CONVERGENCE OF 2-DIMENSIONAL $h$-PROCESSES

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Abstract. Suppose that $D \subset C$ is a simply connected domain and $p$ is a minimal Martin boundary point. Assume that there exists a curve in $D$ which converges to $p$ in the Martin topology and to $z \in C$ in the Euclidean topology. Then the same holds for almost all $h$-paths, where $h$ is a minimal harmonic function represented by $p$. In such a case almost all $h$-paths have finite lifetime. This permits to define a Brownian excursion law in $D$ starting from such a point $p$.

1. Introduction. The purpose of this note is to present probabilistic consequences of a theorem, proved by Jackson [6], which might have escaped the attention of non-specialists in potential theory.

Recently Cranston and McConnell [4] have proved that the expected lifetime of an $h$-process in a 2-dimensional domain with finite area is finite. This has been generalized by Cranston [3] to multidimensional domains with Lipschitz boundaries.

It will be shown that 2-dimensional $h$-paths converge a.s. in a simply connected domain under very mild assumptions. In a sense, one cannot assume less, see Remark 2.1 (ii) (a). It will be proved as a corollary that for a class of harmonic functions $h$, in simply connected domains (of possibly infinite area), almost all $h$-paths have finite lifetime. This implies in turn that for a suitable Martin boundary point $p$ there exists a "standard" Brownian excursion law starting from $p$.

The inquiry resulting in the present note was stimulated by an unpublished example of Michael Cranston, who considered $h$-paths in a domain with infinite area.

The reader is referred to Doob [5] and Ohtsuka [8] for definitions of an $h$-process (i.e. "conditioned Brownian motion"), minimal-fine topology, prime ends and related concepts.

The author is grateful to Professor Christian Pommerenke for the most valuable suggestions and to Professor Makoto Ohtsuka for pointing out the results of Jackson [6].
2. Convergence of h-paths.

Proposition 2.1. Let \( D \subset \mathbb{C} \) be a simply connected domain. If a minimal harmonic function \( h \) is represented by a prime end \( p_h \), then the cluster set of almost every \( h \)-path coincides with the set of all principal points of \( p_h \).

Proof. Almost every \( h \)-path converges to \( p_h \) in the Martin topology ([5], p. 691) and, therefore, its cluster set must contain all principal points of \( p_h \) (cf. [8], p. 270).

Moreover, almost all \( h \)-paths converge to \( p_h \) in the minimal-fine topology ([5], 3 III 3). By Theorem 3.2 (see also Remark 3.1) of Jackson [6] their cluster sets must be equal to the set of all principal points of \( p_h \).

Corollary 2.2. Let \( D \subset \mathbb{C} \) be a simply connected domain. If a minimal harmonic function \( h \) is represented by an accessible prime end \( p_h \), then almost all \( h \)-paths converge (in the Euclidean topology) to the unique principal point of \( p_h \).

Remarks 2.1. (i) The corollary may be restated in the following form:

If at least one (non-random) path in \( D \) converges to \( p_h \) and terminates at \( z \in \partial D \), then the same is true for almost all \( h \)-paths.

(ii) Consider the following generalization of the problem. Let \( D \subset \mathbb{R}^n \), \( n \geq 2 \), be an open set and suppose that a curve \( \Gamma = \{ \Gamma(t), 0 < t < 1 \} \subset D \) converges as \( t \to 1 \) to a minimal Martin boundary point \( p_h \) and it converges also to a point \( z \in \partial D \) in the Euclidean topology. Does it follow that almost all \( h \)-paths converge to \( z \)?

(a) Yes, if \( D \subset \mathbb{C} \) is finitely connected.

(b) Not necessarily for general \( D \). Almost all \( h \)-paths may converge to \( z \) or to \( z_1 \in \partial D, z_1 \neq z \), or they may have no limit in the Euclidean topology. In dimensions \( n \geq 3 \) these kinds of behaviour may occur even if \( D \) is homeomorphic to a ball.

The observation (a) is a relatively easy corollary of Theorem 2.1. Examples illustrating (b) would require a lot of space (cf. [9]).

3. Applications.

Theorem 3.1. Let \( D \subset \mathbb{C} \) be a simply connected domain. If a minimal harmonic function \( h \) is represented by a prime end \( p_h \) and the set of its all principal points is bounded, then almost all \( h \)-paths have finite lifetime. In other words, \( p_h \) is an attainable minimal Martin boundary point.

Proof. Fix the probability measure. Let \( A \) be the set of all principal points of \( p_h \) and

\[ B = \{ z \in \mathbb{C} : |x - z| \geq 1 \text{ for all } x \in A \}. \]

Denote the last exit time from \( B \) as \( L_B \). The lifetime of an \( h \)-path \( X(\cdot) \) will be called \( R \), i.e.

\[ R = \inf \{ t : \lim_{s \to t} X(s) \in \partial D \}. \]
Proposition 2.1 shows that $L_B < R$ a.s.; in particular, $L_B < \infty$ a.s.

The process $\{X(L_B + t), 0 < t < R - L_B\}$ is an $h_1$-process for some harmonic function $h_1$ on the bounded domain $D - B$ (see [7]). Theorem 1 of Cranston and McConnell [4], applied to this process, shows that its lifetime is finite a.s., that is $R - L_B < \infty$ a.s. and, therefore, $R < \infty$ a.s.

**Corollary 3.1.** If a prime end is accessible, then it is attainable.

**Remarks 3.1.** (i) One may assume in Theorem 3.1 that $D$ is finitely connected (see Remark 2.1 (ii) (a)).

(ii) Theorem 3.1 extends Theorem 1 of Cranston and McConnell [4] to some $h$-processes and domains of infinite area. It does not guarantee however that the expected lifetime of $h$-paths is finite.

The next result is a generalization of Lemma 4.1 and Remark 4.2 (ii) of Burdzy [2]. In order to keep this note compact, the reader is referred to that paper for definitions and notation. Excursion laws below have Brownian transition probabilities.

**Proposition 3.1.** Let $D_1, D_2 \subseteq C$ be simply connected regions and $f : D_1 \rightarrow D_2$ be a conformal bijection. Suppose that $p$ is a prime end of $D_1$, $H^p$ is a standard excursion law in $D_1$, and $f(p)$ is a prime end such that the set of its all principal points is bounded. Then the $f$-mapping of $H^p$ is well-defined excursion law in $D_2$.

**Proof.** The proof of Lemma 4.1 in [2] may be repeated with the only change that Theorem 3.1 should be used instead of the result of Cranston and McConnell [4].

**Corollary 3.2.** Let $D \subseteq C$ be a simply connected region and $p$ be a prime end such that the set of its all principal points is bounded. Then there exists a (unique) standard non-null excursion law $H^p$ in $D$. If $p$ is accessible and $x \in C$ is its principal point, then there exists a (unique) standard non-null excursion law $H^x$ in $D$.

**Proof.** Let $f : \{\text{Re}z > 0\} \rightarrow D$, $f(0) = p$, be a conformal bijection and $H^0$ be a standard non-null excursion law in $\{\text{Re}z > 0\}$. Then $H^p (H^0)$ may be defined as the $f$-mapping of $H^0$ by Proposition 3.1. It is easy to see, by Proposition 2.1, that $H^p (H^0)$ has the desired properties.

An alternative proof is supplied by Theorem 4.1 of Burdzy [1] and the fact that $p$ is attainable by Theorem 3.1.

**REFERENCES**


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Received on 11. 11. 1985