ON THE RATE OF CONVERGENCE IN THE CENTRAL LIMIT THEOREM FOR FUNCTIONS OF THE AVERAGE OF INDEPENDENT RANDOM VARIABLES

BY

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Abstract. We give the rate of convergence in the central limit theorem and the random central limit theorem for functions belonging to the class $\mathcal{G}$ of all real differentiable functions $g$ such that $g' \in L(1)$.

1. Introduction and notation. Let $\{X_k, k \geq 1\}$ be a sequence of independent random variables and put $S_n = \sum X_k, k = 1, 2, \ldots, n$. The asymptotical normality of $\{g(S_n/n), n \geq 1\}$, where $g$ is a real function, was considered for instance in [1] (Theorem 4.2.5, p. 76), [8] (Theorem 9.3.1, p. 259), [5], [6], and in [3] for random elements of a Hilbert space. We are interested in the rate convergence in law of the normalized sequence $\{g(S_n/n), n \geq 1\}$.

Throughout this paper we shall use the following notation:

- $\mathcal{G} = \text{the class of all real, differentiable functions } g \text{ such that } g' \text{ satisfies the Lipschitz condition, i.e.}$

\begin{equation}
|g'(x) - g'(y)| < L|x - y|,
\end{equation}

where $L$ is a positive constant;

- $\Phi = \text{the class of all functions } \varphi \text{ defined on } R \text{ for which }$

(a) $\varphi$ is nonnegative, even, and nondecreasing on $[0, \infty]$,  
(b) $x/\varphi(x)$ is defined for all $x$ and nondecreasing $[0, \infty]$;

- $\mathcal{D} = \text{the class of all sequences } \{d_n, n \geq 1\} \text{ of positive numbers such that } d_n \to \infty, n \to \infty,$

\[ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt, \]

$C$ denotes a positive constant.

Moreover, we shall often use the following results.
Lemma 1.1 ([7], p. 28). Assume that $X$ and $Y$ are random variables and $F(x) = P[X < x]$, $G(x) = P[X + Y < x]$. Then, for any $\varepsilon > 0$, $x \in \mathbb{R}$, and any distribution function $H$,

\begin{align*}
(i) \quad |G(x) - H(x)| & \leq \sup_{x} |F(x) - H(x)| + \\
& + \max \{ |H(x - \varepsilon) - H(x)|, |H(x + \varepsilon) - H(x)| \} + P[|Y| \geq \varepsilon].
\end{align*}

From (i) we get

Corollary 1.1. For any given $\varepsilon > 0$

\begin{align*}
(ii) \quad \sup_{x} |G(x) - \Phi(x)| & \leq \sup_{x} |F(x) - \Phi(x)| + \varepsilon/\sqrt{2\pi} + P[|Y| \geq \varepsilon].
\end{align*}

2. Uniform estimates. In what follows we need the following

Lemma 2.1. Let $Z$ be a random variable, and let $b, c \in \mathbb{R}$, $c \neq 0$. Then for every $d > 0$ and every $g \in \mathcal{G}$ with $g'(b/c) \neq 0$

\begin{align*}
2) \quad \sup_{x} |P \left\{ \frac{c}{g'(b/c)} \left[ g \left( \frac{Z}{c} \right) - g \left( \frac{b}{c} \right) \right] < x \right\} - \Phi(x)| \\
& \leq 5 \sup_{x} |P[Z - b < x] - \Phi(x)| + \frac{4}{d} \exp \{ -d^2/2 \} + \frac{Ld^2}{|cg'(b/c)| \sqrt{2\pi}}.
\end{align*}

Proof. Put

\begin{align*}
h(x) &= \begin{cases} 
\frac{g(x) - g(b/c)}{(x - b/c)g'(b/c)} & \text{if } x \neq b/c, \\
1 & \text{if } x = b/c.
\end{cases}
\end{align*}

We see that

\begin{align*}
\frac{c}{g'(b/c)} \left[ g \left( \frac{Z}{c} \right) - g \left( \frac{b}{c} \right) \right] = (Z - b) h \left( \frac{Z}{c} \right).
\end{align*}

Hence, by (ii), for any given $\varepsilon > 0$, we have

\begin{align*}
3) \quad \sup_{x} |P \left\{ \frac{c}{g'(b/c)} \left[ g \left( \frac{Z}{c} \right) - g \left( \frac{b}{c} \right) \right] < x \right\} - \Phi(x)| \\
& = \sup_{x} |P \{ Z - b + (Z - b)(h(Z/c) - 1) < x \} - \Phi(x)| \\
& \leq \sup_{x} |P[Z - b < x] - \Phi(x)| + \varepsilon/\sqrt{2\pi} + P[|Z - b| (h(Z/c) - 1) \geq \varepsilon].
\end{align*}

Note now that for any $d > 0$

\begin{align*}
4) \quad P \left[ |(Z - b)(h(Z/c) - 1)| \geq \varepsilon \right] & \leq P \left[ |Z - b| > d \right] + P \left[ |h(Z/c) - 1| \geq \varepsilon/d \right] \\
& \leq 2 \sup_{x} |P[Z - b < x] - \Phi(x)| + 2(1 - \Phi(d)) + P[|h(Z/c) - 1| \geq \varepsilon/d].
\end{align*}
On the rate of convergence

Taking into account the definition of \( h \) and (1), we get

\[
(5) \quad P[|h(Z/c) - 1| > \varepsilon/d] = P\left[ \left| \frac{g(Z/c) - g(b/c)}{Z/c - b/c} - 1 \right| \geq \varepsilon/d \right]
\]

\[
= P\left[ \left| \frac{g'(b/c + \theta(Z/c - b/c))}{g(b/c)} - 1 \right| \geq \varepsilon/d \right] \leq P\left[ |Z - b| \geq (\varepsilon/d) L^{-1} |cg'(b/c)| \right]
\]

\[
\leq 2 \sup_x P[|Z - b| < x] - \Phi(x) + 2 \left( 1 - \Phi \left( (\varepsilon/d) L^{-1} |cg'(b/c)| \right) \right)
\]

as \( 0 < \theta < 1 \).

Combining (3)-(5) we obtain

\[
(6) \quad \sup_x P\left\{ \frac{c}{g'(b/c)} \left[ g(Z/c) - g(b/c) \right] < x \right\} - \Phi(x) \leq 5 \sup_x P[|Z - b| < x] - \Phi(x) + 2 \left( 1 - \Phi \left( (\varepsilon/d) L^{-1} |cg'(b/c)| \right) \right) + \frac{\varepsilon}{\sqrt{2\pi}}.
\]

Putting, in (6), \( \varepsilon = \frac{Ld^2}{|cg'(b/c)|} \) we get (2).

**Corollary 2.1.** Let \( \{X_k, k \geq 1\} \) be a sequence of random variables, and let \( S_n = \sum X_k \) \((k = 1, 2, \ldots, n)\). Suppose that \( \{a_k, k \geq 1\} \), \( \{b_k, k \geq 1\} \) and \( \{c_k, k \geq 1\} \) are sequences of real numbers such that \( a_k > 0, c_k \neq 0, k \geq 1 \). Then for every \( d > 0 \) and every \( g \in \mathcal{G} \) with \( g'(b/c) \neq 0, k \geq 1 \),

\[
(7) \quad \sup_x P\left\{ \frac{c_k}{g'(b/c_n)} \left[ g\left( \frac{S_n}{a_k c_n} \right) - g\left( \frac{b_k}{c_n} \right) \right] < x \right\} - \Phi(x) \leq 5 \sup_x P\left[ \frac{S_n - b_k}{a_k c_n} < x \right] - \Phi(x) + \frac{4}{d \sqrt{2\pi}} \exp \left\{ -d^2/2 \right\} + \frac{Ld^2}{|c_n g'(b/c)|} \sqrt{2\pi}.
\]

**Corollary 2.2.** Let \( \{X_k, k \geq 1\} \) be a sequence of independent random variables with finite expectations \( EX_k \) and variances \( \sigma^2 X_k, k \geq 1 \). Then for every \( d > 0 \) and every \( g \in \mathcal{G} \) with \( g'(\mu_n) \neq 0 \), where \( \mu_n = n^{-1} \sum EX_k \) \((k = 1, 2, \ldots, n)\),

\[
(8) \quad \sup_x P\left\{ \frac{n}{s_n g'(\mu_n)} \left[ g\left( \frac{S_n}{n} \right) - g(\mu_n) \right] < x \right\} - \Phi(x) \leq 5 \sup_x P\left[ \frac{S_n - ES_n}{s_n} < x \right] - \Phi(x) + \frac{Ld^2 s_n}{n |g'(\mu_n)|} + \frac{4}{d \sqrt{2\pi}} \exp \left\{ -d^2/2 \right\},
\]

\[s_n^2 = \sum_{k=1}^{n} \sigma^2 X_k.\]
COROLLARY 2.3. Let \( \{X_k, k \geq 1\} \) be a sequence of independent identically distributed random variables with \( E X_1 = \mu, \sigma^2 X_1 = \sigma^2 < \infty \). Then for every \( d > 0 \) and every \( g \in \mathcal{G} \) with \( g'(\mu) \neq 0 \)

\[
\sup_x \left| P \left\{ \frac{\sqrt{n}}{\sigma} \left[ g \left( \frac{S_n}{n} \right) - g(\mu) \right] < x \right\} - \Phi(x) \right| 
\leq 5 \sup_x \left| P \left[ \frac{S_n - n\mu}{\sigma \sqrt{n}} < x \right] - \Phi(x) \right| + \frac{Ld^2 \sigma}{\sqrt{n |g'(\mu)|}} + \frac{4}{d \sqrt{2\pi}} \exp \left(-\frac{d^2}{2}\right).
\]

Put

\[
\mu_n = n^{-1} \sum_{k=1}^n EX_k, \quad s^2_n = \sum_{k=1}^n \sigma^2 X_k, \quad X^0_k = X_k - EX_k, \quad k \geq 1.
\]

Estimates (7)-(9) and the known estimates the convergence rate in the central limit theorem allow to obtain, among other things, the following results.

THEOREM 2.4. Let \( \{X_k, k \geq 1\} \) be a sequence of independent random variables such that \( E (X^0_k)^2 \varphi(X^0_k) < \infty, k \geq 1, \) for some \( \varphi \in \Phi. \)

Then for every \( g \in \mathcal{G} \) with \( g'(\mu) \neq 0, n \geq 1, \) and any sequence \( \{d_n, n \geq 1\} \in \mathcal{G} \)

\[
\sup_x \left| P \left\{ \frac{n}{s_n g'(\mu)} \left[ g \left( \frac{S_n}{n} \right) - g(\mu) \right] < x \right\} - \Phi(x) \right| 
= O \left( \sum_{k=1}^n E (X^0_k)^2 \varphi(X^0_k) \right) \left( \frac{s_n d^2_n}{s_n^2 \varphi(s_n)} + \frac{s_n d^2_n}{n |g'(\mu)|} d_n^{-1} \exp \left(-\frac{d^2_n}{2}\right) \right).
\]

If \( E |X^0_k|^3 < \infty, k \geq 1, \) then for every \( g \in \mathcal{G} \) with \( g'(\mu) \neq 0, n \geq 1, \) and any sequence \( \{d_n, n \geq 1\} \in \mathcal{G} \)

\[
\sup_x \left| P \left\{ \frac{n}{s_n g'(\mu)} \left[ g \left( \frac{S_n}{n} \right) - g(\mu) \right] < x \right\} - \Phi(x) \right| 
= O \left( \sum_{k=1}^n E |X^0_k|^3 \right) \left( \frac{s_n d^2_n}{s_n^3} + \frac{s_n d^2_n}{n |g'(\mu)|} d_n^{-1} \exp \left(-\frac{d^2_n}{2}\right) \right).
\]

COROLLARY 2.5. If \( \{X_k, k \geq 1\} \) is a sequence of independent identically distributed random variables, then under the assumptions of Theorem 2.4 for every \( g \in \mathcal{G} \) with \( g'(\mu) \neq 0, \) and any sequence \( \{d_n, n \geq 1\} \in \mathcal{G}, \) we have

\[
\sup_x \left| P \left\{ \frac{\sqrt{n}}{\sigma g'(\mu)} \left[ g \left( \frac{S_n}{n} \right) - g(\mu) \right] < x \right\} - \Phi(x) \right| 
= O \left( \frac{1}{\varphi(\sqrt{n}) + \frac{d^2_n}{\sqrt{n}}} + d_n^{-1} \exp \left(-\frac{d^2_n}{2}\right) \right),
\]
(11') \[ \sup_x \left| P \left\{ \sqrt{\frac{n}{\sigma g'(\mu)}} \left[ g \left( \frac{S_n}{n} \right) - g(\mu) \right] < x \right\} - \Phi(x) \right| = O \left( \frac{d_n^2}{\sqrt{n}} + d_n^{-1} \exp \{-d_n^2/2\} \right), \]

respectively.

Note now that putting in (10)
\[ d_n = \left\{ 2 \ln \left( 1 + \frac{s_n^2 \varphi(s_n)}{\sum_{k=1}^n E(X_k^0)^2 \varphi(X_k^0)} \right) \right\}^{1/2}, \]
and in (11)
\[ d_n = \left\{ 2 \ln \left( 1 + \frac{s_n^3}{\sum_{k=1}^n E|X_k^0|^3} \right) \right\}^{1/2}, \]
one can get the following estimates:

**Corollary 2.6.** Under the assumptions of Theorem 2.4 for every \( g \in \mathcal{G} \) with \( g'(\mu) \neq 0, \ n \geq 1, \)

(13) \[ \sup_x \left| P \left\{ \sqrt{\frac{n}{s_n g'(\mu)}} \left[ g \left( \frac{S_n}{n} \right) - g(\mu) \right] < x \right\} - \Phi(x) \right| = O \left( \frac{\sum_{k=1}^n E(X_k^0)^2 \varphi(X_k^0)}{s_n^2 \varphi(s_n) + \frac{s_n \ln \varphi(s_n)}{ng'(\mu)\sqrt{n}}} \right), \]

and

(14) \[ \sup_x \left| P \left\{ \sqrt{\frac{n}{s_n g'(\mu)}} \left[ g \left( \frac{S_n}{n} \right) - g(\mu) \right] < x \right\} - \Phi(x) \right| = O \left( \frac{\sum_{k=1}^n E|X_k^0|^3}{s_n^3} + \frac{s_n \ln s_n}{ng'(\mu)\sqrt{n}} \right), \]

respectively.

From (13) and (14) we get

**Corollary 2.7.** Under the assumptions of Corollary 2.5 we have

(10') \[ \sup_x \left| P \left\{ \sqrt{\frac{n}{\sigma g'(\mu)}} \left[ g \left( \frac{S_n}{n} \right) - g(\mu) \right] < x \right\} - \Phi(x) \right| = O \left( \frac{1}{\varphi(\sigma \sqrt{n})} + \frac{\ln \varphi(\sigma \sqrt{n})}{\sigma \sqrt{n}} \right), \]

(11') \[ \sup_x \left| P \left\{ \sqrt{\frac{n}{\sigma g'(\mu)}} \left[ g \left( \frac{S_n}{n} \right) - g(\mu) \right] < x \right\} - \Phi(x) \right| = O \left( \frac{\ln n}{\sqrt{n}} \right). \]
The estimate (9) allows us to give a generalization of a result given in paper [2]:

**Theorem 2.8.** Let \( \{X_k, k \geq 1\} \) be a sequence of independent identically distributed random variables with \( EX_1 = \mu, \sigma^2 X_1 = \sigma^2 < \infty, \) and \( E|X_1|^{2+\delta} < \infty, \) \( 0 < \delta < 1. \)

Then for every \( g \in \mathcal{G} \) with \( g'(\mu) \neq 0 \)

\[
\sum_{n=1}^{\infty} n^{-1+\delta/2} \sup_x P \left\{ \frac{\sqrt{n}}{\sigma g'(\mu)} \left[ g \left( \frac{S_n}{n} \right) - g(\mu) \right] < x \right\} - \Phi(x) < \infty.
\]

If \( E(X_1 - \mu)^2 \log(1 + |X_1 - \mu|^2) < \infty, \) then (15) converges with \( \delta = 0. \)

**Proof.** From (9) with \( d = \sqrt{\ln n}, \) we get

\[
\sup_x P \left\{ \frac{\sqrt{n}}{\sigma g'(\mu)} \left[ g \left( \frac{S_n}{n} \right) - g(\mu) \right] < x \right\} - \Phi(x) \leq C \left( \sup_x P \left[ \frac{S_n - n\mu}{\sigma \sqrt{n}} < x \right] - \Phi(x) + \frac{\ln n}{\sqrt{n}} \right).
\]

Moreover, we know [2] that

\[
\sum_{n=1}^{\infty} n^{-1+\delta/2} \sup_x P \left[ \frac{S_n - n\mu}{\sigma \sqrt{n}} < x \right] - \Phi(x) < \infty,
\]

which together with the obvious fact

\[
\sum_{n=1}^{\infty} (n^{-1+\delta/2}(\ln n)/\sqrt{n}) < \infty, \quad 0 \leq \delta < 1,
\]

allow us to obtain (15).

**3. Partial sums with random indices.** Following the consideration of Section 1 one can prove the following

**Lemma 3.1.** Let \( \{X_k, k \geq 1\} \) be a sequence of independent identically distributed random variables with \( EX_1 = \mu, \sigma^2 X_1 = \sigma^2 < \infty. \) Suppose that \( \{N_n, n \geq 1\} \) is a sequence of positive integer-valued random variables. Then for every \( d > 0, \varepsilon > 0, \) and every \( g \in \mathcal{G} \) with \( g'(\mu) \neq 0 \)

\[
\sup_x P \left\{ \frac{\sqrt{N_n}}{g'(\mu) \sigma} \left[ g \left( \frac{S_{N_n}}{N_n} \right) - g(\mu) \right] < x \right\} - \Phi(x) \leq 3 \sup_x P \left[ \frac{S_{N_n} - N_n\mu}{\sigma \sqrt{N_n}} < x \right] - \Phi(x) + P \left\{ \frac{|g'(\mu)|}{\sigma L}(\varepsilon/d)\sqrt{N_n} \right\} +
\]

\[
+ \frac{2}{d \sqrt{2\pi}} \exp \left\{ -d^2/2 \right\} + \varepsilon/\sqrt{2\pi}.
\]
Using Lemma 4.1 we can give the following results:

**Theorem 3.2.** Let \( \{X_k, k \geq 1\} \) be a sequence of independent identically distributed random variables with \( EX_1 = \mu \), \( \sigma^2 X_1 = \sigma^2 \), and \( E|X_1|^3 < \infty \). Suppose that \( \{N_n, n \geq 1\} \) is a sequence of positive integer-valued random variables such that

\[
P \left[ \frac{N_n - 1}{na} \geq \varepsilon_n \right] = O(\sqrt{\varepsilon_n}),
\]

where \( a \) is a positive constant, and \( 1/n \leq \varepsilon_n \to 0, n \to \infty \).

Then for every \( g \in \mathcal{G} \) with \( g'(\mu) \neq 0 \), and any sequence \( \{d_n, n \geq 1\} \)

\[
\sup_x P \left\{ \frac{\sqrt{N_n}}{\sigma g'(\mu)} \left[ g \left( \frac{S_{N_n}}{N_n} \right) - g(\mu) \right] < x \right\} - \Phi(x) = O(\sqrt{\varepsilon_n} + d_n^2/\sqrt{n} + d_n^{-1} \exp \{-d_n^2/2\}).
\]

**Proof.** Following the considerations of the proof of Lemma 2.1 and using (16) together with assumption (17) one can get

\[
\sup_x P \left\{ \frac{\sqrt{N_n}}{\sigma g'(\mu)} \left[ g \left( \frac{S_{N_n}}{N_n} \right) - g(\mu) \right] < x \right\} - \Phi(x) \leq C \sup_x P \left[ \frac{S_{N_n} - N_n \mu}{\sigma \sqrt{N_n}} < x \right] - \Phi(x) + d_n^2/\sqrt{n} + d_n^{-1} \exp \{-d_n^2/2\}
\]

for any sequence \( \{d_n, n \geq 1\} \in \mathcal{G} \), where \( C \) is a positive constant. But it has been proved in [4] that

\[
\sup_x P \left[ \frac{S_{N_n} - N_n \mu}{\sigma \sqrt{N_n}} < x \right] - \Phi(x) = O(\sqrt{\varepsilon_n}),
\]

hence we obtain (18).

**Corollary 3.3.** Under the assumptions of Theorem 3.2

\[
\sup_x P \left\{ \frac{\sqrt{N_n}}{\sigma g'(\mu)} \left[ g \left( \frac{S_{N_n}}{N_n} \right) - g(\mu) \right] < x \right\} - \Phi(x) = O \left( \sqrt{\varepsilon_n} + \frac{\ln n}{\sqrt{n}} \right).
\]

**Corollary 3.4.** If (7) hold, \( \varepsilon_n = (\ln^2 n)/n \), then

\[
\sup_x P \left\{ \frac{\sqrt{N_n}}{\sigma g'(\mu)} \left[ g \left( \frac{S_{N_n}}{N_n} \right) - g(\mu) \right] < x \right\} - \Phi(x) = O \left( \frac{\ln n}{\sqrt{n}} \right).
\]

**Theorem 3.5.** Let \( \{X_n, n \geq 1\} \) be a sequence of independent identically distributed random variables such that \( EX_1 = \mu, \sigma^2 X_1 = \sigma^2, E|X_1|^3 < \infty \), and \( \{\eta_n, n \geq 1\} \) be a sequence with \( n^{-1} \leq \eta_n \to \infty, n \to \infty \). Suppose that \( \{N_n, n \geq 1\} \) is a sequence of positive integer-valued random variables such that there
exist positive constants $c_1$, $c_2$ for which

\begin{equation}
P\left[\left|\frac{N_n}{\sqrt{\lambda_n}} - 1\right| > c_1 \eta_n\right] = O(\sqrt{n}),
\end{equation}

\begin{equation}
P\left[\lambda < \frac{c_2}{n \eta_n}\right] = O(\sqrt{n}),
\end{equation}

$\lambda$ being a random variable taking values in $(0, \infty)$ and independent of $\{X_k, k \geq 1\}$.

Then for every $g \in G$ with $g'(\mu) \neq 0$ and any sequence $\{d_n, n \geq 1\} \in \mathcal{D}$

\begin{equation}
\sup_x \left| P\left\{\sqrt{N_n} \left[ g\left(\frac{S_{N_n}}{N_n}\right) - g(\mu) \right] < x \right\} - \Phi(x) \right|
= O(\sqrt{n} d_n^2 + d_n^{-1} \exp \left\{-d_n^2/2\right\}).
\end{equation}

**Proof.** From (16) we have

\begin{equation}
\sup_x \left| P\left\{\sqrt{N_n} \left[ g\left(\frac{S_{N_n}}{N_n}\right) - g(\mu) \right] < x \right\} - \Phi(x) \right|
\leq 3 \sup_x \left| P\left[\frac{S_{N_n} - N_n \mu}{\sigma \sqrt{N_n}} < x \right] - \Phi(x) \right|
+ P\left[\left|\frac{S_{N_n} - N_n \mu}{\sigma \sqrt{N_n}} \right| \geq \frac{|g'(\mu)|}{\sigma L} \left(\epsilon_n d_n \sqrt{N_n} \right) \right]
+ \frac{2}{d_n \sqrt{2\pi}} \exp \left\{-d_n^2/2\right\} + \epsilon_n \sqrt{2\pi}
\end{equation}

for any given $\epsilon_n > 0$ and $\{d_n, n \geq 1\} \in \mathcal{D}$.

Note now that by (19) and (20) we have

\begin{equation}
P\left[\left|\frac{S_{N_n} - N_n \mu}{\sigma \sqrt{N_n}} \right| \geq \frac{|g'(\mu)|}{\sigma L} \left(\epsilon_n d_n \sqrt{N_n} \right) \right]
\leq C\left\{ P\left[\left|\frac{S_{N_n} - N_n \mu}{\sigma \sqrt{N_n}} \right| \geq \frac{|g'(\mu)|}{\sigma L} \left(\epsilon_n \sqrt{(1 - C_1 \eta_n) [c_2/\eta_n]/d_n} \right) + \sqrt{\eta_n}\right]\right\}
\leq C\left\{2 \sup_x \left| P\left[\frac{S_{N_n} - N_n \mu}{\sigma \sqrt{N_n}} < x \right] - \Phi(x) \right| + 2(1 - \Phi\left(\frac{|g'(\mu)|}{\sigma L} \epsilon_n \sqrt{(1 - C_1 \eta_n) [c_2/\eta_n]/d_n} \right) + \sqrt{\eta_n}\right\}.
\end{equation}

Putting

$$\epsilon_n = d_n^2 \left(\frac{|g'(\mu)|}{\sigma L} \sqrt{(1 - C_1 \eta_n) [c_2/\eta_n]} \right)$$
On the rate of convergence

and combining (22) and (23), we obtain

\[
\sup_x \left| P \left\{ \frac{\sqrt{N_n}}{\sigma g'(\mu)} \left[ g\left( \frac{S_n}{N_n} \right) - g(\mu) \right] < x \right\} - \Phi(x) \right| 
\leq C \left( \sup_x \left| P \left[ \frac{S_n - N_n \mu}{\sigma \sqrt{N_n}} < x \right] - \Phi(x) \right| + \sqrt{\eta_n} d_n^2 + d_n^{-1} \exp\left\{ -d_n^2/2 \right\} \right).
\]

Using now [4] the estimate

\[
\sup_x \left| P \left[ \frac{S_n - N_n \mu}{\sigma \sqrt{N_n}} < x \right] - \Phi(x) \right| = O(\sqrt{\eta_n}),
\]

we obtain (21)

**Corollary 3.6.** Under the assumptions of Theorem 3.5 for every \( g \in \mathcal{G} \) with \( g'(\mu) \neq 0 \)

\[
\sup_x \left| P \left\{ \frac{\sqrt{N_n}}{\sigma g'(\mu)} \left[ g\left( \frac{S_n}{N_n} \right) - g(\mu) \right] < x \right\} - \Phi(x) \right| = O(\sqrt{\eta_n} \ln(1/\eta_n)).
\]

**References**


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