SKOROKHOD PROBLEM – ELEMENTARY PROOF OF THE AZEMA-YOR FORMULA

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Abstract. Let \( \mu \) be a centered probability measure with the finite second moment. Let the stopping time \( T \) for the Brownian motion \( W \) be defined as

\[
T = \inf \{ t \geq 0; \Psi(W_t) \leq \sup_{0 \leq s \leq t} W_s \},
\]

where \( \Psi \) is a barycenter function of measure \( \mu \). Azema and Yor [1] have shown that \( W_T \) has then the distribution \( \mu \) and \( \mathbb{E}T = \int x^2 \mu(dx) \).

This paper contains an elementary proof of this result.

Introduction. Skorokhod [8] has shown that for the centered probability measure \( \mu \) with a continuous distribution function there exists the Brownian motion \( W \) and the stopping time \( T \) so that the distribution of \( W_T \) is \( \mu \). Moreover, if \( \mu \) has the finite second moment, then

\[
\mathbb{E}T = \int_{-\infty}^{\infty} x^2 \mu(dx).
\]

That construction was improved by Monroe [5]. However, the stopping time \( T \), given by the Skorokhod’s construction is regarded with respect to the filtration essentially bigger that the natural filtration of Brownian motion. Dubins [4], Rost [7], Chacon–Walsh [3] and Azema–Yor [1] gave new constructions of stopping times with the desired property but which are stopping times with respect to the natural filtration of Brownian motion. Construction given by Azema–Yor is the best one in some respects. It is an explicit formula and not a result of a limit procedure. Pierre [6] gave a new proof of the Azema–Yor formula but with assumptions of regularity of a
measure \mu. The proof of this formula given in our paper is based on the following

**PROPERTY.** Let \( W \) be the Brownian motion. If \( W_0 \equiv 0, a < 0 < b, \)
\[ \tau_x = \inf \{ t \geq 0: W_t = x \}, \quad x \in \mathbb{R}, \]
then
\[ P(\tau_a < \tau_b) = \frac{b}{b-a}. \]

**Definition 1.** If \( \mu \) is a probability distribution such that
\[ \int_{-\infty}^{\infty} |x| \mu(dx) < \infty, \]
then the *barycenter function* \( \Psi \) of the measure \( \mu \) is defined as
\[ \Psi(a) = \begin{cases} \frac{1}{\mu([a, \infty))} \int_{[a, \infty)} x \mu(dx) & \text{if } \mu([a, \infty)) > 0, \\ a & \text{otherwise}. \end{cases} \]

**Notation.** We write
\[ S_t = \sup_{0 \leq s \leq t} W_s. \]

**Theorem.** If \( T = \inf \{ t \geq 0; \Psi(W_t) \leq S_t \} \), then \( W_T \) has a distribution \( \mu \). Moreover, if
\[ \int_{-\infty}^{+\infty} x^2 \mu(dx) < \infty, \]
then
\[ ET = \int_{-\infty}^{+\infty} x^2 \mu(dx). \]

**Remark 1.** \( T \) is a stopping time with respect to the natural filtration of Brownian motion.

**Remark 2.** To show that \( W_T \) has a distribution \( \mu \) it is enough to prove the following implication:
\[ \mu([a, \infty)) > 0 \Rightarrow \mu([a, \infty)) = P(\{ W_T \geq a \}). \]

Indeed, if \( \mu([a, \infty)) = 0 \) and \( \exists a_n \rightarrow a \,(n = 1, 2, \ldots) \), then either there exists an \( N \) such that \( \mu([a_n, \infty)) = 0 \) for \( n \geq N \) and then, for \( b \geq 0 \), \( \Psi(a_n + b) = \)
$a_n + b$ so that $W_T \leq a_N$ and $P(\{W_T \geq a\}) = 0$ or, for every $n = 1, 2, \ldots$, $\mu([a_n, \infty)) > 0$ and then

$$\mu([a, \infty)) = \lim_{n \to \infty} \mu([a_n, \infty)) = \lim_{n \to \infty} P(\{W_T \geq a\}) = P(\{W_T \geq a\}).$$

**Notation.** We write

$$P_a = P(\{W_T \geq a\}), \quad \phi(a) = \inf_{x \leq a} \{\Psi(x) - x\}, \quad v(x) = \inf \{y: \Psi(y) > x\},$$

$$K_i^n = i 2^{-n} \Psi(a) \quad (0 \leq i \leq 2^n), \quad K = K_2^\infty,$$

$$a_i^n = v(K_i^n) \quad (0 \leq i \leq 2^n), \quad a_0 = a_0^0, \quad \frac{-\infty}{-\infty} = 1.$$

**Remark 3.** Since the set where function $\Psi$ is not continuous is at least denumerable, we can make an assumption that $a$ is such that $\Psi(a_i^n) = K_i^n$ for $0 \leq i \leq 2^n$.

**Remark 4.** If $\Psi(a) > a$, then $\phi(a) > 0$.

**Proof.** Suppose that $\phi(a) = 0$. Since $\phi(a) = \inf_{x \leq a} \{\Psi(x) - x; \phi(a) - a \leq x \leq a\}$, there exists an $(x_n)_{n=0}^{\infty} \in [\phi(a) - a, a]$ such that

$$\phi(a) = \lim_{n \to \infty} (\Psi(x_n) - x_n) \quad \text{and} \quad x_n \to x_0.$$

If we can find a subsequence $x_{n_k} \to x_0$, then — by the left continuity of the function $\phi$ — we have $\Psi(x_0) = x_0$. Also, if there exist subsequences $x_{n_k} \to x_0$, then

$$x_0 \leq \Psi(x_0) \leq \lim_{k \to \infty} \Psi(x_{n_k}) = x_0.$$

Since $x_0 \leq a$, we obtain a contradiction: $\Psi(a) \leq a$.

**Remark 5.** Let

$$\Gamma = \bigcup_{y \in \mathbb{R}} [\Psi(y), \lim_{z \to y} \Psi(z)]$$

and let $Z_t = (W_t, S_t)$ be the process with values in $\mathbb{R}^2$. Then, by the definition of the stopping time $T$, we infer that $T$ is a first entrance time of process $Z$ to the closed set $\Gamma$.

**Lemma 1.** If $v(x) \leq y \leq x$, then

$$\frac{y - v(y)}{x - v(y)} \geq P(S_T \geq x | S_T \geq y) \geq \frac{y - v(x)}{x - v(x)}.$$
Proof. We have
\[ P(S_T \geq x | S_T \geq y) \]
\[ = P(\{\text{after exit from (y, y) } Z \text{ achieves (x, x) before it enters } \Gamma \}) , \]
\[ \frac{y - v(y)}{x - v(y)} \]
\[ = P(\{\text{after exit from (y, y) } Z \text{ achieves (x, x) before it enters } \{v(y) \times R\}) \}) , \]
\[ \frac{y - v(x)}{x - v(x)} \]
\[ = P(\{\text{after exit from (y, y) } Z \text{ achieves (x, x) before it enters } \{v(x) \times R\}) \}). \]

**Lemma 2.** If \( \Psi(a) > a \), then

\[ \lim_{n \to \infty} \prod_{i=1}^{2^n} \frac{K_i^n - a_i^n - a_i^n}{K_i^n - a_i^n - a_i^n} \geq P_a \geq \lim_{n \to \infty} \prod_{i=1}^{2^n} \frac{K_i^n - a_i^n - a_i^n}{K_i^n - a_i^n - a_i^n}. \]

Proof. Let \( n \) be so large that \( 2^{-n} \Psi(a) < \phi(a) \). From the definition, \( \{W_T \geq a\} = \{S_T \geq \Psi(a)\} \), whence

\[ P(\{W_T \geq a\}) = P(\{S_T \geq \Psi(a)\}) = \prod_{i=1}^{2^n} P(S_T \geq K_i^n | S_T \geq K_i^n). \]

Since

\[ K_i^n - a_i^n \geq \Psi(a_i^n) - a_i^n \geq \phi(a) > 2^{-n} \Psi(a) = K_i^n - K_i^n \]

\[ 0 < i \leq 2^n, \]

we have \( a_i^n < K_i^n - K_i^n \) and from Lemma 1 it follows that

\[ \prod_{i=1}^{2^n} \frac{K_i^n - a_i^n - a_i^n}{K_i^n - a_i^n - a_i^n} \geq P_a \geq \prod_{i=1}^{2^n} \frac{K_i^n - a_i^n - a_i^n}{K_i^n - a_i^n - a_i^n}. \]

**Lemma 3.** We have

\[ \lim_{n \to \infty} \prod_{i=1}^{2^n} \frac{K_i^n - a_i^n - a_i^n}{K_i^n - a_i^n - a_i^n} = \lim_{n \to \infty} \prod_{i=1}^{2^n} \frac{K_i^n - a_i^n - a_i^n}{K_i^n - a_i^n - a_i^n}. \]

Proof. We can write

\[ 1 \geq \left( \prod_{i=1}^{2^n} \frac{K_i^n - a_i^n}{K_i^n - a_i^n} \right) \left( \prod_{i=1}^{2^n} \frac{K_i^n - a_i^n - a_i^n}{K_i^n - a_i^n - a_i^n} \right)^{-1} \]

\[ = \prod_{i=1}^{2^n-1} \frac{(K_i^n - a_i^n)(K_i^n - a_i^n)}{(K_i^n - a_i^n)^2} \frac{K_i^n - a_i^n}{K_i^n - a_i^n} \frac{K_i^n - a_i^n}{K_i^n - a_i^n} \]

\[ = \prod_{i=1}^{2^n-1} \frac{(K_i^n - a_i^n - 2^{-n} \Psi(a))(K_i^n - a_i^n + 2^{-n} \Psi(a))}{(K_i^n - a_i^n)^2}. \]
Skorokhod problem

LEMMA 4. If \( 2^{-n} \Psi(a) < \phi(a) \), then

\[
\prod_{i=1}^{2^n-1} \left[ 1 - \left( \frac{\Psi(a)}{2^n (K_i^n - a_i^n)} \right)^2 \right] \frac{\Psi(a) - v(\Psi(a)) - 2^{-n} \Psi(a)}{\Psi(a) - v(\Psi(a))} \frac{2^{-n} \Psi(a) - v(0)}{-v(0)} \to 1,
\]

when \( n \to \infty \).

Proof. Since

\[
\mu([a, \infty)) = \prod_{i=1}^{2^n} \frac{\mu([a_i^n, \infty))}{\mu([a_{i-1}^n, \infty))},
\]

it is enough to show that

\[
\frac{K_{i-1}^n - a_{i-1}^n}{K_i^n - a_i^n} \geq \frac{\mu([a_i^n, \infty))}{\mu([a_{i-1}^n, \infty))} \geq \frac{K_{i-1}^n - a_i^n}{K_i^n - a_i^n} \quad \text{for} \quad 0 < i \leq 2^n.
\]

From the definition of function \( \Psi \) and from Remark 3 we get

\[
\frac{K_{i-1}^n - a_{i-1}^n}{K_i^n - a_i^n} \geq \frac{\mu([a_i^n, \infty))}{\mu([a_{i-1}^n, \infty))} \geq \frac{K_{i-1}^n - a_i^n}{K_i^n - a_i^n} \quad \text{for} \quad 0 < i \leq 2^n
\]

and

\[
\frac{K_{i-1}^n - a_i^n}{K_i^n - a_i^n} = \frac{\mu([a_i^n, \infty))}{\mu([a_{i-1}^n, \infty))} \int_{[a_i^n, \infty)} (x - a_i^n) \mu(dx) \leq \frac{\mu([a_i^n, \infty))}{\mu([a_{i-1}^n, \infty))} \int_{[a_i^n, \infty)} (x - a_i^n) \mu(dx).
\]

Lemmas 2, 3, and 4 now easily imply that \( W_T \) has the distribution \( \mu \).

Remark 6. If \( T \) is a stopping time such that \( ET < \infty \), then \( EW_T = 0 \) and \( EW_T^2 = ET \).

To complete the proof of the theorem it must be shown that if

\[
\int_{-\infty}^{+\infty} x^2 \mu(dx) < \infty,
\]
then

\[ ET = \int_{-\infty}^{+\infty} x^2 \mu(dx). \]

From Remark 6 it follows that it is enough to show that \( ET < \infty \). Let \( \Psi_n \) and \( T_n (n = 1, 2, \ldots) \) be defined as follows:

\[
\Psi_n(x) = \begin{cases} 
0 & \text{if } x \leq -n, \\
\Psi(x) & \text{if } -n < x \leq n, \\
\Psi(n) & \text{if } n < x \leq \Psi(n), \\
x & \text{if } \Psi(n) < x,
\end{cases} \quad T_n = \inf \{ t \geq 0; \Psi_n(W_t) \leq S_t \}.
\]

From the definition, \( T_n \to T \) and \( ET_n < \infty \) (\( ET_n \) means exit time of Brownian motion from \([-n, \Psi(x)]\)). So \( ET_n = EW_{T_n}^2 \).

To obtain from Fatou Lemma that \( ET < \infty \) it is enough to show that

\[
\limsup_{n \to \infty} EW_{T_n}^2 < \infty.
\]

Let \( A = \{ W_t \in [n, \Psi(n)] \} \). Then

\[
EW_{T_n}^2 = E_{Z,A} W_{T_n}^2 + E_{Z,A} W_{T_n}^2 \leq E_{Z,A} W_T^2 + E_{Z,A} W_T^2
\]

\[
= EW_T^2 + E_{Z,A} (W_{T_n}^2 - W_T^2) \leq EW_T^2 + 2 \Psi(n) E_{Z,A} (W_{T_n} - W_T)
\]

\[
= EW_T^2 + 2 \Psi(n) \int_{[n, \Psi(n)]} (\Psi(n) - x) \mu(dx)
\]

\[
= EW_T^2 + 2 \Psi(n) \int_{[\Psi(n), \infty)} (x - \Psi(n)) \mu(dx)
\]

\[
\leq EW_T^2 + 2 \int_{[\Psi(n), \infty)} x^2 \mu(dx) \leq EW_T^2 + \varepsilon \quad \text{for } n \geq n_0,
\]

which completes the proof.

REFERENCES


[6] M. Pierre, Le problème de Skorokhod: une remarque sur la démonstration d'Azema–Yor,
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SUPPLEMENTARY REFERENCES


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