A CLASSIFICATION OF RANDOM MEASURES

BY

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Abstract. Modifying the definition of $\alpha$-times $(0 < \alpha \leq \infty)$ self-decomposable (selfdec.) distributions on linear spaces due to N. V. Thu, we define $\alpha$-times selfdec. random measures (r.m.) on a Polish space. We prove representation theorems for such r.m. and study some related limit problems.

Throughout the paper we preserve the terminology and notation of [2]. Recall some of them. Let $\sigma$ be a Polish space, $\mathcal{B}$ — the ring of all bounded Borel subsets of $\sigma$, $\mathcal{F}_c$ — the class of all continuous functions $f: \sigma \to \mathbb{R}_+ = [0; \infty)$ with compact support and $M$ — the class of all Radon measures on $\sigma$. We shall consider $M$ as a Polish space with the vague topology. By a random measure (r.m.) on $\sigma$ we mean a Borel probability measure on $M$. By $M_0$ we denote the class of all infinitely divisible random measures (i.d.r.m.) on $\sigma$ (cf. [2]).

Let $L_p$ denote the Laplace transform of an i.d.r.m. $P$ on $\sigma$. By virtue of Theorem 6.1 in [2] we get the formula

\[ -\log L_p(f) = m(f) + \lambda(1 - e^{-\beta f}), \quad f \in \mathcal{F}_c, \]

where $m \in M$, $\lambda$ is a measure on $M' = M \setminus \{0\}$ satisfying the condition

\[ \lambda(1 - e^{-\beta B}) < \infty, \quad B \in \mathcal{B}. \]

In what follows $(m, \lambda)$ will be called canonical measure of $P$ and we write $P = (m, \lambda)$. Further, by $L_0$ we denote the class of all measures $\lambda$ on $M'$ satisfying condition (2).
For every $\alpha > 0$ and $k = 0, 1, \ldots$ we put

$$r_{\alpha,k} = \binom{\alpha+k-1}{k} = \begin{cases} 1 & \text{if } k = 0, \\ \frac{\alpha(\alpha+1)\ldots(\alpha+k-1)}{k!} & \text{if } k = 1, 2, \ldots \end{cases}$$

Given a number $c > 0$ and an r.m. $P$ on $\sigma$, we define an r.m. $T_cP$ on $\sigma$ by

$$T_cP(E) = P \{ \mu : c\mu \in E \}$$

for every Borel subset $E$ of $M$.

The concept of $\alpha$-times selfdec. probability measures on linear spaces was introduced and studied by Thu [5, 6]. In the same way one can define $\alpha$-times selfdec. r.m. Namely, an r.m. $P$ on $\sigma$ is said to be $\alpha$-times selfdec. if for every $c \in (0, 1)$ there exists an i.d.r.m. $P_{\alpha,c}$ such that

$$P = \star \lim_{k \to \infty} T_k A(P_{\alpha,c}; r_{\alpha,k}),$$

where for an i.d.r.m. $Q$ and $t > 0$ the symbol $A(Q; t)$ denotes $Q^{*t}$ and $\star$ is the convolution operation.

Further, if (3) holds for some fixed $c \in (0, 1)$ and $P_{\alpha,c} \in M_0$, then we say that $P$ is $\alpha$-times $c$-decomposable ($c$-dec., cf. [4]).

By $M_\alpha$ (resp. $M_{\alpha,c}$), $0 < \alpha < \infty$, we denote the class of all $\alpha$-times selfdec. (resp. $c$-dec.) r.m. on $\sigma$. Further, the r.m. in

$$M_\infty = \bigcap_{\alpha > 0} M_\alpha \quad (\text{resp. } M_{\infty,c} = \bigcap_{\alpha > 0} M_{\alpha,c})$$

are called completely selfdec. (resp. completely $c$-dec.).

An r.m. $P \in M_\alpha$ is said to be $\alpha$-differentiable if the following limit exists in the weak sense:

$$D^{(\alpha)} P = \lim_{t \to 0} A(P_{\alpha,c}; t^{-r});$$

$P_{\alpha,c}$ is determined in (3) with $c = e^{-t}$ (cf. [6]). For every $r > 0$ and $B \in \mathcal{B}$ we put $M_r(B) = \{ \mu \in M : \mu B > r \}$.

The following theorem is an analogon of Theorem 2.1 in [4] and its proof will be omitted:

**Theorem 1.** The following statements are equivalent:

(i) The infinite convolution $\star \lim_{k \to \infty} T_k A(P; r_{\alpha,k})$ is weakly convergent.

(ii) $\int_{M_1(B)} \log^* \mu B P(d\mu) < \infty, \ B \in \mathcal{B}.$

(iii) $\int_{M_1(B)} \log^* \mu B \lambda(d\mu) < \infty, \ B \in \mathcal{B}.$
Let $M_{0,\alpha}$ denote the class of all $P \in M_0$ satisfying condition (ii) of Theorem 1. Further, by $L_{0,\alpha}$ we denote the class of all $\lambda \in L_0$ such that $P = (0, \lambda) \in M_{0,\alpha}$.

**Theorem 2.** The following statements are equivalent:

(i) $P \in M_\alpha$.

(ii) $P \in M_\alpha$ and $\{A(P_{xc}; t^{-s}), \ t > 0, \ c = e^{-t}\}$ is relatively compact in the weak sense.

(iii) There exist an $m_\alpha \in M$ and a $\lambda_\alpha \in L_{0,\alpha}$ such that

$$-\log L^*_P(f) = m_\alpha(f) + \frac{1}{\Gamma(x)} \int_0^\infty T_{e^{-t}} \lambda_\alpha(1 - e^{-xf}) t^{x-1} \ dt, \quad f \in \mathcal{F}_c.$$ 

(iv) $P$ is $\alpha$-differentiable and $D^{(\alpha)} P \in M_{0,\alpha}$.

**Proof.** Suppose first that (i) holds, i.e. $P \in M_\alpha$. By an elementary argument we get $1 - e^{-\alpha t} \geq c(1 - e^{-y})$ for every $c \in (0, 1)$ and $y > 0$. Consequently,

$$L^*_P(f) \leq \{L^*_{P_{xc}}(f)\}^{(1 - c)^{-\alpha}}, \quad f \in \mathcal{F}_c.$$ 

By the last inequality and Lemma 4.5 in [2] we can show that $\{A(P_{xc}; t^{-s}), \ t > 0, \ c = e^{-t}\}$ is relatively compact, which proves (ii).

Now we assume that (ii) holds. Let $P_\alpha = (m_\alpha, \lambda_\alpha)$ be a limit point of $A(P_{xc}; t^{-s})$ as $t \to 0$. By Theorem 2, X.9, in [1] and by the fact that

$$r_{a,k} = \frac{1}{k! \Gamma(x)} \int_0^\infty e^{-t} t^{x+k-1} \ dt$$

it follows that $m_\alpha = m$ and

$$\lambda(1 - e^{-xf}) = \frac{1}{\Gamma(x)} \int_0^\infty T_{e^{-t}} \lambda_\alpha(1 - e^{-xf}) t^{x-1} \ dt, \quad f \in \mathcal{F}_c,$$

which implies (iii).

Finally, if (iii) holds, then by (4), (5) and Theorem 2, X.9, in [1] it follows that

$$\lambda = \lim_{s \to 0} \sum_{k=0}^\infty r_{a,k} T_{e^{-ks}}(s^x \lambda_\alpha).$$

Putting, for $t > 0$, $t_n = t/2^n$, $c_n = e^{-t_n}$ and

$$\lambda_{t,n} = \sum_{k=0}^\infty r_{a,k} T_{e^{-t_n}}(t_n^x \lambda_\alpha), \quad n = 0, 1, 2, \ldots,$$
we get

\[ P_{t,a} = (m_a, \lambda_{t,a}) \in M_{a,c,n}. \]

Note that, for every \( c \in (0, 1) \), \( M_{a,c} \) is closed in the weak topology and \( M \) is contained in \( M_{a,c,2} \). Then (6) together with (7) imply that \( P \in M_{a,c-1}. \) Since \( t > 0 \) is arbitrary, we conclude that \( P \in M_a. \) Hence (ii) holds.

It is easy to show that \( P = (m_a, \lambda_a) \) is uniquely limit point of \( A(P_{x,c}; t^{-x}) \) as \( t \to 0 \). Thus (iv) is proved.

It is clear that (iv) implies (i). Theorem 2 is thus proved.

Let \( S_\infty \) denote the class of all finite convolutions of stable m.s. on \( \sigma \) and their cluster points.

**Theorem 3.** The following statements are equivalent:

(i) \( P \in M_\infty. \)

(ii) \( P \in S_\infty. \)

(iii) There exist an \( m \in M \), a subset \( K \) of \( (0, 1] \times M' \) and a probability measure \( \lambda_\infty \) on \( K \) such that

\[ -\log L_P(f) = m(f) + \int K [\mu(f)]^w \lambda_\infty(dwd\mu), \quad f \in \mathcal{F}_c. \]

**Proof.** By Theorem 1 in [3] one can show that (ii) implies (ii). It is clear that (ii) implies (i). We shall prove that (i) implies (iii). Suppose that \( P = (m, \lambda) \in M_\infty. \) Let \( L_\infty \) be the set of all measures \( \lambda' \in L_0 \) such that \( P' = (0, \lambda') \in M_\infty. \) By the arguments similar to those given in the proof of Proposition 11.5 in [7] one can show that \( L_\infty \) is the union of its caps (see [7], Section 11). Suppose that \( \lambda \) is in a cap \( C \) of \( L_\infty. \) Note that if \( R_+l \) is an extreme ray of \( L_\infty \) (see [7], Section 11), then \( l \) is a canonical measure of a stable r.m. on \( \sigma. \) By Theorem 1 in [3] and Proposition 11.1 in [7] the extreme non-zero points of \( C \) are of the form \( l_{w,*,*} \) with \( w \in (0, 1], \mu \in M', \) such that \( l_{w,*,*}(1-e^{-w}) = [\mu(f)]^w, \quad f \in \mathcal{F}_c. \) By Choquet's theorem ([7], Section 3) there exists a probability measure \( l_\infty \) on the set \( e \times C \) of all extreme points of \( C \) such that

\[ \lambda(1-e^{-w}) = \int e \times C l_{e^{-w}}(dl), \quad f \in \mathcal{F}_c. \]

Let \( \varphi \) be the mapping from \( (0, 1] \times M' \) into the set of all canonical measures of stable r.m. on \( \sigma, \) determined by the formula

\[ \varphi(w, \mu)(1-e^{-w}) = [\mu(f)]^w, \quad f \in \mathcal{F}_c. \]

Put \( k = \varphi^{-1}(a \times C) \) and \( \lambda_\infty = l_\infty \varphi^{-1}. \) We get (iii). The proof of Theorem 3 is completed.

Now, by a minor changing the proof of Theorem 5.1 in [5], one can prove the following
THEOREM 4. (i) Every $M_{\alpha}$ ($0 < \alpha \leq \infty$) is closed under convolution operation shifts changes of scales and passages to weak limits.

(ii) For any $0 \leq \alpha < \beta \leq \infty$,

$$M_\beta \subset M_\alpha, \quad M_\beta = \bigcap_{0 < \gamma < \beta} M_\gamma, \quad M_\alpha = \bigcup_{\gamma > \alpha} M_\gamma,$$

where the bar denotes the closure in the weak topology.

Acknowledgement. The author thanks Dr. Nguyen Van Thu for his kindness and encouragement.

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Received on 20. 11. 1983