x(t + \delta, 1) [x(t + \delta - 0, 1) - x(t - \delta, 1)] - \int_{(t - \delta, t + \delta)} x(s, 1) dx(s, 1) - \int_{(t^*, t + \delta)} x(s, 1) dx(s, 1)
\]

\[
\geq [x(t^*, 1) - x(t - \delta, 1)] [x(t + \delta, 1) - x(t^*, 1)]
\]

\[
\geq [\min \{ x(t^*, 1) - x(t - \delta, 1), x(t + \delta, 1) - x(t^*, 1) \}]^2.
\]

Thus

\[
A^{(1)}_\epsilon(2\delta) \leq [\sup_t w^{(1)}_\delta(t, x)]^{1/2}.
\]

Analogously

\[
A^{(2)}_\epsilon(2\delta) \leq [\sup_t w^{(2)}_\delta(t, x)]^{1/2}.
\]

It is easy to see that from (3) it follows

\[
\lim_{\delta \to 0} \sup_{x \in A} \sup_t w^{(i)}_\delta(t, x) = 0, \quad i = 1, 2.
\]

In fact, if this were not true, we could find \( \epsilon > 0 \), \( t_k \) and \( \delta_k \to 0 \) such that \( w^{(i)}_\delta(t_k, x) > \epsilon \) for some \( x \in A, k = 1, 2, \ldots \) If \( t_k \to t_0 \), then for an arbitrary \( \delta \) the interval \((t_0 - \delta, t_0 + \delta)\) contains \((t_k - \delta_k, t_k + \delta_k)\) for sufficiently large \( k \), and hence

\[
w^{(i)}_\delta(t_0, x) > w^{(i)}_\delta(t_k, x) > \epsilon,
\]

which contradicts (3).

Now by (11)-(13) condition (3) implies (10).
4. Convergence of random measures. By a random measure on $T$ we mean any measurable mapping of some fixed probability space $(\Omega, \mathcal{F}, P)$ into $(M, \mathcal{M})$. An $N$-valued random measure is called the point process.

Given a random measure $\xi$ on $T$ let $T_\epsilon$ consist of those $t \in T$ for which $P(\xi(t) = 0) = 1$. Write $T_\epsilon = T \setminus T_\epsilon$.

For a random measure $\xi$ and $t = (t_1, t_2)$ we set
\[
\xi(t) = \xi([0, t_1] \times [0, t_2]).
\]

We denote by $\xi_n \xrightarrow{d} \xi$ or $\xi_n \xrightarrow{d_v} \xi$ convergence in distribution of random measures $\xi_n$ to $\xi$ in the $s$-topology or $v$-topology, respectively.

**Theorem 3.** Let $\xi, \xi_1, \xi_2, \ldots$ be random measures. Then $\xi_n \xrightarrow{d} \xi$ if and only if $\xi_n \xrightarrow{d_v} \xi$ and the sequence $\{\xi_n\}$ is relatively compact w.r.t. convergence in distribution in the $s$-topology.

The proof follows immediately from the facts that $\xi_n \xrightarrow{d_v} \xi$ is equivalent to $(\xi_n(t_1), \ldots, \xi_n(t_k)) \xrightarrow{d} (\xi(t_1), \ldots, \xi(t_k))$ for all $k \in \mathbb{Z}_+$ and all $t_1, \ldots, t_k \in T_\epsilon$, and that the distribution of any random measure $\xi$ is determined by the distributions of $(\xi(t_1), \ldots, \xi(t_k))$, $k \in \mathbb{Z}_+$ with $t_1, \ldots, t_k$ running over an arbitrary dense subset of $T$.

Given a random measure $\xi$ on $T$, $a > 0$, $t > 0$ and $\delta$, write
\[
w^{(1)}(a, t, \xi) = \int_{(t-\delta, t+\delta)} [\xi(t+\delta, a) - \xi(s, a)] \xi(ds, a)
\]
and
\[
w^{(2)}(a, t, \xi) = \int_{(t-\delta, t+\delta)} [\xi(a, t+\delta) - \xi(a, s)] \xi(a, ds).
\]

**Theorem 4.** Let $\xi, \xi_1, \xi_2, \ldots$ be random measures. Then $\xi_n \xrightarrow{d} \xi$ if and only if the following conditions hold:

- For all $k \in \mathbb{Z}_+$ and $t_1, \ldots, t_k \in T_\epsilon$
  \[
  (\xi_n(t_1), \ldots, \xi_n(t_k)) \xrightarrow{d} (\xi(t_1), \ldots, \xi(t_k))
  \]
- And, for all $a > 0$, $t = (t_1, t_2) \in T_\epsilon$ and $\varepsilon > 0$,
  \[
  \lim_{\delta \to 0} \limsup_{n \to \infty} P\{w^{(1)}(a, t, \xi_n) > \varepsilon\} = 0.
  \]

**Corollary.** If $T_\epsilon = T$, then $\xi_n \xrightarrow{d} \xi$ if and only if $\xi_n \xrightarrow{d_v} \xi$.

The proof follows from Theorems 2 and 3.

5. Convergence of superpositions of point processes. Here we apply Theorem 4 to obtain conditions of convergence of superpositions of point processes to the Poisson process.

A triangular array of point processes $\xi_{n1}, \ldots, \xi_{nk_n} (n = 1, 2, \ldots)$ is called infinitesimal if, for all $t = (t_1, t_2) \in T$,
\[
\lim_{n \to \infty} \max_{1 \leq k \leq k_n} P\{\xi_{nk}(t) > 0\} = 0.
\]
We recall that $\xi(t)$ denotes the number of points of $\xi$ in the closed rectangle $[0, t_1] \times [0, t_2]$.

A point process $\xi$ is called a Poisson process with intensity $\lambda$, $\lambda \in M$, if $\xi A_1, \ldots, \xi A_k$, $k \in \mathbb{Z}^+$, $A_i \in \mathcal{B}(T)$ are independent when $A_1, \ldots, A_k$ are disjoint and $\xi A$ has the Poisson distribution with parameter $\lambda A$ for all bounded $A \in \mathcal{B}(T)$.

For $\lambda \in M$ let $T_\lambda$ be the set of continuity points of $\lambda$. Write $T_\lambda = T \setminus T_\lambda$.

**Theorem 5.** Let $\xi_{n_1}, \ldots, \xi_{n_k}$ $(n \in \mathbb{Z}^+)$ be an infinitesimal array of independent point processes on $T$. Let $\xi$ be a Poisson process with intensity $\lambda$. Then

$$\xi_n = \sum_{k=1}^{h_n} \xi_{n_k} \xrightarrow{d} \xi$$

if and only if the following conditions hold:

for all $t \in T$

$$\sum_{k=1}^{h_n} P \{\xi_{n_k}(t) > 1\} \to 0, \; n \to \infty;$$

for all $t' \in T_\lambda$

$$A_n(t') = \sum_{k=1}^{h_n} P \{\xi_{n_k}(t') > 0\} \to \lambda(t), \; n \to \infty,$$

and for all $a > 0$ and $u = (u_1, u_2) \in T_\lambda$,

$$\lim_{\delta \to 0} \sup_{n \to \infty} w_0^{(i)}(a, u_i, A_n) = 0, \; i = 1, 2,$$

where

$$w_0^{(1)}(a, u_1, A_n) = \int_{(u_1 - \delta, u_1 + \delta)} \left[ A_n(u_1 + \delta, a) - A_n(s, a) \right] A_n(ds, a),$$

$w_0^{(2)}(a, u_2, A_n)$ being defined analogously.

**Corollary.** If $\lambda$ is continuous, then $\xi_n \xrightarrow{d} \xi$ if and only if for all $t \in T$

$$\lim_{n \to \infty} \sum_{k=1}^{h_n} P \{\xi_{n_k}(t) > 1\} = 0$$

and

$$\lim_{n \to \infty} \sum_{k=1}^{h_n} P \{\xi_{n_k}(t) > 0\} = \lambda(t).$$

**Proof.** By [5], $\xi_n \xrightarrow{d} \xi$ if and only if (16) and (17) hold. So by Theorem 4 it suffices to show that condition (14) of Theorem 4 implies (18), and vice versa.

Let $(u, v) \in T_\lambda$ and $a > 0$ be given. Write

$$\tau_{nk} = \min \{t: \xi_{nk}(t, a) \geq 1\}, \; k = 1, \ldots, k_n.$$
We have
\[ P_{\lambda} w_{\lambda}^{(1)}(a, u, \xi_n) > 0 \] \[ \geq P_{\lambda} \left( \bigcup_{k \neq l} (u - \delta < \tau_{nk} < \tau_{nl} \leq u + \delta, \tau_{nm} > a, m \neq k, l) \right) \]
\[ \geq P_{\lambda} (\xi_n(a, a) = 0) \sum_{k \neq l} [P_{\lambda} (\xi_n(u + \delta, a) > 0) - \]
\[ - P_{\lambda} (\xi_n(s, a) > 0) dP_{\lambda} (\xi_n(s, a) > 0) \]
\[ = P_{\lambda} (\xi_n(a, a) = 0) w_{\lambda}^{(1)}(a, u, A_n) + o(1) \quad (n \to \infty). \]

Analogously,
\[ P_{\lambda} w_{\lambda}^{(2)}(a, v, \xi_n) > 0 \] \[ \geq P_{\lambda} (\xi_n(a, a) = 0) w_{\lambda}^{(2)}(a, u, A_n) + o(1). \]

Thus (14) implies (18).

Now we prove the reverse implication. We have
\[ P_{\lambda} \left( w_{\lambda}^{(1)}(a, u, \xi_n) \geq 1 \right) \leq P_{\lambda} \left( \bigcup_{k \neq l} (u - \delta < \tau_k < \tau_{nl} \leq u + \delta) \right) \]
\[ + P_{\lambda} \left( \bigcup_{k} (\xi_n(u + \delta, a) > 1) \right) \]
\[ \leq w_{\lambda}^{(1)}(a, u, A_n) + o(1) \quad (n \to \infty). \]

From (19) and the analogous inequality for \( w_{\lambda}^{(2)}(a, v, \xi_n) \) it follows that (18) implies (14). This proves the theorem.

By Lemma, condition (17) together with (18) means that \( A_n(\cdot) \) s-converges to \( \lambda(\cdot) \). Thus we can reformulate Theorem 5 in the following way:

**Theorem 6.** Let \( \xi_{n1}, \ldots, \xi_{nk_n}, n \in \mathbb{Z}_+ \), be an infinitesimal array of independent point processes satisfying (16). Let \( \xi \) be a Poisson process with intensity \( \lambda \). Then
\[ \xi_n = \sum_{k=1}^{k_n} \xi_{nk} \overset{d}{\to} \xi \]

if and only if \( A_n(\cdot) \) s-converges to \( \lambda(\cdot) \).

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Convergence of random measures


Institute of Mathematics and Cybernetics
Academy of Sciences of the Lithuanian SSR
Vilnius, K. Pozelos, 54, USSR

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