FUNCTIONAL RANDOM CENTRAL LIMIT THEOREMS
FOR RANDOM WALKS CONDITIONED TO STAY POSITIVE

BY

A. SZUBARGA AND D. SZYNAL (LUBLIN)

Abstract. The aim of this note is to investigate the limiting behaviour of the random function $Y_n$ conditioned on $[T > N_n]$, where $\{N_n, n \geq 0\}$ is a sequence of positive integer-valued random variables. The results obtained are extensions of results [7] under the additional assumption that $E|X_1|^3 < +\infty$, and $X_1$ is non-lattice or integer-valued with span 1.

1. Introduction. Let $\{X_k, k \geq 1\}$ be a sequence of independent, identically distributed random variables (i. i. d. r. v.) with

$$E X_1 = 0, \quad E X_1^2 = \sigma^2, \quad 0 < \sigma^2 < \infty.$$ 

Define the random function $Y_n$ by

$$Y_n(t) = S_{[nt]} / \sigma \sqrt{n}, \quad 0 \leq t \leq 1,$$

where $S_0 = 0, S_n = X_1 + \ldots + X_n, n \geq 1$. Next, let $T$ be the hitting time of the set $(-\infty, 0]$ by the random walk $\{S_n, n \geq 1\}$,

$$T = \inf \{n > 0: S_n < 0\},$$

where the infimum of the empty set is taken to be $+\infty$.

We assume that $\{X_k, k \geq 1\}$ are the coordinate functions defined on the product space $(\Omega, \mathcal{A}, P)$. Let $A_n$ stand for $[T > n], D \equiv D[0, 1]$ stand for the space of real-valued right continuous functions on $[0, 1]$ having left limits and $\mathcal{D}$ stand for the $\sigma$-field of Borel sets generated by the open sets of the Skorokhod $\mathcal{J}_1$-topology. For $g, f \in D$ let

$$\rho(f, g) = \sup_{0 \leq x \leq 1} |f(x) - g(x)| \quad \text{and} \quad K(\theta, \sqrt{\varepsilon}) = \{f \in D: \rho(\theta, f) < \sqrt{\varepsilon}\},$$

where $\theta(x) \equiv 0$ for $x \in [0, 1]$ and $\varepsilon > 0$. 
Put $D_+ = \{ f \in D : f \geq \theta \}$ and $\mathcal{D}_+ = D_+ \cap \mathcal{D}$. The measurable mapping $Y_n^+ : (A_n, A_n \cap \mathscr{A}) \to (D_+, \mathcal{D}_+)$ is defined by

$$Y_n^+(\cdot, \omega) = S_{[n]}(\omega)/\sigma \sqrt{n}, \quad \omega \in A_n.$$  

In [3] it is given a complete proof of the functional conditioned central limit theorem, i.e., it is shown that $Y_n^+ \Rightarrow W^+, \ n \to \infty$, if $E|X_1|^3 < \infty$, and $X_1$ is nonlattice or integer-valued with span 1, where $W^+$ is Brownian meander.

2. Results.

THEOREM 1. Let $\{X_k, k \geq 1\}$ be a sequence of i.i.d.r.v. with $EX_1 = 0$, $EX_1^2 < +\infty$, $E|X_1|^3 < +\infty$, $X_1$ being nonlattice or integer-valued with span 1.

If $\{N_n, n \geq 0\}$, $N_0 = 0$ a.s., is a sequence of positive integer-valued random variables independent of $\{X_k, k \geq 1\}$ and $\{a_n, n \geq 1\}$ is a sequence of positive real numbers such that, for any given $\varepsilon > 0$,

$$\begin{align*}
\mathbb{P}[|N_n/a_n - \lambda| \geq \varepsilon] &= o\left(\frac{1}{\sqrt{n}}\right) \\
\text{with } a_n &\to \infty \text{ as } n \to \infty, \text{ and } \lambda \text{ is a random variable such that} \\
\mathbb{P}[\lambda \geq a] &= 1 \quad \text{for a constant } a > 0,
\end{align*}$$

then

$$Y_n^+ \Rightarrow W^+, \ n \to \infty.$$

Remark. Note that if $\lambda$ is a degenerate random variable at $a$, then (2) is satisfied. In this case we can use instead of (1) the condition

$$\mathbb{P}[|N_n/a_n - a| \geq \varepsilon] = o\left(\frac{1}{\sqrt{n}}\right).$$

In general, (1') cannot be replaced by the weaker condition

$$N_n/a_n \overset{p}{\to} a, \ n \to \infty$$

($P$ — in probability), which is shown by the following example.

Let $\mathbb{P}[N_n = 1] = 1/\sqrt{n}$, $\mathbb{P}[N_n = [an]] = 1 - 1/\sqrt{n}$ ($n = 1, 2, \ldots$), $a > 0$, where $[x]$ denotes the integral part of $x$. Then, for any given $\varepsilon > 0$,

$$\mathbb{P}[|N_n/a_n - a| \geq \varepsilon] = 1/\sqrt{n} \to 0, \ n \to \infty.$$

Let $\{X_k, k \geq 1\}$ be a sequence of i.i.d.r.v. of Theorem 1, independent of $N_n$, $n \geq 1$. In this case we have

$$EN_n = 1/\sqrt{n} + [an](1 - 1/\sqrt{n}),$$

and, for sufficiently large $n$,

$$\mathbb{P}[S_1 > 0, \ldots, S_n > 0] \sim \mathbb{P}[S_1 > 0]/\sqrt{n} + c(1 - 1/\sqrt{n})/\sqrt{[an]}.$$
as (see [6])

\[ P[S_1 > 0, \ldots, S_n > 0] \sim c/\sqrt{n}, \quad n \to \infty. \]

Therefore, taking into account that in this case (3) reduces to

\( \lim_{n \to \infty} P[S_n/\sqrt{n} < x | S_1 > 0, \ldots, S_n > 0] = 1 - \exp \left( -\frac{x^2}{2} \right), \quad x \geq 0, \)

we have

\[ P[S_n/\sqrt{n} < x | S_1 > 0, \ldots, S_n > 0] = \frac{P[S_1 < \sigma x, S_1 > 0]/\sqrt{n}}{P[S_1 > 0]/\sqrt{n} + P[S_1 > 0, \ldots, S_{[an]} > 0] T_n} + \frac{P\left[ S_{[an]}/\sigma < x | S_1 > 0, \ldots, S_{[an]} > 0 \right] P[S_1 > 0, \ldots, S_{[an]} > 0] T_n}{P[S_1 > 0]/\sqrt{n} + P[S_1 > 0, \ldots, S_{[an]} > 0] T_n} \]

\[ \to \frac{P[X_1 < \sigma x, X_1 > 0] + \frac{c}{\sqrt{a}} \left( 1 - \exp \left( -\frac{x^2}{2} \right) \right)}{P[X_1 > 0] + c/\sqrt{a}} \neq 1 - \exp \left( -\frac{x^2}{2} \right), \]

where \( T_n = 1 - 1/\sqrt{n} \). Obviously, in this case (2) is trivially satisfied.

We now show that, in general, assumption (2) cannot be omitted in proving (3) when \( \lambda \) is a nondegenerate random variable. Assume that \( (\langle 0, 1 \rangle, \mathcal{B}(\langle 0, 1 \rangle), P) \) is a probability space, where \( P \) is the Lebesgue measure and \( \mathcal{B}(\langle 0, 1 \rangle) \) is the \( \sigma \)-field of Borel subsets of \( \langle 0, 1 \rangle \). Assume that \( \{X_k, k \geq 1\} \) is a sequence of random variables satisfying the assumptions of Theorem 1, independent of \( \{N_n, n \geq 1\} \), where \( \{N_n, n \geq 1\} \) is defined by

\[ N_n(\omega) = \begin{cases} 1 & \text{if } \omega \in \langle 0, 1/\sqrt{n} \rangle, \\ n+1 & \text{if } \omega \in (1/\sqrt{n}, 1/n-1/\sqrt{n}-\lfloor \sqrt{n} \rfloor/n], \\ k & \text{if } \omega \in (k-1)/n, k/n), \quad k = \lfloor \sqrt{n} \rfloor + 2, \ldots, n. \end{cases} \]

It is not difficult to see that for any \( \varepsilon > 0 \) there exists an \( n_0 \) such that

\[ P[|N_n/n - \lambda| > \varepsilon] = 0 \quad \text{for } n \geq n_0 > [1/\sqrt{\varepsilon}] + 1, \]

where \( \lambda \) is uniformly distributed on \( \langle 0, 1 \rangle \). Thus (1) is satisfied but (2) does not hold.
Next we have

\[
P[S_1 > 0, \ldots, S_{N_n} > 0] = P[S_1 > 0]/\sqrt{n} + \sum_{k = [\sqrt{n}] + 2}^{n} P[S_1 > 0, \ldots, S_k > 0]/n + \\
+ P[S_1 > 0, \ldots, S_{[\sqrt{n}] + 1 > 0} \left( \frac{1}{n} - \left( \frac{1}{\sqrt{n}} - \left[ \frac{\sqrt{n}}{n} \right] \right) \right)
\]

\[
\approx \frac{1}{\sqrt{n}} P[S_1 > 0] + \\
+ \frac{1}{\sqrt{n}} \int_{([\sqrt{n}] + 2)/n}^{1} \frac{c}{\sqrt{n}} dx + c(\sqrt{\left[ \frac{\sqrt{n}}{n} \right] + 1}) \left( \frac{1}{n} - \left( \frac{1}{\sqrt{n}} - \left[ \frac{\sqrt{n}}{n} \right] \right) \right)
\]

\[
= T_n = O(1/\sqrt{n}) \quad \text{as} \quad n \to \infty.
\]

Hence, using (3'), we get

\[
P[Y_n(l) < x | S_1 > 0, \ldots, S_{N_n} > 0] \approx \frac{P[X_i < \sigma x, X_1 > 0]}{nT_n} + \\
+ \sum_{k = [\sqrt{n}] + 1}^{\sqrt{n}} \frac{P[Y_k(l) < x | S_1 > 0, \ldots, S_k > 0] P[S_1 > 0, \ldots, S_k > 0]}{nT_n}
\]

\[
+ P[Y_{[\sqrt{n}]+1}(l) < x | S_1 > 0, \ldots, S_{[\sqrt{n}]+1 > 0}] \times \\
\times P[S_1 > 0, \ldots, S_{[\sqrt{n}]+1 > 0} \left( \frac{1}{T_n} - \left( \frac{1}{\sqrt{n}} - \left[ \frac{\sqrt{n}}{n} \right] \right) \right)
\]

\[
P[X_1 < \sigma x, X_1 > 0] + \left( 1 - \exp \left( \frac{-x^2}{2} \right) \right) 2c
\]

\[
\rightarrow \frac{1 - \exp \left( \frac{-x^2}{2} \right)}{P[X_1 > 0] + 2c} = 1 - \exp \left( \frac{-x^2}{2} \right).
\]

We have seen that, in general, (2) cannot be omitted in proving (3). However, we are able to give more general conditions than (1) and (2), under which (3) holds.

**Theorem 2.** Let \( \{X_n, n \geq 1\} \) be a sequence of i.i.d.r.v. of Theorem 1. Suppose that \( \{N_n, n \geq 1\} \) is a sequence of positive integer-valued random variables independent of \( \{X_k, k \geq 1\} \) and \( \{\alpha_n, n \geq 1\} \) is a sequence of positive real numbers such that \( \lim_{n \to \infty} \alpha_n = \infty \).
Central limit theorems

If $\lambda$ is a positive random variable such that

$$
\Pr \left[ |N_n/\alpha_n - \lambda| > \varepsilon \right] = o\left( E(1/\sqrt{N_n}) \right),
$$

(5)

$$
\Pr \left[ \lambda - 2\alpha_n \right] = o\left( E(1/\sqrt{N_n}) \right),
$$

(6)

where $\{\varepsilon_n, n \geq 1\}$ is a sequence of positive real numbers such that $0 < \varepsilon_n \to 0$, $\alpha_n \varepsilon_n \to \infty$ as $n \to \infty$, then (3) holds.

Note that assumptions similar to (5) and (6) were used in [4] to give the rate of convergence in the functional central limit theorem.

A functional random central limit theorem for random walks conditioned to stay positive without the assumption of independence $\{X_k, k \geq 1\}$ and $\{N_n, n \geq 1\}$ is given in the following

**Theorem 3.** Let $\{X_k, k \geq 1\}$ be a sequence of i.i.d.r.v. of Theorem 1.

If $\{N_n, n \geq 1\}$ is a sequence of positive integer-valued random variables and $\{\alpha_n, n \geq 1\}$ is a sequence of positive real numbers such that, for any given $\varepsilon > 0$,

$$
\Pr \left[ |N_n/\alpha_n - a| \geq \varepsilon \right] = o(1/\sqrt{\alpha_n})
$$

(7)

with $\alpha_n \to \infty$ as $n \to \infty$, where $a$ is a positive constant, then (3) holds.

3. Proofs of the results.

**Proof of Theorem 1.** Let $\varepsilon$, $0 < \varepsilon < 1$, be fixed, and $a_n = [(a-\varepsilon)\alpha_n]$. By (1), (2) and the assumption $\alpha_n \to \infty$ we can choose an $n$ such that

$$
0 \leq \sum_{k=1}^{a_n} \Pr[S_1 > 0, \ldots, S_k > 0] \Pr[N_n = k] \leq \sum_{k=1}^{a_n} \Pr[N_n = k]
$$

$$
\leq \Pr \left[ |N_n/\alpha_n - \lambda| \geq \varepsilon \right] = o\left( E(1/\sqrt{N_n}) \right)
$$

and, at the same time, by (4),

$$
\sum_{k=a_n+1}^{\infty} \Pr[S_1 > 0, \ldots, S_k > 0] \Pr[N_n = k] \sum_{k=a_n+1}^{\infty} (c/\sqrt{k}) \Pr[N_n = k]
$$

$$
= c\left( E(1/\sqrt{N_n}) \right) - \sum_{k=1}^{a_n} (c/\sqrt{k}) \Pr[N_n = k].
$$

But

$$
0 \leq c \sum_{k=1}^{a_n} \frac{1}{\sqrt{k}} \Pr[N_n = k] \leq c \sum_{k=1}^{a_n} \Pr[N_n = k] \leq c \Pr \left[ \frac{|N_n/\alpha_n - \lambda| \geq \varepsilon} \right]
$$

$$
= o\left( E(1/\sqrt{N_n}) \right).
$$

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Hence
\[
P[S_1 > 0, \ldots, S_{N_n} > 0] \approx cE(1/\sqrt{N_n})
\]

Put now
\[
C_{n,k} = \frac{P[S_1 > 0, \ldots, S_k > 0]P[N_n = k]}{P[S_1 > 0, \ldots, S_{N_n} > 0]} \quad (k \geq 1, n \geq 1).
\]

We see that \(\sum_{k=1}^{\infty} C_{n,k} = 1\) and, for fixed \(k\), by (1) and (8),
\[
0 \leq C_{n,k} \leq \frac{\sum_{k=1}^{\infty} P[N_n = k]}{P[S_1 > 0, \ldots, S_{N_n} > 0]} \approx \frac{o(E(1/\sqrt{N_n}))}{cE(1/\sqrt{N_n})} \rightarrow 0, \quad n \rightarrow \infty,
\]
which proves that \([C_{n,k}]\) is a Toeplitz matrix. Therefore, by [5], p. 472, and (3'), we have
\[
P[Y_n(1) < x | S_1 > 0, \ldots, S_{N_n} > 0]
= \sum_{k=1}^{\infty} C_{n,k} P[Y_k(1) < x | S_1 > 0, \ldots, S_i > 0] \rightarrow 1 - \exp\left(-\frac{x^2}{2}\right),
\quad n \rightarrow \infty, x \geq 0.
\]

Now we need the notations
\[
g(t, x_1, x_2) = (2\pi t)^{-1/2} [\exp(-(x_2 - x_1)^2/2t) - \exp(-(x_1 + x_2)^2/2t)],
\]
\[
x_1, x_2 > 0, \quad 0 < t \leq 1,
\]
\[
p(0, 0, t, x) = t^{-3/2} x \exp(-x^2/2t)|N|(x(1-t)^{1/2}),
\]
where
\[
|N|(x) = (2/\pi)^{1/2} \int_0^x \exp\left(-\frac{u^2}{2}\right) du,
\]
and
\[
p(t_1, x_1, t_2, x_2) = g(t_2 - t_1, x_1, x_2)|N|(x_2/(1-t_2)^{1/2})|N|(x_1/(1-t_1)^{1/2}),
\]
\[
x_1, x_2 > 0, \quad 0 < t_1 < t_2 \leq 1.
\]

It is known [3] that for \(x \geq 0\)
\[
\lim_{n \rightarrow \infty} P[Y_n(t) < x | T > n] = \int_0^x p(0, 0, t, y) dy,
\]
\[
(11)
\]
whence

\[ P \left[ Y_n(t) < x \mid S_1 > 0, \ldots, S_{N_n} > 0 \right] \]

\[ = \sum_{k=1}^{n-m} C_{n,k} P \left[ Y_k(t) < x \mid S_1 > 0, \ldots, S_k > 0 \right] \int_{0}^{x} \int \ldots \int_{0}^{x} \int_{0}^{y} \ldots \int_{0}^{y} dy \ldots dy_k \]

\[ \lim_{n \to \infty} \text{as (11) holds and } [C_{n,k}] \text{ is a Toeplitz matrix. Moreover, since} \]

\[ \lim_{n \to \infty} P \left[ Y_n(t_1) < x_1, Y_n(t_2) < x_2, \ldots, Y_n(t_k) < x_k \mid T > n \right] \]

\[ = \int_{0}^{x_1} \int_{0}^{x_k} \int_{0}^{y_1} \int_{0}^{y_2} \ldots \int_{0}^{y_{k-1}} \int_{0}^{y_{k-1}} \int_{0}^{y_k} dy_1 \ldots dy_k \]

for all \( k \geq 1, x_1, \ldots, x_k > 0 \) and \( 0 < t_1 < t_2 < \ldots < t_k \leq 1 \), we have

\[ P \left[ Y_n(t_1) < x_1, Y_n(t_2) < x_2, \ldots, Y_n(t_k) < x_k \mid S_1 > 0, \ldots, S_{N_n} > 0 \right] \]

\[ = \sum_{j=1}^{\infty} C_{n,j} P \left[ Y_j(t_1) < x_1, \ldots, Y_j(t_k) < x_k \mid S_1 > 0, \ldots, S_j > 0 \right] \]

\[ \int_{0}^{x_1} \int_{0}^{x_k} \int_{0}^{y_1} \int_{0}^{y_2} \ldots \int_{0}^{y_{k-1}} \int_{0}^{y_{k-1}} \int_{0}^{y_k} dy_1 \ldots dy_k \]

for all \( k \geq 1, x_1, \ldots, x_k > 0 \) and \( 0 < t_1 < t_2 < \ldots < t_k \leq 1 \).

We now prove that \( \{Y_{N_n}^+\} \) is tight.

Taking into account that for \( \epsilon > 0 \)

\[ \lim_{n \to \infty} \sup_{\delta > 0} P \left[ \omega_{Y_n}(\delta, 0, 1) \geq \epsilon \mid S_1 > 0, \ldots, S_n > 0 \right] = 0, \]

where

\[ \omega_{Y_n}(\delta, a, b) = \sup_{x,s,t} |x(s) - x(t)| : 0 \leq a \leq b \leq 1, 0 < \delta < 1, a \leq s \leq t \leq b, |t-s| < \delta, \]

we obtain, by the above arguments,

\[ \lim_{n \to \infty} \sup_{\delta > 0} P \left[ \omega_{Y_n}(\delta, 0, 1) \geq \epsilon \mid S_1 > 0, \ldots, S_{N_n} > 0 \right] = 0, \]

which, by theorems 15.1 and 15.5 of [1], proves that \( \{Y_{N_n}^+\} \) is tight. Therefore, by (14) and (15), we have proved (3).

**Proof of Theorem 2.** By (4) and (5) we have, for sufficiently large \( n \),

\[ P \left[ S_1 > 0, \ldots, S_{N_n} > 0 \right] = \sum_{k=1}^{[\epsilon N_n]} P \left[ S_1 > 0, \ldots, S_k > 0 \right] P \left[ N_s = k \right] + \sum_{k=[\epsilon N_n]+1}^{\infty} P \left[ S_1 > 0, \ldots, S_k > 0 \right] P \left[ N_s = k \right] \approx cE \left( \frac{1}{\sqrt{N_n}} \right). \]
Now we note that, for \( n \geq 1 \) and \( j \geq 1 \),
\[
C_{n,j} = \frac{P[S_1 > 0, \ldots, S_j > 0] P[N_n = j]}{P[S_1 > 0, \ldots, S_{N_n} > 0]}
\]
is a Toeplitz matrix. Indeed, we have
\[
C_{n,j} \geq 0, \quad \sum_{j=1}^{\infty} C_{n,j} = 1,
\]
\[
\sum_{k=1}^{[\varepsilon_n \varepsilon_n]} P[N_n = k] \frac{P \left( \left| \frac{N_n - \lambda}{\varepsilon_n} \right| \geq \varepsilon_n \right) + P[\lambda < 2\varepsilon_n]}{cE(1/\sqrt{N_n})} \leq \frac{1}{cE(1/\sqrt{N_n})} \to 0,
\]
n\( \to \infty \), by (5), (6) and (16), since \( j < \varepsilon_n \varepsilon_n \) for sufficiently large \( n \).

Following the considerations of the proof of Theorem 1 we get (3).

**Proof of Theorem 3.** Let \( \varepsilon, 0 < \varepsilon < \alpha \), be fixed and put \( a_n = [(a-\varepsilon)\alpha_n], \ b_n = [(a+\varepsilon)\alpha_n], \ c_n = b_n - a_n, \ u_n = (a_n/b_n)^{1/2}, \ A_n = \{k: a_n \leq k \leq b_n\} \)
and \( A_n^c \) is the complement of \( A_n \).

Set
\[
r_k = P[S_1 > 0, \ldots, S_k > 0], \quad \hat{r}_n = P[S_1 > 0, \ldots, S_{N_n} > 0].
\]

From (4) and (7) we get
\[
\frac{c}{\sqrt{b_n}} - o(1/\sqrt{b_n}) \leq \hat{r}_n \leq \frac{c}{\sqrt{a_n}} + o(1/\sqrt{a_n}). \tag{17}
\]

We see that
\[
P[Y_n^1(1) < x | S_1 > 0, \ldots, S_{N_n} > 0]
\]
\[
= P[Y_n^1(1) < x, S_1 > 0, \ldots, S_{N_n} > 0]/\hat{r}_n
\]
\[
\sim \sum_{k \in A_n^c} P[Y_k(1) < x, S_1 > 0, \ldots, S_k > 0, N_n = k]/\hat{r}_n +
\]
\[
+ P[Y_n^1(1) < x, S_1 > 0, \ldots, S_{N_n} > 0, N_n \in A_n^c]/\hat{r}_n.
\]

But, by (7) and (17) we have
\[
P[Y_n^1(1) < x, S_1 > 0, \ldots, S_{N_n} > 0, N_n \in A_n^c]/\hat{r}_n \to 0, \quad n \to \infty.
\]

Therefore, to prove that for \( x \geq 0 \)
\[
P[Y_n^1(1) < x | S_1 > 0, \ldots, S_{N_n} > 0] \to 1 - \exp\left(-\frac{x^2}{2}\right), \quad n \to \infty,
\]
it is enough to consider
\[
P[Y_n^1(1) < x, S_1 > 0, \ldots, S_{N_n} > 0, N_n \in A_n]/\hat{r}_n.
\]
Put now, for $0 \leq t \leq 1$,

$$Z_n(t) = \max_{k \in A_n} \frac{S_{[kt]} - S_{[(a+\varepsilon)kn]}}{\sigma \sqrt{b_n}}, \quad Z^*_n(t) = \max_{k \in A_n} \frac{S_{[kt]} - S_{[(a-\varepsilon)kn]}}{\sigma \sqrt{c_n}}.$$

Then we have

\begin{align*}
   (20) & \quad P[Y_n(1) < x, S_1 > 0, \ldots, S_{N_n} > 0, N_n \in A_n] \\
         & \quad \geq P[Y_n(1) < x, S_1 > 0, \ldots, S_{B_n} > 0, N_n \in A_n] \\
         & \quad \geq P[Y_n(1) + Z_n(1) < xu_n, S_1 > 0, \ldots, S_{B_n} > 0, N_n \in A_n] \\
         & \quad \geq P[Y_n(1) + Z_n(1) < xu_n, S_1 > 0, \ldots, S_{B_n} > 0, Z_n(1) < 4\sqrt{\varepsilon} - P[N_n \in A_n] \\
         & \quad \geq P[Y_n(1) < xu_n - 4\sqrt{\varepsilon} | S_1 > 0, \ldots, S_{B_n} > 0] r_n - P[Z_n(1) \geq 4\sqrt{\varepsilon}] r_n - P[N_n \in A_n],
\end{align*}

as $Z_n(1)$ does not depend on $S_1, S_2, \ldots, S_{a_n}$.

The similar evaluations lead us to

\begin{align*}
   (21) & \quad P[Y_n(1) < x, S_1 > 0, \ldots, S_{N_n} > 0, N_n \in A_n] \\
         & \quad \leq P[Y_n(1) < x/u_n - 4\sqrt{\varepsilon} | S_1 > 0, \ldots, S_{a_n} > 0] r_n + P[Z_n(1) \geq 4\sqrt{\varepsilon}] r_n.
\end{align*}

Note now that, by Kolmogorov's inequality,

\begin{align*}
   (22) & \quad P[Z_n(t) \geq 4\sqrt{\varepsilon}] \leq P \left[ \max_{1 \leq k \leq [(a+\varepsilon)tn] - [(a-\varepsilon)tn]} \left| \frac{S_k}{\sigma \sqrt{b_n}} \right| \geq 4\sqrt{\varepsilon} \right] \\
        & \quad \leq \frac{2t \sqrt{\varepsilon}}{a+\varepsilon} \quad \text{as } n \to \infty
\end{align*}

and

\begin{align*}
   (23) & \quad P[Z_n^*(t) \geq 4\sqrt{\varepsilon}] \leq \frac{[(a+\varepsilon)\alpha_n t] - [(a-\varepsilon)\alpha_n t]}{\sqrt{\varepsilon} a_n} \rightarrow \frac{2t \sqrt{\varepsilon}}{a-\varepsilon} \quad \text{as } n \to \infty.
\end{align*}

Therefore, by (18), (20)-(23), we obtain

\begin{align*}
   (24) & \quad P[Y_n(1) < xu_n - 4\sqrt{\varepsilon} | S_1 > 0, \ldots, S_{B_n} > 0] (r_n/\bar{r}_n)^- \\
         & \quad - \frac{c_n}{\sqrt{\varepsilon} b_n} (r_n/\bar{r}_n) - P[N_n \in A_n]^c/\bar{r}_n \leq P[Y_n(1) < x | S_1 > 0, \ldots, S_{N_n} > 0] \\
         & \quad \leq P[Y_n(1) < x/u_n + 4\sqrt{\varepsilon} | S_1 > 0, \ldots, S_{a_n} > 0] (r_n/\bar{r}_n) + \frac{c_n}{\sqrt{\varepsilon} b_n} (r_n/\bar{r}_n) + \\
         & \quad + P[N_n \in A_n]^c/\bar{r}_n.
\end{align*}
But by (3') we have

\[(25) \quad P[Y_n(1) < xu_n - \frac{4}{\sqrt{\varepsilon}} | S_1 > 0, \ldots, S_{n-1} > 0] = \exp\left(-\frac{x \sqrt{\frac{a+\varepsilon}{a-\varepsilon} - \frac{4}{\sqrt{\varepsilon}}}}{2}\right)\]

and

\[(26) \quad P[Y_n(1) < xu_n - \frac{4}{\sqrt{\varepsilon}} | S_1 > 0, \ldots, S_{n-1} > 0] = \exp\left(-\frac{x \sqrt{\frac{a-\varepsilon}{a+\varepsilon} + \frac{4}{\sqrt{\varepsilon}}}}{2}\right)\]

as \( n \to \infty. \)

Moreover, by (4) and (17) we get

\[(27) \quad \liminf_{n \to \infty} (r_{b_n} / \hat{r}_n) = \frac{\sqrt{a-\varepsilon}}{a+\varepsilon}\]

and

\[(28) \quad \limsup_{n \to \infty} (r_{a_n} / \hat{r}_n) = \frac{\sqrt{a+\varepsilon}}{a-\varepsilon}.\]

Therefore, for any given \( \varepsilon, \ 0 < \varepsilon < a, \) by (20)-(28) we get, for \( x \geq 0, \)

\[-2 \sqrt{\varepsilon} \frac{a+\varepsilon}{a-\varepsilon} + \frac{a-\varepsilon}{a+\varepsilon} \left( -\exp\left(-\frac{x \sqrt{\frac{a-\varepsilon}{a+\varepsilon} - \frac{4}{\sqrt{\varepsilon}}}}{2}\right) \right)\]

\[\leq \liminf_{n \to \infty} P[Y_n(1) < x | S_1 > 0, \ldots, S_{n-1} > 0] \leq \limsup_{n \to \infty} P[Y_n(1) < x | S_1 > 0, \ldots, S_{n-1} > 0] \leq 2 \sqrt{\varepsilon} \frac{a+\varepsilon}{a-\varepsilon} + \frac{a+\varepsilon}{a-\varepsilon} \left( 1 - \exp\left(-\frac{x \sqrt{\frac{a-\varepsilon}{a+\varepsilon} + \frac{4}{\sqrt{\varepsilon}}}}{2}\right) \right).\]

Letting now \( \varepsilon \to 0 \) for \( x \geq 0, \) we obtain (19).

Note that \( Z_n(t) \) does not depend on \( S_1, \ldots, S_{[a-\varepsilon]n}. \) This fact and the same arguments as above show that, for \( x \geq 0, \)
Letting now $n \to \infty$, next $\varepsilon \to 0$ for $x \geq 0$ and $t \in (0, 1)$ we obtain

\begin{equation}
(29) \quad \lim_{n \to \infty} \frac{[(a+\varepsilon)\alpha_n t] - [(a-\varepsilon)\alpha_n t]}{\sqrt{\varepsilon} b_n} (r_{\lfloor \alpha_n t \rfloor}/\varepsilon) - \frac{P[N_n \in A_n^*]}{\varepsilon} r_n - P[N_n \in A_n^*] P[Y_n(t) < x | S_1 > 0, \ldots, S_n > 0] \leq P[Y_n(t) < x | S_1 > 0, \ldots, S_n > 0]
\end{equation}

\begin{equation}
(30) \quad \lim_{n \to \infty} P[Y_n(t) < x] = \int_0^x p(0, 0, t, y) dy.
\end{equation}

In the same way for all $k \geq 1$, $x_1, \ldots, x_k > 0$ and $0 < t_1 < \ldots < t_k \leq 1$ we have

\begin{equation}
(31) \quad P[Y_n(t_1) < x_1 u_n - \frac{4}{\sqrt{\varepsilon}}, \ldots, Y_n(t_k) < x_k u_n - \frac{4}{\sqrt{\varepsilon}} | S_1 > 0, \ldots, S_n > 0] (r_{\lfloor \alpha_n t \rfloor}/\varepsilon) - P[Z(t_1) > \frac{4}{\sqrt{\varepsilon}}] X \times P[S_1 > 0, \ldots, S_{\lfloor \alpha_n t \rfloor} > 0] / \varepsilon - \ldots - P[Z(t_k) > \frac{4}{\sqrt{\varepsilon}}] X \times P[S_1 > 0, \ldots, S_{\lfloor \alpha_n t \rfloor} > 0] / \varepsilon - P[N_n \in A_n^*] / \varepsilon
\end{equation}

\begin{equation}
(32) \quad \lim_{n \to \infty} P[Y_n(t_1) < x_1, \ldots, Y_n(t_k) < x_k | S_1 > 0, \ldots, S_n > 0] = \int_0^x \cdots \int_0^x p(0, 0, t_1, y_1) p(t_1, y_1, t_2, y_2) \cdots p(t_k-1, y_{k-1}, t_k, y_k) dy_1 \cdots dy_k
\end{equation}

for all $k \geq 1$, $x_1, \ldots, x_k > 0$ and $0 < t_1 < t_2 < \ldots < t_k < 1$.

To complete the proof of the weaker convergence of $\{Y_n\}$ to $W^+$ it suffices (cf. Theorems 15.1 and 15.3 of [1]) to show that for every $v > 0$

\begin{equation}
(33) \quad \lim \limsup_{\varepsilon \to 0, n \to \infty} P[\omega_Y(t_1, 0, 1) > v | S_1 > 0, \ldots, S_n > 0] = 0.
\end{equation}
We can see that $\omega_f(\delta, 0, 1) < 2 \frac{\sqrt[4]{\varepsilon}}{\varepsilon}$ whenever $f \in K(\theta, \frac{\sqrt[4]{\varepsilon}}{\varepsilon})$. Hence, for a fixed $\varepsilon > 0$ such that $2 \frac{\sqrt[4]{\varepsilon}}{\varepsilon} < \nu$, we have

\[ (34) \quad P[\omega_{Y_{N_n}}(\delta, 0, 1) \geq \nu, S_1 > 0, \ldots, S_{N_n} > 0]/\hat{r}_n \]

\[ \leq P[\omega_{Y_{\alpha_n}}(\delta, 0, 1) + \omega_{(Y_{N_n} - Y_{\alpha_n})}(\delta, 0, 1) \geq \nu, S_1 > 0, \ldots, S_{N_n} > 0, N_n \in A_n, (Y_{N_n} - Y_{\alpha_n}) \in K(\theta, \frac{\sqrt[4]{\varepsilon}}{\varepsilon})]/\hat{r}_n + P[(Y_{N_n} - Y_{\alpha_n}) \in K(\theta, \frac{\sqrt[4]{\varepsilon}}{\varepsilon}), S_1 > 0, \ldots, S_{\alpha_n} > 0, N_n \in A_n, (Y_{N_n} - Y_{\alpha_n}) \in K(\theta, \frac{\sqrt[4]{\varepsilon}}{\varepsilon})]/\hat{r}_n \]

\[ \leq P[\omega_{Y_{\alpha_n}}(\delta, 0, 1) \geq \nu - \frac{\sqrt[4]{\varepsilon}}{\varepsilon} | S_1 > 0, \ldots, S_{\alpha_n} > 0, (r_{\alpha_n}/\hat{r}_n) + P[\omega_{(Y_{N_n} - Y_{\alpha_n})}(\delta, 0, 1) \geq \nu - \frac{\sqrt[4]{\varepsilon}}{\varepsilon} | S_1 > 0, \ldots, S_{\alpha_n} > 0, N_n \in A_n, (Y_{N_n} - Y_{\alpha_n}) \in K(\theta, \frac{\sqrt[4]{\varepsilon}}{\varepsilon})]/\hat{r}_n + P[(Y_{N_n} - Y_{\alpha_n}) \in K(\theta, \frac{\sqrt[4]{\varepsilon}}{\varepsilon}), S_1 > 0, \ldots, S_{\alpha_n} > 0, N_n \in A_n, (Y_{N_n} - Y_{\alpha_n}) \in K(\theta, \frac{\sqrt[4]{\varepsilon}}{\varepsilon})]/\hat{r}_n \]

\[ \leq P[\omega_{Y_{\alpha_n}}(\delta, 0, 1) \geq \nu - \frac{\sqrt[4]{\varepsilon}}{\varepsilon} | S_1 > 0, \ldots, S_{\alpha_n} > 0, (r_{\alpha_n}/\hat{r}_n) + P[\max_{k \in A_n} S_k < \frac{1}{\sqrt{\alpha_n}} \frac{S_{[\alpha_n]} - S_{[k]}}{\sqrt{a_n}}, \theta > 0, \ldots, S_{\alpha_n} > 0] / \hat{r}_n + P[N_n \in A_n, (Y_{N_n} - Y_{\alpha_n}) \in K(\theta, \frac{\sqrt[4]{\varepsilon}}{\varepsilon})]/\hat{r}_n \]

Knowing that

\[ (35) \quad \lim\sup_{\delta \to 0} \lim_{n \to \infty} P[\omega_{Y_{\alpha_n}}(\delta, 0, 1) \geq \nu - 2 \frac{\sqrt[4]{\varepsilon}}{\varepsilon} | S_1 > 0, \ldots, S_{\alpha_n} > 0, (r_{\alpha_n}/\hat{r}_n) = 0 \]

and

\[ (36) \quad \lim_{n \to \infty} P[N_n \in A_n, (Y_{N_n} - Y_{\alpha_n}) \in K(\theta, \frac{\sqrt[4]{\varepsilon}}{\varepsilon})]/\hat{r}_n = 0 \]

and taking into account that

\[ (37) \quad P[\max_{k \in A_n} \frac{S_k}{\sqrt{a_n}} - \frac{S_{[\alpha_n]}}{\sqrt{a_n}} \geq K(\theta, \frac{\sqrt[4]{\varepsilon}}{\varepsilon}, S_1 > 0, \ldots, S_{\alpha_n} > 0)] / \hat{r}_n \]

\[ \leq P[\max_{k \in A_n} \frac{S_k - S_{\alpha_n}}{\sqrt{a_n}} > \frac{\sqrt[4]{\varepsilon}}{\varepsilon}, S_1 > 0, \ldots, S_{\alpha_n} > 0] / \hat{r}_n \]

\[ = P[Z_{\alpha_n}(1) > 2 \frac{\sqrt[4]{\varepsilon}}{\varepsilon}, S_1 > 0, \ldots, S_{\alpha_n} > 0] / \hat{r}_n \]

we conclude, by (37), (22), (35) and (36), that (33) holds. This completes the proof of Theorem 3.
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Mathematical Institute
Maria Curie-Skłodowska University
ul. Nowotki 10
20-031 Lublin, Poland

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