Abstract. Let \( \{Y_n, n \geq 1\} \) be a sequence of independent positive random variables, defined on a probability space \((\Omega, \mathcal{A}, P)\), with a common distribution function \( F \). Put
\[
Y_n^* = \inf(Y_1, Y_2, \ldots, Y_n), \quad m \geq 1 \quad \text{and} \quad S_n = \sum_{m=1}^{n} Y_n^*, \quad n \geq 1.
\]
In this paper mixing limit theorem for the sums \( S_n, n \geq 1 \), is given and the random central limit theorem is proved.

1. Introduction and results. Let \( \{Y_n, n \geq 1\} \) be a sequence of independent positive random variables with a common distribution function \( F \). Let us put
\[
Y_n^* = \inf(Y_1, Y_2, \ldots, Y_m), \quad m \geq 1, \quad \text{and} \quad S_n = \sum_{m=1}^{n} Y_n^*, \quad n \geq 1.
\]
The three convergences: in probability, almost sure and in law were established in [4]-[7] for sums \( S_n \) of infima of independent random variables uniformly distributed on \([0, 1]\). The almost sure invariance principle was investigated in [8].

Now, let \( \{Y_n, n \geq 1\} \) be a sequence of independent positive random variables with a common distribution function \( F \) such that
\[
\int_{0}^{b} \left| F(x) - \frac{x}{b} \right| x^{-2} \, dx < \infty \quad \text{for} \quad 0 < b < \infty.
\]

T. Höglund proved in [9] the following central limit theorem:

**Theorem 0.** Under assumption (1)
\[
\lim_{n \to \infty} P(Z_n < x) = \Phi(x),
\]
where

\[ Z_n = \frac{S_n - b \log n}{b \sqrt{2 \log n}}, \quad n > 1, \]

(2)

\[ S_n = \sum_{k=1}^{n} Y_k^*, \quad Y_k^* = \inf(Y_1, Y_2, \ldots, Y_k), \quad k \geq 1, \quad n \geq 1, \]

and \( \Phi \) is the standard normal distribution function.

In this paper we give a mixing limit theorem and a random central limit theorem for \( \{Z_n, n > 1\} \).

**Theorem 1.** (i) Under the assumptions of Theorem 0 the sequence \( \{Z_n, n > 1\} \) is mixing, i.e.

\[ \lim_{n \to \infty} P(Z_n < x | B) = \Phi(x) \]

for any event \( B \in \mathcal{A} \) such that \( P(B) > 0 \).

(ii) Let \( \{N_n, n \geq 1\} \) be a sequence of positive integer-valued random variables such that

\[ N_n/a_n \overset{p}{\to} \lambda \quad \text{as } n \to \infty, \]

(3)

where \( \lambda \) is a positive random variable dependent only on finitely many \( Y_n, n \geq 1 \), and \( \{a_n, n \geq 1\} \) is a sequence of positive numbers tending to \( +\infty \). Then

\[ \lim_{n \to \infty} P(Z_{N_n} < x) = \Phi(x). \]

2. **Proofs of results.** In the proof of Theorem 1 we apply some lemmas given by Deheuvels [5] and Höglund [9]. For the sake of completeness we present them in Section 3.

**Proof of Theorem 1.** (i) Let \( \{Z_n, n > 1\} \) be defined by (2) and let \( Y_{m,n}^* = \inf(Y_{m+1}, \ldots, Y_n) \) for \( n > m \). Denote by \( A_k \) the event \( \{Z_k < x\} \) for \( k \geq n_0 \), where \( n_0 \) is such that \( P(A_k) > 0 \) for all \( k \geq n_0 \). We prove that the sequence \( \{Z_n, n > 1\} \) is mixing.

By Theorem 1 ([10], p. 406) it is sufficient to show that

\[ \lim_{n \to \infty} P(A_n | A_k) = \Phi(x), \quad k \geq n_0, \]

(5)

as, by Theorem 0, \( \lim_{n \to \infty} P(A_n | \Omega) = \Phi(x) \). Since

\[ Z_n = \frac{S_k}{b \sqrt{2 \log n}} + \sum_{i=k+1}^{n} \frac{(Y_i^* - Y_{i-1}^*)}{b \sqrt{2 \log n}} + \sum_{i=k+1}^{n} \frac{Y_i^*}{b \sqrt{2 \log n}}, \]


we have $S_k/b \sqrt{2 \log n} \to 0$ a.s. as $n \to \infty$, and, by Lemma 3.4,

$$
\sum_{i=k+1}^{n} (Y_i - Y_i^*)/b \sqrt{2 \log n} \to 0 \text{ a.s. as } n \to \infty.
$$

The random variables $\sum Y_i^*$ are independent of $S_k$ for every $k \geq n_0$, so, by Theorem 0, we immediately obtain (5) and the proof of (i) is completed.

(ii) To prove that $P(Z_{n_k} < x) \to \Phi(x)$ as $n \to \infty$ for every $\{N_n, n \geq 1\}$ satisfying (3), it is sufficient to note that the sequence $\{Z_n, n \geq 1\}$ satisfies assumptions of Theorem 3 in [3].

By (i) and since the random variable $\lambda$ depends only on finitely many $Y_{n_k}, n \geq 1$, we have

$$
\lim_{n \to \infty} P(Z_n < x | A) = \Phi(x)
$$

for all $A \in \mathcal{F}_{\lambda}$, where $\mathcal{F}_{\lambda}$ is the $\sigma$-field generated by the random variable $\lambda$.

Now we show that $\{Z_n, n \geq 1\}$ satisfies the generalized Anscombe's condition with the norming sequence $\{k_n = n, n \geq 1\}$, i.e. that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$
\limsup_{n \to \infty} P_A(\max_{(1-\delta)n < i < (1+\delta)n} |Z_n - Z_i| \geq \varepsilon) \leq \varepsilon P(A)
$$

holds for every $A \in \mathcal{F}_{\lambda}$, where $P_A(B) = P(A \cap B)$.

If we write $D_{\delta}(\varepsilon) = \{i: (1-\delta)n < i < (1+\delta)n\}$, then by a simple estimation we obtain

$$
\max_{i \in D_{\delta}(\varepsilon)} |Z_n - Z_i| = \max_{i \in D_{\delta}(\varepsilon)} \left| \frac{S_n - b \log n}{b \sqrt{2 \log n}} - \frac{S_i - b \log i}{b \sqrt{2 \log i}} \right|
\leq \max_{i \in D_{\delta}(\varepsilon)} \left| \frac{S_n}{b \sqrt{2 \log n}} - \frac{S_i}{b \sqrt{2 \log i}} \right| + \max_{i \in D_{\delta}(\varepsilon)} \left| \frac{\log n}{\sqrt{2 \log n}} - \frac{\log i}{\sqrt{2 \log i}} \right|
\leq \max_{i \in D_{\delta}(\varepsilon)} \left( \frac{S_n}{b \sqrt{2 \log n}} - \frac{S_i}{b \sqrt{2 \log i}} + \frac{\log n}{\sqrt{2 \log n}} - \frac{\log i}{\sqrt{2 \log i}} \right)
+ \frac{1}{\sqrt{2}} \max_{i \in D_{\delta}(\varepsilon)} \left( \sqrt{\log n} - \sqrt{\log i}, \sqrt{\log i} - \sqrt{\log n} \right)
\leq \max \left( \frac{S_n}{b \sqrt{2 \log n}} - \frac{S_{[n(1-\delta)]}}{b \sqrt{2 \log n}} + \frac{S_{[n(1+\delta)]}}{b \sqrt{2 \log (n(1+\delta))}} - \frac{S_n}{b \sqrt{2 \log n}} \right)
+ \frac{1}{\sqrt{2}} \max \left( \sqrt{\log n} - \sqrt{\log (n(1-\delta))}, \sqrt{\log (n(1+\delta))} - \sqrt{\log n} \right)
\begin{align*}
&\leq \max \left( S_{[m(1-\delta)]} \left( \frac{1}{b \sqrt{2 \log n}} - \frac{1}{b \sqrt{2 \log n(1+\delta)}} \right) + \sum_{k=[m(1-\delta)]+1}^{n} \frac{Y_k^*}{b \sqrt{2 \log n(1-\delta)}} \right) + \\
&\quad \left( \frac{1}{b \sqrt{2 \log n(1-\delta)}} - \frac{1}{b \sqrt{2 \log n(1+\delta)}} \right) \sum_{k=n+1}^{[n(1+\delta)]} \frac{Y_k^*}{b \sqrt{2 \log n(1-\delta)}} + \max \left( \frac{S_{[m(1-\delta)]} - b_n}{b \log n(1-\delta)}, \frac{S_n - b_n'}{b \log n(1-\delta)} \right) + c_n,
\end{align*}

where

\begin{align*}
b_n &= \log n(1-\delta) \left[ \frac{1}{\sqrt{2 \log n}} - \frac{1}{\sqrt{2 \log n(1+\delta)}} \right], \\
b_n' &= \log n \left[ \frac{1}{\sqrt{2 \log n(1-\delta)}} - \frac{1}{\sqrt{2 \log n}} \right], \\
c_n &= \frac{1}{\sqrt{2}} (\sqrt{\log n(1+\delta)} - \sqrt{\log n(1-\delta)}).
\end{align*}

It is easy to see that \( b_n \to 0, b_n' \to 0 \) and \( c_n \to 0 \) as \( n \to \infty \).

Now let \( \{X_n, n \geq 1\} \) be a sequence of independent random variables uniformly distributed on \([0, 1]\).

Put \( G(t) = \inf \{x \geq 0: F(x) \geq t\} \). Then, by [6], the sequences \( \{G(X_n), n \geq 1\} \) and \( \{Y_n, n \geq 1\} \) are the same in law.

Furthermore, the sequence \( S_n = \sum_{k=1}^{n} Y_k^* \) may be represented as \( \bar{S}_n \)

\[ \bar{S}_n = \sum_{k=1}^{n} G(X_k^*), \] where \( X_k^* = \inf(X_1, X_2, \ldots, X_k), k \geq 1. \)

On the other hand, Höglund [9] proved that

\[ \frac{\sum_{k=1}^{n} G(X_k^*) - b \log n}{b \sqrt{2 \log n}} = \frac{\sum_{k=1}^{n} X_k^* - \log n}{\sqrt{2 \log n}} + r_n \]

holds in law, where \( r_n \to^p 0 \) as \( n \to \infty \). Therefore, by Lemma 3.1,

\begin{equation}
\frac{\bar{S}_{[m(1-\delta)]}}{b \log n(1-\delta)} - b_n = \frac{\bar{S}_{[m(1-\delta)]}}{b \log n(1-\delta)} b_n + r_n b_n \to 0, \text{ a.s. as } n \to \infty
\end{equation}
and

\[ \frac{S_n}{b \log n} b_n' = \frac{S_n}{\log n} b_n' + r_n b_n' \to 0 \text{ a.s. as } n \to \infty, \]

where \( S_n = \sum_{k=1}^{n} X_k^* \), \( n \geq 1 \). So, by (8)-(10) we get

\[ \left[ \max_{i \in D_n(\delta)} |Z_n - Z_i| \geq \varepsilon \right] \leq \left[ \frac{\sum_{k=[n(1 - \delta)]+1}^{[n(1 + \delta)]} Y^*_k}{b \sqrt{2 \log n(1 - \delta)}} \geq \frac{\varepsilon}{2} \right] \]

for any \( \varepsilon > 0 \) and sufficiently large \( n \).

Observe that

\[ \sum_{k=[n(1 - \delta)]+1}^{[n(1 + \delta)]} Y^*_k = \sum_{k=[n(1 - \delta)]+1}^{[n(1 + \delta)]} (Y^*_k - Y^*_{[n(1 - \delta)]}) + \sum_{k=[n(1 - \delta)]+1}^{[n(1 + \delta)]} Y^*_{[n(1 - \delta)]} \cdot \]

By Lemma 3.4 and the fact that the random variables \( \lambda \) and

\[ \sum_{k=[n(1 - \delta)]+1}^{[n(1 + \delta)]} Y^*_{[n(1 - \delta)]} \]

are independent for sufficiently large \( n \), one can check that condition (7) is a consequence of the following well-known Anscombe condition:

\[ \limsup_{n \to \infty} P \left( \max_{i \in D_n(\delta)} |Z_n - Z_i| \geq \delta \right) \leq \varepsilon. \]

By (11), Lemma 3.3, the Markoff inequality and Lemma 3.2 we obtain

\[ P \left( \max_{i \in D_n(\delta)} |Z_n - Z_i| \geq \varepsilon \right) \leq \frac{\sum_{k=[n(1 - \delta)]+1}^{[n(1 + \delta)]} Y^*_k}{b \sqrt{2 \log n(1 - \delta)}} \geq \frac{\varepsilon}{2} \]

\[ \leq P \left[ \sqrt{2 \log n(1 - \delta)} \geq \frac{E \left( \sum_{k=[n(1 - \delta)]+1}^{[n(1 + \delta)]} X^*_k \right)}{3 \varepsilon \sqrt{2 \log n(1 - \delta)}} \right] = \frac{O(1)}{\sqrt{2 \log n(1 - \delta)}} \to 0 \text{ as } n \to \infty. \]

Hence, from Theorem 3 of [3], we immediately obtain (4) for every \( \{N_n, n \geq 1\} \) satisfying (3). This completes the proof of Theorem 1.

3. Lemmas. In this section we present some lemmas we needed in the proofs of Theorem 1.
lemma 3.1. Let \( \{X_n, n \geq 1\} \) be a sequence of independent random variables uniformly distributed on \([0, 1]\). Then \( S_n \log n \to 1 \) a.s. as \( n \to \infty \), where \( S_n = \sum_{k=1}^{n} X^*_k \) and \( X^*_k = \inf(X_1, X_2, \ldots, X_k) \), \( k \geq 1, n \geq 1 \).

lemma 3.2. \( EX^*_k = (k+1)^{-1} \) \( (k \geq 1) \), \( E S_n - \log n = O(1) \).

lemma 3.3. Under the assumptions of Theorem 0

\[
\frac{n}{b \sqrt{2 \log n}} = \frac{X^*_k - \log n}{\sqrt{2 \log n}} + r_n \text{ in law,}
\]

where \( r_n \to 0 \) as \( n \to \infty \), and

\[
\sum_{k=1}^{n} y_k |G(X^*_k) - b X^*_k| < \sqrt{\log n} \to 0 \text{ as } n \to \infty,
\]

where, for \( 0 < \delta < 1 \), \( y_k = 1 \) if \( X^*_k \leq \delta \) and \( y_k = 0 \) if \( X^*_k > \delta \), and \( G(t) = \inf\{x \geq 0 : F(x) \geq t\} \).

lemma 3.4. Let \( \{Y_n, n \geq 1\} \) be a sequence of positive independent random variables with the common distribution function \( F \) such that \( F(x) = 0 \) for \( x \leq 0 \), \( F(x) > 0 \) for \( x > 0 \). Let us put \( Y^*_n = \inf(Y_1, \ldots, Y_n) \), \( Y^*_{m,n} = \inf(Y_{m+1}, \ldots, Y_n) \), \( n > m, n \geq 1 \).

Then the sum \( \sum_{m+1}^{n} (Y^*_{m,n} - Y^*_n) \) converges almost surely.

Proof. We observe that

\[
0 \leq Y^*_{m,n} - Y^*_n \leq \begin{cases} 0 & \text{if } Y^*_{m,n} \leq Y^*_n, \\ Y^*_n & \text{if } Y^*_{m,n} > Y^*_n. 
\end{cases}
\]

Then

\[
\sum_{n=m+1}^{\infty} (Y^*_{m,n} - Y^*_n) \leq \sum_{n=m+1}^{\infty} \sum_{n=m+1}^{\infty} Y^*_{m,n} \mathbb{1}_{[Y^*_{m,n} > Y_n]}.
\]

Now, it is sufficient to show that

\[
\lim_{K \to \infty} P(\sum_{n=m+1}^{\infty} Y^*_{m,n} \mathbb{1}_{[Y^*_{m,n} > Y_n]} \geq K) = 0.
\]

Indeed,

\[
\lim_{K \to \infty} P(\sum_{n=m+1}^{\infty} Y^*_{m,n} \mathbb{1}_{[Y^*_{m,n} > Y_m]} \geq K)
= \left( \lim_{K \to \infty} P(\sum_{n=m+1}^{\infty} Y^*_{m,n} \mathbb{1}_{[Y^*_{m,n} > c]} \geq K) \right) P_{Y^*_m}(dC) = 0
\]
by

\[
\lim_{K \to \infty} \mathbb{P}\left( \sum_{n=m+1}^{\infty} Y_{m,n}^* \mathbb{1}_{Y_{m,n}^* > C} \geq K \right) = 0 \quad \text{for every } C > 0,
\]

and \( \mathbb{P}(Y_m = C) = 0 \) for \( C = 0 \).

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REFERENCES


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